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UNIQUENESS RESULTS FOR DIFFERENTIAL POLYNOMIALS WEIGHTED SHARING A VALUE OR A SET OF ROOTS OF UNITY

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Abstract. We study the uniqueness results of meromorphic functions f and g if differential polynomials of the type $(P(f)^n)^{(k)}$ and $(P(g)^n)^{(k)}$ weighted share a set of roots of unity or a value, where P is a polynomial of one variable. The results of the paper generalize some earlier results due to Khoai and An [Advanced Studies Euro-Tbilisi Math. J., **15** (2022), 39–51] and Sahoo and Sultana [An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.), Tomul LXX, 2024, f.1].

1. Introduction and main results

In this paper, by meromorphic function we shall always mean a meromorphic function in the complex plane. We adopt the standard notations in the Nevanlinna Theory of meromorphic functions as explained in [2, 16, 17]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence.

For any non-constant meromorphic function f, we denote by $S(r,f)$ any quantity satisfying $S(r,f) = \circ(T(r,f))$ as $r \to \infty$, $r \notin E$. A meromorphic function a is said to be small with respect to f if $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions with respect to f. Clearly $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ is a field over the set of complex numbers.

For any two non-constant meromorphic functions f and g, and $a \in S(f) \cap S(g)$, we say that f and g share a CM(IM) provided that $f-a$ and $g-a$ have the same zeros counting (ignoring) multiplicities.

In 1997 Yang and Hua [3] studied the uniqueness problem for meromorphic functions when $(f^n)'$ and $(g^n)'$ share a non-zero value. Bhoosnurmath and Dyavanal [1] extended the Yang-Hua's [3] result and proved the following theorem.

THEOREM 1. [1] *Let f and g be two non-constant meromorphic functions and n, k be positive integers with $n > 3k+8$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then one of the following holds:*

1. *$f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three non-zero constants such that $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$.*
2. *$f = tg$ for some $t \in \mathbb{C}$ such that $t^n = 1$.*

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To state the next result we need the following set sharing concept of functions.

For a set $S \subseteq \mathbb{C}$ and a meromorphic function f, we define $E_f(S) = \bigcup_{a \in S}\{z|f(z) - a = 0, \text{counting multiplicities}\}$, $\overline{E}_f(S) = \bigcup_{a \in S}\{z|f(z) - a = 0, \text{ignoring multiplicities}\}$. If $E_f(S) = E_g(S)$ $\left(\overline{E}_f(S) = \overline{E}_g(S)\right)$, then we say that f and g share S CM (IM). Apparently, if S contains only one element, then it coincides with the standard definition of CM (respectively, IM) shared value.

In 2018, An and Khoai [6] extended the result of Yang and Hua [3] by considering set sharing instead of value sharing and proved the following theorem for uniqueness of meromorphic functions.

THEOREM 2. [6] *Let f and g be two non-constant meromorphic functions. Let k, d, n be three positive integers with $n > 2k + \frac{2k+8}{d}$, $d \geqslant 2$ and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share S CM, then one of the following holds:*

1. $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three non-zero constants such that $(-1)^{kd}(c_1 c_2)^{nd}(nc)^{2kd} = 1$.

2. $f = tg$ for some $t \in \mathbb{C}$ such that $t^{nd} = 1$.

Regarding Theorem 2, a natural question to ask is the following:

QUESTION 1. Can CM be replace by IM keeping the same conclusion?

In 2020 Dilip et al. [12] answer the Question 1 positively and proved the following theorem.

THEOREM 3. [12] *Let f and g be two non-constant meromorphic functions. Let k, d, n be three positive integers with $n > 2k + \frac{8k+14}{d}$, $d \geqslant 2$ and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share S IM, then one of the following holds:*

1. $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three non-zero constants such that $(-1)^{kd}(c_1 c_2)^{nd}(nc)^{2kd} = 1$.

2. $f = tg$ for some $t \in \mathbb{C}$ such that $t^{nd} = 1$.

Now, we recall the idea of weighted sharing, which appeared in the literature in ([8, 9]). This concept encourages more open discussions about sharing. We explain this in the following definition.

DEFINITION 1. [8, 9] Let m be a non-negative integer or infinity and $a \in S(f)$. We denote by $E_m(a, f)$ the set of all zeros of $f - a$, where a zero of multiplicity k is counted k times if $k \leqslant m$ and $m+1$ times if $k > m$. If $E_m(a, f) = E_m(a, g)$, we say that f, g share the function a with weight m and we write f and g share (a, m). Since $E_m(a, f) = E_m(a, g)$ implies that $E_s(a, f) = E_s(a, g)$ for any integer s $(0 \leqslant s < m)$, if f, g share (a, m), then f, g share (a, s), $(0 \leqslant s < m)$. Moreover, we note that f and g share the function a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) respectively.

DEFINITION 2. Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and m be a non-negative integer. We denote by $E_f(S,m)$ the set $E_f(S,m)=\bigcup_{a\in S}E_m(a,f)$. We say that f and g share the set S with weight m if $E_f(S,m)=E_g(S,m)$ and we write f and g share (S,m).

The following theorem was established in 2020 by Lahiri and Sinha [11], who also addressed some gaps in Theorems 2.

THEOREM 4. [11] *Let f and g be two non-constant meromorphic functions sharing $(\infty,0)$. Let k, d, n be three positive integers with $n>\max\{3,2k+\frac{2k+8}{d}\}$, $d\geqslant 2$ and $S=\{a\in\mathbb{C}:a^d=1\}$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share $(S,2)$, then one of the following holds:*

1. *$f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three non-zero constants satisfying $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd}=1$.*
2. *$f=tg$ for some $t\in\mathbb{C}$ such that $t^{nd}=1$.*

The scenario where two differential polynomials share the set of roots of unity with relax weighted sharing was explored in 2022 by Pramanik and Roy [13]. They demonstrated the following theorem.

THEOREM 5. [13] *Let f and g be two non-constant meromorphic functions sharing $(\infty,0)$. Let k $(\geqslant 1)$, m $(\geqslant 0)$, d $(\geqslant 2)$, n $(\geqslant 1)$ be integers, and $S=\{a\in\mathbb{C}:a^d=1\}$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share (S,m) with one of the following conditions:*

(i) $m\geqslant 2$ and

$$n>\max\left\{3,2k+\frac{2k+8}{d}\right\},$$

(ii) $m=1$ and

$$n>\max\left\{3,2k+\frac{3k+9}{d}\right\},$$

(iii) $m=0$ and

$$n>\max\left\{3,2k+\frac{8k+14}{d}\right\},$$

then one of the following holds:

1. *$f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three non-zero constants such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd}=1$.*
2. *$f=tg$ for some $t\in\mathbb{C}$ such that $t^{nd}=1$.*

Let P be a polynomial of degree q,

$$P(z)=(z-a_1)^{m_1}(z-a_2)^{m_2}\ldots\ldots(z-a_l)^{m_l}, \tag{1}$$

where $a_1,a_2,\ldots,a_l$ are distinct zeros of P, and $m_1+m_2+\ldots+m_l=q$.

DEFINITION 3. [5] A polynomial $P(z)$ is called a strong uniqueness polynomial for meromorphic (entire) functions if for any two non-constant meromorphic (entire) functions f and g, and a non-zero constant c, the condition $P(f) = cP(g)$ implies $f = g$.

The following findings were recently proved by Khoai and An [7], who did not accept the premise that f and g share ∞, as suggested by [11].

