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STATISTICAL AND ROUGH STATISTICAL CONVERGENCE IN AN S–METRIC SPACE

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Abstract. In this paper, using the concept of natural density, we have introduced the ideas of statistical and rough statistical convergence in an S-metric space. We have investigated some of their basic properties. We have defined statistical Cauchyness and statistical boundedness of sequences and then some results related to these ideas have been studied. We have defined the set of rough statistical limit points of a sequence in an S-metric space and have proved some relevant results associated with such type of convergence.

1. Introduction

The notion of statistical convergence is a generalization of ordinary convergence. The definition of statistical convergence was introduced by H. Fast [17] and H. Steinhaus [25] independently in the year of 1951. The formal definitions are as follows:

For $B \subset \mathbb{N}$ let $B_n = \{k \in B : k \leqslant n\}$. Then the natural density of B, denoted by $\delta(B)$, is defined by $\delta(B) = lim_{n\to\infty} \frac{|B_n|}{n}$, if the limit exists, where $|B_n|$ stands for the cardinality of B_n. A real sequence $\{\xi_n\}$ is said to be statistically convergent to ξ if for every $\varepsilon > 0$ the set $B(\varepsilon) = \{k \in \mathbb{N} : |\xi_n - \xi| \geqslant \varepsilon\}$ has natural density zero. In this case, ξ is called the statistical limit of the sequence $\{\xi_n\}$ and we write $st\text{-}lim\ \xi_n = \xi$. In many directions works were done on statistical convergence by many authors [1, 12, 13, 14, 20, 27].

Many authors tried to give generalization of the concept of metric spaces in several ways. For example probabilistic metric spaces [15], C^*-algebra valued metric spaces [16] etc. and related works were carried out by several authors [4, 7, 8]. In 2012, Sedghi et al. [26] introduced the idea of S-metric spaces as a new structure of metric spaces.

The idea of rough convergence of sequences in a normed linear space was introduced by H. X. Phu [23] in 2001. Also, Phu [24] extended this concept in an infinite dimensional normed space in 2003. After that several works [2, 3, 18, 19, 21] were done in many generalized spaces. For example the idea of rough convergence in a metric space was studied by S. Debnath and D. Rakhshit [11], in a cone metric space it was studied by A. K. Banerjee and R. Mondal [6] and in a partial metric space by A. K. Banerjee and S. Khatun [9, 10].

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In this paper, we have introduced the idea of statistical convergence in an S-metric space. We have investigated some basic properties of statistical convergence in this space. We have defined statistical Cauchyness and statistical boundedness. Also we have introduced the idea of rough statistical convergence which is an extension work of rough convergence in the S-metric space. We have defined the rough limit set of a sequence and investigated some basic properties of rough limit set in this space. Also we have proved some relevant theorems in this space.

2. Preliminaries

DEFINITION 1. [26] In a non-empty set X, a function $S : X^3 \longrightarrow [0,\infty)$ is said to be an S-metric on X, if the following three conditions hold for every $x,y,z,a \in X$:
(i) $S(x,y,z) \geqslant 0$,
(ii) $S(x,y,z) = 0$ if and only if $x = y = z$,
(iii) $S(x,y,z) \leqslant S(x,x,a) + S(y,y,a) + S(z,z,a)$.
Then the pair (X,S) is said to be an S-metric space.

Properties and examples of S-metric spaces have been thoroughly discussed in [26].

DEFINITION 2. [26] In an S-metric space (X,S) the open and closed ball of radius $r > 0$ and center $x \in X$ respectively are as follows:

$$B_S(x,r) = \{y \in X : S(y,y,x) < r\},$$
$$B_S[x,r] = \{y \in X : S(y,y,x) \leqslant r\}.$$

In an S-metric space (X,S) the collection τ of all open balls in X forms a base of a topology on X called topology induced by the S-metric. The open ball $B_S(x,r)$ of radius r and centre x is an open set in X.