THEOREM 6. [7] *Let f and g be two non-constant meromorphic functions, P be a polynomial of degree q (> 0) of the form* (1) *and n $(\geqslant 1)$, k $(\geqslant 1)$ be integers with $n > 3k + \frac{4l+4}{q}$. If $(P(f)^n)^{(k)}$ and $(P(g)^n)^{(k)}$ share 1 CM, then one of the following holds:*

1. $l = 1$; $f = a_1 + c_1e^{cz}$ and $g = a_1 + c_2e^{-cz}$ for three non-zero constants c_1, c_2, c such that $(-1)^k(c_1c_2)^{nq}(nqc)^{2k} = 1$.

2. $l = 1$; $f - a_1 = t(g - a_1)$ with $t^{nq} = 1$, $t \in \mathbb{C}$.

3. $l \geqslant 2$; $P(f) = tP(g)$ with $t^n = 1$, $t \in \mathbb{C}$. Further if P is a strong uniqueness polynomial then, $f = g$.

In particular, when $P(z) = z^q + az^{q-p} + b$ with $a, b \neq 0$ and $q \geqslant 2p + 9$, we have $g = hf$ for a constant h such that $h^q = 1$ and $h^p = 1$ if $p \geqslant 2$; $f = g$ if $p \geqslant 2$, $(q, p) = 1$.

THEOREM 7. [7] *Let f and g be two non-constant meromorphic functions, P be a polynomial of degree q (> 0) of the form* (1)*. Let n $(\geqslant 1)$, k $(\geqslant 1)$ be integers with $n > \max\{\frac{4k+5l-2}{2q}, 2k + \frac{4l+4+2kq}{qd}\}$, $d \geqslant 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $(P(f)^n)^{(k)}$ and $(P(g)^n)^{(k)}$ share S CM then one of the following holds:*

1. $l = 1$; $f = a_1 + c_1e^{cz}$ and $g = a_1 + c_2e^{-cz}$ for three non-zero constants c_1, c_2, c such that $(-1)^k(c_1c_2)^{nq}(nqc)^{2k} = 1$.

2. $l = 1$; $f - a_1 = t(g - a_1)$ with $t^{nq} = 1$, $t \in \mathbb{C}$.

3. $l \geqslant 2$; $P(f) = tP(g)$ with $t^n = 1$, $t \in \mathbb{C}$. Further if P is a strong uniqueness polynomial then, $f = g$.

In particular, when $P(z) = z^q + az^{q-p} + b$ with $a, b \neq 0$ and $q \geqslant 2p + 9$, we have $g = hf$ for a constant h such that $h^q = 1$ and $h^p = 1$ if $p \geqslant 2$; $f = g$ if $p \geqslant 2$, $(q, p) = 1$.

Sahoo and Sultana [14] investigated the uniqueness problem of meromorphic functions in 2024, when two differential polynomials share a value or a set of roots of unity with finite weight. The outcomes of Sahoo and Sultana are listed below.

THEOREM 8. [14] *Let f and g be two non-constant meromorphic functions, P be a polynomial of degree q (> 0) of the form* (1) *and n $(\geqslant 1)$, k $(\geqslant 1)$ be integers with $n > 3k + \frac{4l+4}{q}$. If $(P(f)^n)^{(k)}$ and $(P(g)^n)^{(k)}$ share $(1, 2)$, then one of the following holds:*

1. $l = 1$; $f = a_1 + c_1e^{cz}$ and $g = a_1 + c_2e^{-cz}$ for three non-zero constants c_1, c_2, c such that $(-1)^k(c_1c_2)^{nq}(nqc)^{2k} = 1$.

2. $l = 1$; $f - a_1 = t(g - a_1)$ with $t^{nq} = 1$, $t \in \mathbb{C}$.

3. *$l \geqslant 2$; $P(f) = tP(g)$ with $t^n = 1$, $t \in \mathbb{C}$. Further if P is a strong uniqueness polynomial then, $f = g$.*

In particular, when $P(z) = z^q + az^{q-p} + b$ with $a, b \neq 0$ and $q \geqslant 2p+9$, we have $g = hf$ for a constant h such that $h^q = 1$ and $h^p = 1$ if $p \geqslant 2$; $f = g$ if $p \geqslant 2$, $(q, p) = 1$.

THEOREM 9. [14] *Let f and g be two non-constant meromorphic functions, P be a polynomial of degree q (> 0) of the form (1). Let n $(\geqslant 1)$, k $(\geqslant 1)$ be integers with $n > \max\{\frac{2k+2l-1}{q}, 2k + \frac{4l+4+2kq}{qd}\}$, $d \geqslant 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $(P(f)^n)^{(k)}$ and $(P(g)^n)^{(k)}$ share $(S, 2)$, then one of the following holds:*

1. *$l = 1$; $f = a_1 + c_1 e^{cz}$ and $g = a_1 + c_2 e^{-cz}$ for three non-zero constants c_1, c_2, c such that $(-1)^k (c_1 c_2)^{nq} (nqc)^{2k} = 1$.*

2. *$l = 1$; $f - a_1 = t(g - a_1)$ with $t^{nq} = 1$, $t \in \mathbb{C}$.*

3. *$l \geqslant 2$; $P(f) = tP(g)$ with $t^n = 1$, $t \in \mathbb{C}$. Further if P is a strong uniqueness polynomial then, $f = g$.*

In particular, when $P(z) = z^q + az^{q-p} + b$ with $a, b \neq 0$ and $q \geqslant 2p+9$, we have $g = hf$ for a constant h such that $h^q = 1$ and $h^p = 1$ if $p \geqslant 2$; $f = g$ if $p \geqslant 2$, $(q, p) = 1$.

Regarding Theorems 6–9, it is natural to ask the following question:

QUESTION 2. Is it possible to relax the nature of sharing of value or set?

We establish the following theorems in order to respond the Question 2.

THEOREM 10. *Let f and g be two non-constant meromorphic functions and P be a polynomial of degree q (> 0) of the form (1). Let n $(\geqslant 1)$, k $(\geqslant 1)$ be integers and m be non-negative integer or ∞. If $(P(f)^n)^{(k)}$ and $(P(g)^n)^{(k)}$ share $(1, m)$ with one of the following conditions:*

(i) $m \geqslant 2$ and

$$n > 3k + \frac{4l+4}{q}, \tag{2}$$

(ii) $m = 1$ and

$$n > \frac{7k}{2} + \frac{k+9l+15}{2q}, \tag{3}$$

(iii) $m = 0$ and

$$n > 6k + \frac{3k+7l+7}{q}, \tag{4}$$

then one of the following holds:

1. *$l = 1$; $f = a_1 + c_1 e^{cz}$ and $g = a_1 + c_2 e^{-cz}$ for three non-zero constants c_1, c_2, c such that $(-1)^k (c_1 c_2)^{nq} (nqc)^{2k} = 1$.*

2. *$l = 1$; $f - a_1 = t(g - a_1)$ with $t^{nq} = 1$, $t \in \mathbb{C}$.*

3. *$l \geqslant 2$; $P(f) = tP(g)$ with $t^n = 1$, $t \in \mathbb{C}$. Further if P is a strong uniqueness polynomial then, $f = g$.*

In particular, when $P(z) = z^q + az^{q-p} + b$ *with* $a, b \neq 0$ *and* $q \geqslant 2p+9$, *we have* $g = hf$ *for a constant* h *such that* $h^q = 1$ *and* $h^p = 1$ *if* $p \geqslant 2$; $f = g$ *if* $p \geqslant 2$, $(q, p) = 1$.