DEFINITION 3. [26] In an S-metric space (X,S) a subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for every $x,y \in A$.

DEFINITION 4. [26] In an S-metric space (X,S) a sequence $\{x_n\}$ is said to be convergent to x in X if for every $\varepsilon > 0$ there exists a natural number k such that $S(x_n,x_n,x) < \varepsilon$ for every $n \geqslant k$.

DEFINITION 5. [26] In an S-metric space (X,S) a sequence $\{x_n\}$ is said to be a Cauchy sequence in X if for every $\varepsilon > 0$ there exists a natural number k such that $S(x_n,x_n,x_m) < \varepsilon$ for every $n,m \geqslant k$.

DEFINITION 6. [26] The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.

LEMMA 1. [26] *In an S-metric space (X,S), we have $S(x,x,y) = S(y,y,x)$ for every $x,y \in X$.*

DEFINITION 7. [21] A sequence $\{x_n\}$ in an S-metric space (X,S) is said to be rough convergent or simply r-convergent to p if for every $\varepsilon > 0$ there exists a natural number k such that $S(x_n,x_n,p) < r+\varepsilon$ holds for all $n \geqslant k$.

Here r is called the roughness degree of a sequence $\{x_n\}$. Note that rough limit of a sequence may not be unique. The set of all rough limits of a sequence $\{x_n\}$ is denoted by $LIM^r x_n$. If $r=0$, then the idea of rough convergence reduces to ordinary convergence.

3. Statistical convergence in an S-metric space

DEFINITION 8. A sequence $\{x_n\}$ in an S-metric space (X,S) is said to be statistically convergent to $x \in X$ if for any $\varepsilon > 0$, $\delta(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{n \in \mathbb{N} : S(x_n,x_n,x) \geqslant \varepsilon\}$ and we write $st\text{-}lim_{n\to\infty} x_n = x$.

THEOREM 1. *Every convergent sequence is statistically convergent in an S-metric space (X,S).*

Proof. Let (X,S) be an S-metric space and let $\{x_n\}$ be a convergent sequence converging to x in X. Then for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $S(x_n,x_n,x) < \varepsilon$ $\forall n \geqslant N$. We consider the set $A(\varepsilon) = \{n \in \mathbb{N} : S(x_n,x_n,x) \geqslant \varepsilon\} \subset \{1,2,3,\dots(N-1)\} = P$(say). Since P is a finite set, so $\delta(P) = 0$. Hence $\delta(A(\varepsilon)) = 0$. Therefore, $\{x_n\}$ is statistically convergent to x in (X,S). So, $st\text{-}lim_{n\to\infty} x_n = x$. $\square$

REMARK 1. The converse of the above theorem may not be true as shown in the following example.

EXAMPLE 1. Let $X = \mathbb{R}^2$ and $||.||$ be the Euclidean norm on X, then $S(x,y,z) = ||x-z|| + ||y-z||$, $\forall x,y,z \in X$ is an S-metric on X.

Let $\{x_n\}$ be a sequence in $\mathbb{R}^2$ defined by

$$x_n = \begin{cases} (k,k) & \text{if } n = k^2 \text{ for some } k \in \mathbb{N}, \\ (0,0) & \text{otherwise.} \end{cases}$$

Let $\varepsilon > 0$ be given. Then we will show that $\{x_n\}$ is statistically convergent to $x = (0,0)$.

Now, $A(\varepsilon) = \{n \in \mathbb{N} : S(x_n,x_n,x) \geqslant \varepsilon\} \subset \{1^2,2^2,3^2\dots\} = P$ (say).

As $\delta(P) = 0$, so $\delta(A(\varepsilon)) = 0$. Hence $st\text{-}lim_{n\to\infty} x_n = x$.

So, $\{x_n\}$ is statistically convergent to $x = (0,0)$.