THEOREM 11. *Let* f *and* g *be two non-constant meromorphic functions and* P *be a polynomial of degree* q (> 0) *of the form* (1). *Let* n $(\geqslant 1)$, k $(\geqslant 1)$ *be integers,* m *be non-negative integer or* ∞, *and* $S = \{a \in \mathbb{C} : a^d = 1\}$. *If* $(P(f)^n)^{(k)}$ *and* $(P(g)^n)^{(k)}$ *share* (S, m) *with one of the following conditions:*

(i) $m \geqslant 2$ *and*

$$n > \max\left\{2k + \frac{4l+4+2kq}{qd}, \frac{2k+2l-1}{q}\right\}, \tag{5}$$

(ii) $m = 1$ *and*

$$n > \max\left\{2k + \frac{3kq+k+9l+9}{2dq}, \frac{2k+2l-1}{q}\right\}, \tag{6}$$

(iii) $m = 0$ *and*

$$n > \max\left\{2k + \frac{5kq+3k+7l+7}{dq}, \frac{2k+2l-1}{q}\right\}, \tag{7}$$

then one of the following holds:

1. $l = 1$; $f = a_1 + c_1 e^{cz}$ *and* $g = a_1 + c_2 e^{-cz}$ *for three non-zero constants* c_1, c_2, c *such that* $(-1)^k (c_1 c_2)^{nq} (nqc)^{2k} = 1$.
2. $l = 1$; $f - a_1 = t(g - a_1)$ *with* $t^{nq} = 1$, $t \in \mathbb{C}$.
3. $l \geqslant 2$; $P(f) = tP(g)$ *with* $t^n = 1$, $t \in \mathbb{C}$. *Further if* P *is a strong uniqueness polynomial then,* $f = g$.

In particular, when $P(z) = z^q + az^{q-p} + b$ *with* $a, b \neq 0$ *and* $q \geqslant 2p+9$, *we have* $g = hf$ *for a constant* h *such that* $h^q = 1$ *and* $h^p = 1$ *if* $p \geqslant 2$; $f = g$ *if* $p \geqslant 2$, $(q, p) = 1$.

REMARK 1. We can easily see that Theorem 10 and Theorem 11 generalize and improve Theorem 6, Theorem 8 and Theorem 7, Theorem 9 respectively.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G, $\tilde{F}$ and $\tilde{G}$ be non-constant meromorphic functions and H, $\tilde{H}$ be two functions which are defined as follows:

$$H := \left(\frac{F^{(2)}}{F^{(1)}} - 2\frac{F^{(1)}}{F-1}\right) - \left(\frac{G^{(2)}}{G^{(1)}} - 2\frac{G^{(1)}}{G-1}\right) \tag{8}$$

and

$$\tilde{H} := \left(\frac{\tilde{F}^{(2)}}{\tilde{F}^{(1)}} - 2\frac{\tilde{F}^{(1)}}{\tilde{F}-1}\right) - \left(\frac{\tilde{G}^{(2)}}{\tilde{G}^{(1)}} - 2\frac{\tilde{G}^{(1)}}{\tilde{G}-1}\right).$$

LEMMA 1. [15, 17] *Let* f *be a non-constant meromorphic function and let* $a_0, a_1, \ldots, a_n$ $(\not\equiv 0)$ *be small functions with respect to* f. *Then*

$$T\left(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_0\right) = nT(r,f) + S(r,f).$$

LEMMA 2. [17] *Let* f *be a non-constant meromorphic function and let* k *be a positive integer. Then*

$$T(r, f^{(k)}) \leqslant (k+1)T(r,f) + S(r,f).$$

LEMMA 3. [14] *Let* f *be a non-constant meromorphic function and* k, n *be positive integers with* $n \geqslant k+2$, *and let* $P(z)$ *be a polynomial of degree* q (>0). *If* $a \in \mathbb{C} \setminus \{0\}$, *then*

$$\frac{n-k-2}{n+k} T(r,f) \leqslant \overline{N}\left(r, \frac{1}{(P(f)^n)^{(k)} - a}\right) + S(r,f).$$

LEMMA 4. [7] *Let* f *be a non-constant meromorphic function and* k, n *be positive integers with* $n > 2k$, *and let* $P(z)$ *be a polynomial of degree* q (>0). *Then*
(i)

$$(n-2k)qT(r,f) + kN(r,P(f)) + N\left(r, \frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right) \leqslant T\left(r, (P(f)^n)^{(k)}\right) + S(r,f).$$

(ii)

$$\begin{aligned} N\left(r, \frac{P(f)^{n-k}}{(P(f)^n)^{(k)}}\right) &\leqslant kqT(r,f) + k\overline{N}(r,P(f)) + S(r,f) \\ &= kqT(r,f) + k\overline{N}(r,f) + S(r,f). \end{aligned}$$

LEMMA 5. [10] *Let* f *and* g *be two non-constant meromorphic functions and let* n, k *be integers with* $n > 2k$. *If* $(f^n)^{(k)}.(g^n)^{(k)} = 1$, *then* f *and* g *are transcendental entire functions such that* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are three non-zero constants such that* $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$.

LEMMA 6. [13] *Let* m *be a non-negative integer or* ∞. F *and* G *be non-constant meromorphic functions sharing* $(1,m)$ *and* H *as defined in* (8) *such that* $H \not\equiv 0$.
(i) *If* $m \geqslant 2$, *then*

$$T(r,F) \leqslant 2\overline{N}(r,F) + 2\overline{N}(r,G) + N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + S(r,F) + S(r,G).$$

(ii) *If* $m = 1$, *then*

$$\begin{aligned} T(r,F) \leqslant &\frac{5}{2}\overline{N}(r,F) + 2\overline{N}(r,G) + N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) \\ &+ \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + S(r,F) + S(r,G). \end{aligned}$$

(iii) If $m=0$, then

$$T(r,F) \leqslant 4\overline{N}(r,F)+3\overline{N}(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right) +2\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G).$$

The same inequality holds for $T(r,G)$.

LEMMA 7. [14] *Let f and g be two non-constant meromorphic functions, and $P(z)$ be a polynomial of degree q (>0) of the form (1), and k, n be positive integers with $n>\frac{2k+2l-1}{q}$. If $(P(f)^n)^{(k)}=(P(g)^n)^{(k)}$, then $P(f)=tP(g)$ with $t^n=1$.*

LEMMA 8. [4] *Let p,q be positive integer, $c,d,e,u,v,t\in\mathbb{C}$ be non-zero constants, and let $q\geqslant 2p+4$, and either $p\geqslant 2$, $(p,q)=1$, or $q\geqslant 4$. Suppose that (f,g) is a non-constant meromorphic solution of the equation*

$$cf^q+df^{q-p}+e=ug^q+vg^{q-p}+t.$$

Then $t=e$ and there exist a non-zero constant h, such that $g=hf$, with $h^q=\frac{c}{u}$, $h^{q-p}=\frac{d}{v}$.

3. Proof of the Main Theorems

Proof of Theorem 10. Let

$$F_1:=P(f),\ F:=(P(f)^n)^{(k)} \quad\text{and}\quad G_1:=P(g),\ G:=(P(g)^n)^{(k)}.$$

Then we have $(P(f)^n)^{(k)}=P(f)^{n-k}L(f)$ and $(P(g)^n)^{(k)}=P(g)^{n-k}L(g)$, where L is a differential polynomial. By Lemmas 1, 2 and 4, we have

$$(n-2k)qT(r,f)\leqslant T(r,F)+S(r,f)\leqslant (k+1)nqT(r,f)+S(r,f),$$

and

$$(n-2k)qT(r,g)\leqslant T(r,G)+S(r,g)\leqslant (k+1)nqT(r,g)+S(r,g).$$

Therefore $S(r,F)=S(r,f)$ and $S(r,G)=S(r,g)$.