If $\{x_n\}$ is bounded, then $\exists\, B(>0) \in \mathbb{R}$ such that $S(x_n,x_n,x_m) < B$ $\forall n,m \in \mathbb{N}$ i.e. $2||x_n - x_m|| < B$ $\forall n,m \in \mathbb{N}$ i.e. $||x_n - x_m|| < \frac{B}{2} \dots (1)$ $\forall n,m \in \mathbb{N}$. Choose $n = k^2$ where $k > B$, $k \in \mathbb{N}$ and m is any natural number such that $m \neq k^2$ for $k \in \mathbb{N}$, then $x_m = (0,0)$ and $x_n = (k,k)$. So, $||x_n - x_m|| = \sqrt{(k-0)^2 + (k-0)^2} = \sqrt{2}k > B > \frac{B}{2}$. This contradicts (1). So, it is not bounded and hence $\{x_n\}$ is not ordinary convergent.

By the following theorem we conclude that the statistical limit in an S-metric space is unique.

THEOREM 2. *Let $\{x_n\}$ be a sequence in an S-metric space (X,S) such that $x_n \xrightarrow{st} x$ and $x_n \xrightarrow{st} y$, then $x=y$.*

Proof. Let $\varepsilon>0$ be arbitrary. Since $x_n \xrightarrow{st} x$ and $x_n \xrightarrow{st} y$, for $\varepsilon>0$, $\delta(A_1(\varepsilon))=0$ and $\delta(A_2(\varepsilon))=0$, where $A_1(\varepsilon)=\{n\in\mathbb{N}: S(x_n,x_n,x)\geqslant \frac{\varepsilon}{3}\}$ and $A_2(\varepsilon)=\{n\in\mathbb{N}: S(x_n,x_n,y)\geqslant \frac{\varepsilon}{3}\}$. Let $K(\varepsilon)=A_1(\varepsilon)\cup A_2(\varepsilon)$, then $\delta(K(\varepsilon))=0$. Hence $\delta(K(\varepsilon))^c=1$. Suppose $k\in (K(\varepsilon))^c=(A_1(\varepsilon)\cup A_2(\varepsilon))^c=(A_1(\varepsilon))^c\cap(A_2(\varepsilon))^c$. Then

$$\begin{aligned} S(x,x,y) &\leqslant S(x,x,x_k)+S(x,x,x_k)+S(y,y,x_k)\\ &= S(x_k,x_k,x)+S(x_k,x_k,x)+S(x_k,x_k,y) \text{ (by Lemma 1)}\\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon. \end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we get $S(x,x,y)=0$. Therefore, $x=y$. □

DEFINITION 9. A sequence $\{x_n\}$ in an S-metric space (X,S) is said to be statistically Cauchy if for any $\varepsilon>0$, $\exists\, N\in\mathbb{N}$ such that $\delta(B(\varepsilon))=0$, where $B(\varepsilon)=\{n\in\mathbb{N}: S(x_n,x_n,x_N)\geqslant\varepsilon\}$.

THEOREM 3. *Let $\{x_n\}$ be a statistically convergent sequence in an S-metric space (X,S). Then the sequence $\{x_n\}$ is statistically Cauchy sequence in (X,S).*

Proof. Let the sequence $\{x_n\}$ be statistically convergent to x in (X,S). Then for an arbitrary $\varepsilon>0$, $\delta(A(\varepsilon))=0$, where $A(\varepsilon)=\{n\in\mathbb{N}: S(x_n,x_n,x)\geqslant\frac{\varepsilon}{3}\}$. So, $\delta(A(\varepsilon))^c=1$. Take $N_\varepsilon\in A(\varepsilon)^c$. Then $S(x_{N_\varepsilon},x_{N_\varepsilon},x)<\frac{\varepsilon}{3}$. Now, we show that

$$\left\{n\in\mathbb{N}: S(x_n,x_n,x)<\frac{\varepsilon}{3}\right\}\subset\{n\in\mathbb{N}: S(x_n,x_n,x_{N_\varepsilon})<\varepsilon\}.$$