We see that, each pole of F is of order $\geqslant n+k\geqslant 2$, because it is a pole of $P(f)$. By (ii) of Lemma 4 we get

$$N_2(r,F)=2\overline{N}(r,f)\leqslant 2T(r,f)+S(r,f),$$

and

$$\overline{N}\left(r,\frac{1}{F_1}\right)\leqslant\sum_{i=1}^{l}\overline{N}\left(r,\frac{1}{f-a_i}\right)\leqslant lT(r,f)+S(r,f), \tag{9}$$

$$\begin{aligned}N_2\left(r,\frac{1}{F}\right) &\leqslant N_2\left(r,\frac{1}{F_1^{n-k}}\right)+N\left(r,\frac{1}{L(f)}\right)\\ &\leqslant 2\overline{N}\left(r,\frac{1}{F_1}\right)+N\left(r,\frac{1}{L(f)}\right)+S(r,f)\\ &\leqslant 2lT(r,f)+k\overline{N}(r,F_1)+kqT(r,f)+S(r,f)\\ &=(kq+2l)T(r,f)+k\overline{N}(r,F_1)+S(r,f).\end{aligned} \tag{10}$$

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F}\right) &\leqslant \overline{N}\left(r,\frac{1}{F_1^{n-k}}\right)+N\left(r,\frac{1}{L(f)}\right)\\ &\leqslant \overline{N}\left(r,\frac{1}{F_1}\right)+N\left(r,\frac{1}{L(f)}\right)+S(r,f)\\ &\leqslant lT(r,f)+k\overline{N}(r,F_1)+kqT(r,f)+S(r,f)\\ &=(kq+l)T(r,f)+k\overline{N}(r,F_1)+S(r,f).\end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned}N_2\left(r,\frac{1}{G}\right) &\leqslant 2lT(r,g)+N\left(r,\frac{1}{L(g)}\right)+S(r,g)\\ &\leqslant (kq+2l)T(r,g)+k\overline{N}(r,G_1)+S(r,g),\end{aligned} \tag{12}$$

$$\overline{N}\left(r,\frac{1}{G}\right) \leqslant (kq+l)T(r,g)+k\overline{N}(r,G_1)+S(r,g). \tag{13}$$

Case 1: $H\not\equiv 0$. Next, using Lemma 6, we obtain the subsequent subcases:

Subcase 1.1: If $m\geqslant 2$, then

$$\begin{aligned}T(r,F) \leqslant\ & N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2\overline{N}(r,F)\\ &+2\overline{N}(r,G)+S(r,F)+S(r,G).\end{aligned} \tag{14}$$

By using (9)–(13) in (14) we get

$$\begin{aligned}T(r,F) \leqslant\ & (kq+2l+2)T(r,f)+k\overline{N}(r,F_1)+(2l+2)T(r,g)+N\left(r,\frac{1}{L(g)}\right)\\ &+S(r,f)+S(r,g).\end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned}T(r,G) \leqslant\ & (kq+2l+2)T(r,g)+k\overline{N}(r,G_1)+(2l+2)T(r,f)+N\left(r,\frac{1}{L(f)}\right)\\ &+S(r,f)+S(r,g).\end{aligned} \tag{16}$$

Adding (15) and (16) we obtain

$$T(r,F)+T(r,G)\leqslant (kq+4l+4)(T(r,f)+T(r,g))+k(\overline{N}(r,F_1)+\overline{N}(r,G_1))$$
$$+N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(g)}\right)+S(r,f)+S(r,g). \tag{17}$$

By Lemma 4 we get

$$(n-2k)qT(r,f)+kN(r,F_1)+N\left(r,\frac{1}{L(f)}\right)\leqslant T(r,F)+S(r,f), \tag{18}$$

and

$$(n-2k)qT(r,g)+kN(r,G_1)+N\left(r,\frac{1}{L(g)}\right)\leqslant T(r,G)+S(r,g). \tag{19}$$

From (17), (18) and (19) we get

$$\begin{aligned}&q(n-2k)(T(r,f)+T(r,g))+k(N(r,F_1)+N(r,G_1))\\&+N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(g)}\right)\\ \leqslant\ &(kq+4l+4)(T(r,f)+T(r,g))+k(\overline{N}(r,F_1)+\overline{N}(r,G_1))\\&+N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(g)}\right)+S(r,f)+S(r,g).\end{aligned}$$

Moreover, $\overline{N}(r,F_1)+\overline{N}(r,G_1)\leqslant N(r,F_1)+N(r,G_1)$. Therefore $(n-2k)q\leqslant kq+4l+4$ $\Rightarrow n\leqslant\ 3k+\frac{4l+4}{q}$, which contradicts (2).

Subcase 1.2: $m=1$, then

$$T(r,F)\leqslant N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\frac{5}{2}\overline{N}(r,F)$$
$$+2\overline{N}(r,G)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,F)+S(r,G). \tag{20}$$

By using (9)–(13) in (20), we get

$$\begin{aligned}T(r,F)\leqslant\ &(kq+2l)T(r,f)+k\overline{N}(r,F_1)+2lT(r,g)+N\left(r,\frac{1}{L(g)}\right)\\&+\frac{5}{2}\overline{N}(r,f)+2\overline{N}(r,g)+\frac{(kq+l)}{2}T(r,f)+\frac{k}{2}\overline{N}(r,F_1)+S(r,f)+S(r,g)\\ \leqslant\ &\frac{(3kq+k+5l+5)}{2}T(r,f)+k\overline{N}(r,F_1)+(2l+2)T(r,g)+N\left(r,\frac{1}{L(g)}\right)\\&+S(r,f)+S(r,g).\end{aligned} \tag{21}$$

Similarly,

$$\begin{aligned}T(r,G)\leqslant\ &\frac{(3kq+k+5l+5)}{2}T(r,g)+k\overline{N}(r,G_1)+(2l+2)T(r,f)+N\left(r,\frac{1}{L(f)}\right)\\&+S(r,f)+S(r,g).\end{aligned} \tag{22}$$

Adding (21) and (22) we obtain

$$T(r,F)+T(r,G)\leqslant\left(\frac{(3kq+k+5l+5)}{2}+2l+2\right)(T(r,f)+T(r,g))$$
$$+k(\overline{N}(r,F_1)+\overline{N}(r,G_1))+N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(g)}\right)$$
$$+S(r,f)+S(r,g). \tag{23}$$

From (18), (19) and (23) we get

$$q(n-2k)(T(r,f)+T(r,g))+k(N(r,F_1)+N(r,G_1))$$
$$+N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(g)}\right)$$
$$\leqslant\left(\frac{(3kq+k+5l+5)}{2}+2l+2\right)(T(r,f)+T(r,g))+k(\overline{N}(r,F_1)+\overline{N}(r,G_1))$$
$$+N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(g)}\right)+S(r,f)+S(r,g)$$

$$\Rightarrow (n-2k)q\{T(r,f)+T(r,g)\}\leqslant\left(\frac{(3kq+k+5l+5)}{2}+2l+2\right)\{T(r,f)+T(r,g)\}$$
$$+S(r,f)+S(r,g).$$

Therefore $(n-2k)q\leqslant\left(\frac{(3kq+k+5l+5)}{2}+2l+2\right)\Rightarrow n\leqslant\frac{7k}{2}+\frac{k+9l+15}{2q}$, which contradicts (3).

Subcase 1.3: $m=0$, then

$$T(r,F)\leqslant 4\overline{N}(r,F)+3\overline{N}(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2\overline{N}\left(r,\frac{1}{F}\right)$$
$$+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G) \tag{24}$$

By using (9)–(13) in (20) we get

$$T(r,F)\leqslant(3kq+4l+2k+4)T(r,f)+k\overline{N}(r,F_1)+(3+3l+kq+k)T(r,g)$$
$$+N\left(r,\frac{1}{L(g)}\right)+S(r,f)+S(r,g). \tag{25}$$

Similarly,

$$T(r,G)\leqslant(3kq+4l+2k+4)T(r,g)+k\overline{N}(r,G_1)+(3+3l+kq+k)T(r,f)$$
$$+N\left(r,\frac{1}{L(f)}\right)+S(r,f)+S(r,g). \tag{26}$$

Adding (25) and (26) we obtain

$$T(r,F)+T(r,G) \leqslant (4kq+3k+7l+7)(T(r,f)+T(r,g))+k(\overline{N}(r,F_1)+\overline{N}(r,G_1)) \\ +N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(g)}\right)+S(r,f)+S(r,g). \tag{27}$$

From (18), (19) and (27) we get

$$\begin{aligned} &q(n-2k)(T(r,f)+T(r,g))+k(N(r,F_1)+N(r,G_1)) \\ &+N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(g)}\right) \\ \leqslant\ &(4kq+3k+7l+7)(T(r,f)+T(r,g))+k(\overline{N}(r,F_1)+\overline{N}(r,G_1)) \\ &+N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(g)}\right)+S(r,f)+S(r,g), \end{aligned}$$

$$\Rightarrow (n-2k)q\{T(r,f)+T(r,g)\} \leqslant (4kq+3k+7l+7)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

Therefore $(n-2k)q \leqslant (4kq+3k+7l+7) \Rightarrow n \leqslant 6k+\frac{3k+7l+7}{q}$, which contradicts (4).

Case 2: $H \equiv 0$. So on integrating twice we get

$$\frac{1}{G-1}=\frac{A}{F-1}+B,$$

where $A\ (\neq 0)$ and B are constants.

Thus

$$G=\frac{(B+1)F+(A-B-1)}{BF+(A-B)}, \tag{28}$$

and

$$F=\frac{(B-A)G+(A-B-1)}{BG-(B+1)}. \tag{29}$$

Next we consider the following three subcases:

Subcase 2.1: $B \neq 0,-1$. Then from (29) we have

$$\overline{N}\left(r,\frac{1}{G-\frac{B+1}{B}}\right)=\overline{N}(r,F).$$

By Nevanlinna second fundamental theorem

$$\begin{aligned} T(r,G) &\leqslant \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-\frac{B+1}{B}}\right)+S(r,G) \\ &\leqslant \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+S(r,G) \end{aligned}$$

$$\Rightarrow T(r,G) \leqslant (l+1)T(r,g)+N\left(r,\frac{1}{L(g)}\right)+T(r,f)+S(r,f)+S(r,g). \tag{30}$$

If $A-B-1\neq 0$, then it follows from (28) that

$$\overline{N}\left(r,\frac{1}{F-\frac{-A+B+1}{B+1}}\right)=\overline{N}\left(r,\frac{1}{G}\right).$$

Again by Nevanlinna second fundamental theorem we have

$$\begin{aligned} T(r,F) &\leqslant \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{-A+B+1}{B+1}}\right)+S(r,F)\\ &\leqslant \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G), \end{aligned}$$

$$\begin{aligned} \Rightarrow T(r,F) \leqslant (l+2)T(r,f)+N\left(r,\frac{1}{L(f)}\right)+k\overline{N}(r,G_1)\\ +(kq+l)T(r,g)+S(r,f)+S(r,g). \end{aligned} \tag{31}$$

By (18), (19), (30) and (31) we get

$$\begin{aligned} &q(n-2k)(T(r,f)+T(r,g))+k(N(r,F_1)+N(r,G_1))\\ &+N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(g)}\right)\\ \leqslant\ &(l+3)T(r,f)+(qk+2l+1)T(r,g)+k(\overline{N}(r,F_1)+\overline{N}(r,G_1))\\ &+N\left(r,\frac{1}{L(f)}\right)+N\left(r,\frac{1}{L(g)}\right)+S(r,f)+S(r,g), \end{aligned}$$

which gives

$$\begin{aligned} \left(n-2k-\frac{l+3}{q}\right)T(r,f)+\left(n-3k-\frac{2l+1}{q}\right)T(r,g) \leqslant S(r,f)+S(r,g)\\ \Rightarrow \left(n-3k-\frac{2l+1}{q}\right)(T(r,f)+T(r,g)) \leqslant S(r,f)+S(r,g), \end{aligned}$$

which contradicts our assumptions (2)–(4).

Therefore $A-B-1=0$. Then by (28)

$$\overline{N}\left(r,\frac{1}{F+\frac{1}{B}}\right)=\overline{N}(r,G).$$

By Nevanlinna second fundamental theorem and Lemma 4 we get

$$\begin{aligned} T(r,F) &\leqslant \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F+\frac{1}{B}}\right)+S(r,F)\\ &\leqslant \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+S(r,F) \end{aligned}$$

$$\Rightarrow T(r,F) \leqslant \overline{N}(r,f) + lT(r,f) + N\left(r, \frac{1}{L(f)}\right) + \overline{N}(r,g)$$
$$+S(r,f) + S(r,g). \tag{32}$$

Adding (18), (19) and using (32), (30) we get

$$\begin{aligned} &q(n-2k)(T(r,f)+T(r,g)) + k(N(r,F_1)+N(r,G_1)) \\ &+N\left(r,\frac{1}{L(f)}\right) + N\left(r,\frac{1}{L(g)}\right) \\ \leqslant\ &(l+2)(T(r,f)+T(r,g)) + k(\overline{N}(r,F_1)+\overline{N}(r,G_1)) \\ &+N\left(r,\frac{1}{L(f)}\right) + N\left(r,\frac{1}{L(g)}\right) + S(r,f) + S(r,g). \end{aligned}$$

Therefore we obtain

$$\left(n - 2k - \frac{l+2}{q}\right)\{T(r,f)+T(r,g)\} \leqslant S(r,f) + S(r,g),$$

which violates assumptions (2)–(4).

Subcase 2.2: $B=-1$. Then

$$G = \frac{A}{A+1-F},$$

and

$$F = \frac{(1+A)G - A}{G}.$$

If $A+1 \neq 0$,

$$\overline{N}\left(r, \frac{1}{F-(A+1)}\right) = \overline{N}(r,G),$$

$$\overline{N}\left(r, \frac{1}{G - \frac{A}{A+1}}\right) = \overline{N}\left(r, \frac{1}{F}\right).$$

By similar argument as Subcase 2.1 we have a contradiction.

Therefore $A+1=0$ then $FG=1$. and so we get $(P(f)^n)^{(k)}.(P(g)^n)^{(k)}=1$. Let $l=1$, then $P(z)=(z-a_1)^q$. Therefore we have

$$((f-a_1)^{nq})^{(k)}.((g-a_1)^{nq})^{(k)} = 1.$$

Using Lemma 5 and noting that $n>2k$, we obtain $f=a_1+c_1e^{cz}$ and $g=a_1+c_2e^{-cz}$ for three non-zero constants c_1,c_2,c such that $(-1)^k(c_1c_2)^{nq}(nqc)^{2k}=1$, which is (i). Next suppose that $l \geqslant 2$ and since $n>2k$, by Lemma 5, we obtain $P(f)=c_1e^{cz}$ and $P(g)=c_2e^{-cz}$ for three non-zero constants c_1,c_2,c such that $(-1)^k(c_1c_2)^{nq}(nqc)^{2k}=1$. From this it follows that f and g are entire function. Since $P(z)=(z-a_1)_1^{m_1}(z-a_2)^{m_2}\ldots(z-a_l)^{m_l}$ for $l \geqslant 2$, by Picard's theorem $P(f)$ and $P(g)$ must have a zero, which is a contradiction.

Subcase 2.3: $B=0$. Then (28) and (29) gives $G=\frac{F+A-1}{A}$ and $F=AG+1-A$. If $A-1\neq 0$, $\overline{N}\left(r,\frac{1}{A-1+F}\right)=\overline{N}(r,\frac{1}{G})$ and $\overline{N}\left(r,\frac{1}{G-\frac{A-1}{A}}\right)=\overline{N}(r,\frac{1}{F})$. Proceeding similarly as in Subcase 2.1, we get a contradiction.

Therefore, $A-1=0$ then $F\equiv G$ i.e., $(P(f)^n)^{(k)}=(P(g)^n)^{(k)}$. Using Lemma 7 and $n>\frac{2k+2l-1}{q}$, we obtain $P(f)=tP(g)$ with $t^n=1$. Further, if P is a strong uniqueness polynomial then, $f=g$. If $l=1$, then $(f-a_1)=t_1(g-a_1)$ with $t_1^{nq}=1, t_1\in\mathbb{C}$.

We now consider the case when $P(z)=z^q+az^{q-p}+b$ with $a,b\neq 0$. By the similar argument as in Lemma 7, we see that there exist a t such that $P(f)=tP(g)$, i.e.

$$f^q+af^{q-p}+b=t(g^q+ag^{q-p}+b).$$

Applying Lemma 8 and using the similar argument as in the proof of Theorem 1 [7], we get $f=g$. This completes the proof. □

Proof of Theorem 11. Let

$$F:=(P(f)^n)^{(k)},\ F_1:=P(f),\ \tilde{F}:=F^d \text{ and } G:=(P(g)^n)^{(k)},\ G_1:=P(g),\ \tilde{G}:=G^d.$$

Then we have $(P(f)^n)^{(k)}=P(f)^{n-k}L(f)$ and $(P(g)^n)^{(k)}=P(g)^{n-k}L(g)$, where L is a differential polynomial.

Since

$$\overline{E}(F;1)=\cup_{j=0}^{d-1}\overline{E}((P(f)^n)^{(k)});u^j),$$

where $1,u^1,u^2,\ldots,u^{d-1}$ are distinct roots of $z^d=1$, from Lemma 3 with the value 1, it implies that $\{(P(f)^n)^{(k)}\}^d=1$ has infinitely many solutions. Therefore $E(S,(P(f)^n)^{(k)})\neq\phi$ and $E(S,(P(g)^n)^{(k)})\neq\phi$. Since $(P(f)^n)^{(k)}$ and $(P(g)^n)^{(k)}$, share (S,m), we see that $\tilde{F}$ and $\tilde{G}$ share $(1,m)$.

We see that, each pole of F is of order $\geqslant n+k\geqslant 2$, because it is a pole of $P(f)$. Now, if a is a zero of $(f^n)^{(k)}$ then $F(a)=0$ with multiplicity of zero $\geqslant 2$.

By (ii) of Lemma 4 we get

$$N_2(r,\tilde{F})=2\overline{N}(r,f)\leqslant 2T(r,f)+S(r,f),$$

and

$$\overline{N}\left(r,\frac{1}{F_1}\right)\leqslant\sum_{i=1}^{l}\overline{N}\left(r,\frac{1}{f-a_i}\right)\leqslant lT(r,f)+S(r,f), \tag{33}$$

$$\begin{aligned} N_2\left(r,\frac{1}{\tilde{F}}\right)&=2\overline{N}\left(r,\frac{1}{\tilde{F}}\right)=2\overline{N}\left(r,\frac{1}{F}\right)\\ &\leqslant 2\overline{N}\left(r,\frac{1}{F_1^{n-k}}\right)+2\overline{N}\left(r,\frac{1}{L(f)}\right)\\ &\leqslant 2\overline{N}\left(r,\frac{1}{F_1}\right)+2\overline{N}\left(r,\frac{1}{L(f)}\right)+S(r,f)\\ &\leqslant 2lT(r,f)+2\overline{N}\left(r,\frac{1}{L(f)}\right)+S(r,f). \end{aligned} \tag{34}$$

Similarly,

$$N_2\left(r,\frac{1}{\tilde{G}}\right)=2\overline{N}\left(r,\frac{1}{\tilde{G}}\right)=2\overline{N}\left(r,\frac{1}{G}\right)$$
$$\leqslant 2lT(r,g)+2\overline{N}\left(r,\frac{1}{L(g)}\right)+S(r,g). \tag{35}$$

Case 1: $\tilde{H}\not\equiv 0$. Then by Lemma 6, we get following subcases:

Subcase 1.1: If $m\geqslant 2$, then

$$T(r,\tilde{F})\leqslant N_2\left(r,\frac{1}{\tilde{F}}\right)+N_2\left(r,\frac{1}{\tilde{G}}\right)+2\overline{N}\left(r,\tilde{F}\right)$$
$$+2\overline{N}\left(r,\tilde{G}\right)+S\left(r,\tilde{F}\right)+S\left(r,\tilde{G}\right). \tag{36}$$

From Lemmas 1, 2 and 4, we have $S(r,\tilde{F})=S(r,f)$ and $S(r,\tilde{G})=S(r,g)$. Using (33)–(35) in (36) we get

$$T(r,\tilde{F})\leqslant(2l+2)T(r,f)+2\overline{N}\left(r,\frac{1}{L(f)}\right)+(2l+2)T(r,g)$$
$$+2kqT(r,g)+2k\overline{N}(r,g)+S(r,f)+S(r,g). \tag{37}$$

Similarly,

$$T(r,\tilde{G})\leqslant(2l+2)T(r,g)+2\overline{N}\left(r,\frac{1}{L(g)}\right)+(2l+2)T(r,f)$$
$$+2kqT(r,f)+2k\overline{N}(r,f)+S(r,f)+S(r,g). \tag{38}$$

Adding (37) and (38), we get

$$T(r,\tilde{F})+T(r,\tilde{G})\leqslant(4l+2kq+4)(T(r,f)+T(r,g))$$
$$+2\overline{N}\left(r,\frac{1}{L(g)}\right)+2\overline{N}\left(r,\frac{1}{L(f)}\right)$$
$$+2k\overline{N}(r,f)+2k\overline{N}(r,g)+S(r,f)+S(r,g). \tag{39}$$

Since $n>2k$ and $d\geqslant 2$. Using (18), (19) and (39), we have

$$qd(n-2k)(T(r,f)+T(r,g))$$
$$\leqslant(4l+2kq+4)(T(r,f)+T(r,g))+2k\overline{N}(r,f)+2k\overline{N}(r,g)$$
$$-dk(N(r,F_1)+N(r,G_1))+S(r,f)+S(r,g)$$
$$\leqslant(4l+2+2kq)(T(r,f)+T(r,g))+S(r,f)+S(r,g).$$

Therefore, $(n-2k)dq\leqslant 2kq+4l+4\ \Rightarrow n\leqslant\ 2k+\frac{4l+4+2kq}{qd}$, which contradicts (5).

Subcase 1.2: $m=1$, then

$$T(r,\tilde{F}) \leqslant N_2\left(r,\frac{1}{\tilde{F}}\right)+N_2\left(r,\frac{1}{\tilde{G}}\right)+\frac{5}{2}\overline{N}\left(r,\tilde{F}\right) +2\overline{N}\left(r,\tilde{G}\right)+\frac{1}{2}\overline{N}\left(r,\frac{1}{\tilde{F}}\right)+S\left(r,\tilde{F}\right)+S\left(r,\tilde{G}\right). \tag{40}$$

By using (33)–(35) in (40), we get

$$\begin{aligned}T(r,\tilde{F}) &\leqslant 2lT(r,f)+2\overline{N}\left(r,\frac{1}{L(f)}\right)+2lT(r,g)+2kqT(r,g)+2k\overline{N}(r,g)\\&\quad+\frac{5}{2}\overline{N}(r,f)+2\overline{N}(r,g)+\frac{(kq+l)}{2}T(r,f)+\frac{k}{2}\overline{N}(r,F_1)+S(r,f)+S(r,g)\\&\leqslant \frac{(kq+k+5l+5)}{2}T(r,f)+2\overline{N}\left(r,\frac{1}{L(f)}\right)+(2l+2+2kq)T(r,g)\\&\quad+2k\overline{N}(r,g)+S(r,f)+S(r,g).\end{aligned} \tag{41}$$

Similarly,

$$\begin{aligned}T(r,\tilde{G}) &\leqslant \frac{(kq+k+5l+5)}{2}T(r,g)+2\overline{N}\left(r,\frac{1}{L(g)}\right)+(2l+2+2kq)T(r,f)\\&\quad+2k\overline{N}(r,f)+S(r,f)+S(r,g).\end{aligned} \tag{42}$$

Adding (41) and (42) we obtain

$$\begin{aligned}T(r,\tilde{F})+T(r,\tilde{G}) &\leqslant \frac{(3kq+k+9l+9)}{2}(T(r,f)+T(r,g))+2k(\overline{N}(r,f)+\overline{N}(r,g))\\&\quad+2\overline{N}\left(r,\frac{1}{L(f)}\right)+2\overline{N}\left(r,\frac{1}{L(g)}\right)+S(r,f)+S(r,g).\end{aligned} \tag{43}$$

For $n>2k$ and $d\geqslant 2$. From (18), (19) and (43) we get

$$\begin{aligned}&dq(n-2k)(T(r,f)+T(r,g))+dk(N(r,F_1)+N(r,G_1))\\&+dN\left(r,\frac{1}{L(f)}\right)+dN\left(r,\frac{1}{L(g)}\right)\\&\leqslant \frac{(3kq+k+9l+9)}{2}(T(r,f)+T(r,g))+2k(\overline{N}(r,f)+\overline{N}(r,g))\\&+2\overline{N}\left(r,\frac{1}{L(f)}\right)+2\overline{N}\left(r,\frac{1}{L(g)}\right)+S(r,f)+S(r,g),\end{aligned}$$

$$\begin{aligned}\Rightarrow (n-2k)dq\{T(r,f)+T(r,g)\} &\leqslant \left(\frac{(3kq+k+5l+9)}{2}+2l+4\right)\{T(r,f)+T(r,g)\}\\&\quad+S(r,f)+S(r,g).\end{aligned}$$

Therefore, $(n-2k)\leqslant \frac{(3kq+k+9l+9)}{2dq} \Rightarrow n\leqslant 2k+\frac{3kq+k+9l+9}{2dq}$, which contradicts (6).

Subcase 1.3: $m=0$, then

$$T(r,\tilde{F}) \leqslant 4\overline{N}(r,\tilde{F})+3\overline{N}(r,\tilde{G})+N_2\left(r,\frac{1}{\tilde{F}}\right)+N_2\left(r,\frac{1}{\tilde{G}}\right)+2\overline{N}\left(r,\frac{1}{\tilde{F}}\right)$$
$$+\overline{N}\left(r,\frac{1}{\tilde{G}}\right)+S(r,\tilde{F})+S(r,\tilde{G}). \tag{44}$$

By using (33)–(35) in (44), we obtain

$$T(r,\tilde{F}) \leqslant (4l+2k+2kq+4)T(r,f)+2\overline{N}\left(r,\frac{1}{L(f)}\right)+(3l+k+3)T(r,g)$$
$$+3kqT(r,g)+2k\overline{N}(r,g)+S(r,f)+S(r,g). \tag{45}$$

Similarly,

$$T(r,\tilde{G}) \leqslant (4l+2k+2kq+4)T(r,g)+2\overline{N}\left(r,\frac{1}{L(g)}\right)+(3l+k+3)T(r,f)$$
$$+3kqT(r,f)+2k\overline{N}(r,f)+S(r,f)+S(r,g). \tag{46}$$

Adding (45) and (46) we obtain

$$T(r,\tilde{F})+T(r,\tilde{G}) \leqslant (5kq+3k+7l+7)(T(r,f)+T(r,g))+2k(\overline{N}(r,f)+\overline{N}(r,g))$$
$$+2\overline{N}\left(r,\frac{1}{L(f)}\right)+2\overline{N}\left(r,\frac{1}{L(g)}\right)+S(r,f)+S(r,g). \tag{47}$$

From (18), (19) and (47) we get

$$\begin{aligned}
&dq(n-2k)(T(r,f)+T(r,g))+dk(N(r,F_1)+N(r,G_1))\\
&+dN\left(r,\frac{1}{L(f)}\right)+dN\left(r,\frac{1}{L(g)}\right)\\
\leqslant\ &(5kq+3k+7l+7)(T(r,f)+T(r,g))+2k(\overline{N}(r,f)+\overline{N}(r,g))\\
&+2\overline{N}\left(r,\frac{1}{L(f)}\right)+2\overline{N}\left(r,\frac{1}{L(g)}\right)+S(r,f)+S(r,g).
\end{aligned}$$

$$\Rightarrow (n-2k)dq\{T(r,f)+T(r,g)\} \leqslant (5kq+3k+7l+7)\{T(r,f)+T(r,g)\}$$
$$+S(r,f)+S(r,g).$$

Therefore $(n-2k)dq \leqslant (5kq+3k+7l+7) \Rightarrow n \leqslant 2k+\frac{5kq+3k+7l+7}{dq}$, which contradicts (7).

Case 2: $\tilde{H}\equiv 0$. Thus, when we integrate twice, we get

$$\frac{1}{\tilde{G}-1}=\frac{A}{\tilde{F}-1}+B,$$

where $A\ (\neq 0)$ and B are constants.

Thus

$$\tilde{G} = \frac{(B+1)\tilde{F} + (A-B-1)}{B\tilde{F} + (A-B)}, \tag{48}$$

and

$$\tilde{F} = \frac{(B-A)\tilde{G} + (A-B-1)}{B\tilde{G} - (B+1)}. \tag{49}$$

Now we consider the following three subcases:

Subcase 2.1: $B \neq 0, -1$. Then from (49) we have

$$\overline{N}\left(r, \frac{1}{\tilde{G} - \frac{B+1}{B}}\right) = \overline{N}(r, \tilde{F}).$$

By Nevanlinna second fundamental theorem

$$\begin{aligned} T(r,\tilde{G}) &\leqslant \overline{N}(r,\tilde{G}) + \overline{N}\left(r, \frac{1}{\tilde{G}}\right) + \overline{N}\left(r, \frac{1}{\tilde{G} - \frac{B+1}{B}}\right) + S(r,\tilde{G}) \\ &\leqslant \overline{N}(r,\tilde{G}) + \overline{N}\left(r, \frac{1}{\tilde{G}}\right) + \overline{N}(r,\tilde{F}) + S(r,\tilde{G}), \end{aligned}$$

$$\Rightarrow T(r,\tilde{G}) \leqslant (2l+1)T(r,g) + 2\overline{N}\left(r, \frac{1}{L(g)}\right) + T(r,f) + S(r,f) + S(r,g). \tag{50}$$

If $A-B-1 \neq 0$, then it follows from (48) that

$$\overline{N}\left(r, \frac{1}{\tilde{F} - \frac{-A+B+1}{B+1}}\right) = \overline{N}\left(r, \frac{1}{\tilde{G}}\right).$$

Again by Nevanlinna second fundamental theorem we have

$$\begin{aligned} T(r,\tilde{F}) &\leqslant \overline{N}(r,\tilde{F}) + \overline{N}\left(r, \frac{1}{\tilde{F}}\right) + \overline{N}\left(r, \frac{1}{\tilde{F} - \frac{-A+B+1}{B+1}}\right) + S(r,\tilde{F}) \\ &\leqslant \overline{N}(r,\tilde{F}) + \overline{N}\left(r, \frac{1}{\tilde{F}}\right) + \overline{N}\left(r, \frac{1}{\tilde{G}}\right) + S(r,\tilde{F}) + S(r,\tilde{G}), \end{aligned}$$

$$\begin{aligned} \Rightarrow T(r,\tilde{F}) \leqslant\ & (2l+1)T(r,f) + 2\overline{N}\left(r, \frac{1}{L(f)}\right) + 2k\overline{N}(r,g) \\ & +2(kq+l)T(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{51}$$

Using (18), (19), (50) and (51) we get

$$\begin{aligned} & dq(n-2k)(T(r,f)+T(r,g)) + dk(N(r,F_1)+N(r,G_1)) \\ & +dN\left(r, \frac{1}{L(f)}\right) + dN\left(r, \frac{1}{L(g)}\right) \\ \leqslant\ & (2l+2)T(r,f) + (2qk+4l+1)T(r,g) + 2k\overline{N}(r,f) \\ & +2\overline{N}\left(r, \frac{1}{L(f)}\right) + 2\overline{N}\left(r, \frac{1}{L(g)}\right) + S(r,f) + S(r,g). \end{aligned} \tag{52}$$

For $n>2k$, $d\geqslant 2$ and (52), we get

$$\left(n-2k-\frac{2l+2}{qd}\right)T(r,f)+\left(n-2k-\frac{2kq+4l+1}{qd}\right)T(r,g)\leqslant S(r,f)+S(r,g)$$
$$\Rightarrow\left(n-2k-\frac{2kq+4l+1}{qd}\right)(T(r,f)+T(r,g))\leqslant S(r,f)+S(r,g),$$

which contradicts our assumptions (5)–(7).

Therefore $A-B-1=0$. Then by (48)

$$\overline{N}\left(r,\frac{1}{\tilde{F}+\frac{1}{B}}\right)=\overline{N}(r,\tilde{G}).$$

By Nevanlinna second fundamental theorem and Lemma 4 we get

$$T(r,\tilde{F})\leqslant\overline{N}(r,\tilde{F})+\overline{N}\left(r,\frac{1}{\tilde{F}}\right)+\overline{N}\left(r,\frac{1}{\tilde{F}+\frac{1}{B}}\right)+S(r,\tilde{F})$$
$$\leqslant\overline{N}(r,\tilde{F})+\overline{N}\left(r,\frac{1}{\tilde{F}}\right)+\overline{N}(r,\tilde{G})+S(r,\tilde{F}),$$

$$\Rightarrow T(r,\tilde{F})\leqslant(2l+1)T(r,f)+2\overline{N}\left(r,\frac{1}{L(f)}\right)+\overline{N}(r,g)+S(r,f)+S(r,g). \tag{53}$$

Adding (18), (19) and using (53), (30) we get

$$\begin{aligned}&dq(n-2k)(T(r,f)+T(r,g))+dk(N(r,F_1)+N(r,G_1))\\&+dN\left(r,\frac{1}{L(f)}\right)+dN\left(r,\frac{1}{L(g)}\right)\\\leqslant\ &(2l+2)(T(r,f)+T(r,g))+2\overline{N}\left(r,\frac{1}{L(f)}\right)+2\overline{N}\left(r,\frac{1}{L(g)}\right)\\&+S(r,f)+S(r,g).\end{aligned}$$

Therefore we obtain

$$\left(n-2k-\frac{2l+2}{qd}\right)\{T(r,f)+T(r,g)\}\leqslant S(r,f)+S(r,g),$$

which violates assumptions (5)–(7).

Subcase 2.2: $B=-1$. Then

$$\tilde{G}=\frac{A}{A+1-\tilde{F}},$$

and

$$\tilde{F}=\frac{(1+A)\tilde{G}-A}{\tilde{G}}.$$

If $A+1 \neq 0$,

$$\overline{N}\left(r, \frac{1}{\tilde{F}-(A+1)}\right) = \overline{N}(r, \tilde{G}),$$

$$\overline{N}\left(r, \frac{1}{\tilde{G}-\frac{A}{A+1}}\right) = \overline{N}\left(r, \frac{1}{\tilde{F}}\right).$$

By similar argument as Subcase 2.1 we have a contradiction.

Therefore $A+1=0$ then $\tilde{F}\tilde{G}=1$ and so we get $(P(f)^n)^{(k)}.(P(g)^n)^{(k)}=h$, with $h^d=1$. Let $l=1$, then $P(z)=(z-a_1)^q$. Therefore we have

$$((f-a_1)^{nq})^{(k)}.((g-a_1)^{nq})^{(k)}=1.$$

Using Lemma 5 and $n>2k$, we obtain $f=a_1+c_1e^{cz}$ and $g=a_1+c_2e^{-cz}$ for three non-zero constants c_1,c_2,c such that $(-1)^k(c_1c_2)^{nq}(nqc)^{2k}=1$, which is (i). If $l \geqslant 2$, then proceeding similarly as in the proof of subcase 2.2 of Theorem 10, we obtain a contradiction.

Subcase 2.3: $B=0$. Then (48) and (49) gives $\tilde{G}=\frac{\tilde{F}+A-1}{A}$ and $\tilde{F}=A\tilde{G}+1-A$. If $A-1\neq 0$, $\overline{N}\left(r, \frac{1}{A-1+\tilde{F}}\right)=\overline{N}\left(r, \frac{1}{\tilde{G}}\right)$ and $\overline{N}\left(r, \frac{1}{\tilde{G}-\frac{A-1}{A}}\right)=\overline{N}\left(r, \frac{1}{\tilde{F}}\right)$. Proceeding similarly as in Subcase 2.1 we get a contradiction.

Therefore, $A-1=0$ then $\tilde{F}\equiv\tilde{G}$ i.e., $(P(f)^n)^{(k)}=h(P(g)^n)^{(k)}$, with $h^d=1$. Using Lemma 7 and $n>\frac{2k+2l-1}{q}$, we obtain $P(f)=tP(g)$ with $t^{nd}=1$. Further, if P is a strong uniqueness polynomial then, $f=g$. If $l=1$, then $(f-a_1)=t_1(g-a_1)$ with $t_1^{nqd}=1, t_1\in\mathbb{C}$.

We now consider the case when $P(z)=z^q+az^{q-p}+b$ with $a,b\neq 0$. By the similar argument as in Lemma 7, we see that there exist a t such that $P(f)=tP(g)$, i.e.

$$f^q+af^{q-p}+b=t(g^q+ag^{q-p}+b).$$

Applying Lemma 8 and using the similar argument as in the proof of Theorem 10, we obtain $f=g$. This completes the proof. □

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