Let $k\in\{n\in\mathbb{N}: S(x_n,x_n,x)<\frac{\varepsilon}{3}\}=A(\varepsilon)^c$. Then $S(x_k,x_k,x)<\frac{\varepsilon}{3}$. Now,

$$\begin{aligned} S(x_k,x_k,x_{N_\varepsilon}) &\leqslant S(x_k,x_k,x)+S(x_k,x_k,x)+S(x_{N_\varepsilon},x_{N_\varepsilon},x)\\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon. \end{aligned}$$

So, $k\in\{n\in\mathbb{N}: S(x_n,x_n,x_{N_\varepsilon})<\varepsilon\}$. Hence $A(\varepsilon)^c\subset B(N_\varepsilon)^c$, where $B(N_\varepsilon)=\{n\in\mathbb{N}: S(x_n,x_n,x_{N_\varepsilon})\geqslant\varepsilon\}$. This implies that $B(N_\varepsilon)\subset A(\varepsilon)$. Since $\delta(A(\varepsilon))=0$, it follows that $\delta(B(N_\varepsilon))=0$. Hence the theorem follows. □

The following Theorem 4 is a direct consequence of Theorem 12 as discussed in Remark 3. However, we give the details proof of Theorem 4 for the sake of completeness.

THEOREM 4. *Let $\{x_n\}$ be a statistically convergent sequence in an S-metric space (X,S). Then there is a convergent sequence $\{y_n\}$ in X such that $x_n = y_n$ for almost all $n \in \mathbb{N}$.*

Proof. Let $\{x_n\}$ be a statistically convergent to x. Then for any $\varepsilon > 0$, we have $\delta(\{n \in \mathbb{N} : S(x_n,x_n,x) \geqslant \varepsilon\}) = 0$. So,

$$\delta(\{n \in \mathbb{N} : S(x_n,x_n,x) < \varepsilon\}) = lim_{n\to\infty} \frac{|\{k \leqslant n : S(x_k,x_k,x) < \varepsilon\}|}{n} = 1.$$

So, for every $k \in \mathbb{N}$, $\exists\ n_k \in \mathbb{N}$ such that $\forall\ n > n_k$

$$\frac{|\{k \leqslant n : S(x_k,x_k,x) < \frac{1}{2^k}\}|}{n} > 1 - \frac{1}{2^k}.$$

Choose

$$y_m = \begin{cases} x_m, & \text{if } 1 \leqslant m \leqslant n_1 , \\ x_m, & \text{if } n_k < m \leqslant n_{k+1}, S(x_m,x_m,x) < \frac{1}{2^k}, \\ x, & \text{otherwise.} \end{cases}$$

Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ so that $\frac{1}{2^k} < \varepsilon$. Now, for each $m > n_k$, $S(y_m,y_m,x) = S(x_m,x_m,x) < \frac{1}{2^k} < \varepsilon$. Hence $\{y_m\}$ converges to x. So, $lim_{n\to\infty} y_m = x$.

Let $n \in \mathbb{N}$ be fixed and let $n_k < n \leqslant n_{k+1}$, then we have

$$\{m \leqslant n : y_m \neq x_m\} \subset \{1,2,\ldots n\} - \left\{m \leqslant n : S(x_m,x_m,x) < \frac{1}{2^k}\right\}.$$

So,

$$\begin{aligned}\frac{1}{n}|\{m \leqslant n : y_m \neq x_m\}| &\leqslant 1 - \frac{1}{n}\left|\left\{m \leqslant n : S(x_m,x_m,x) < \frac{1}{2^k}\right\}\right| \\ &< \frac{1}{2^k} < \varepsilon.\end{aligned}$$

Hence $lim_{n\to\infty} \frac{1}{n}|\{m \leqslant n : y_m \neq x_m\}| = 0$. So, $\delta(\{m \in \mathbb{N} : y_m \neq x_m\}) = 0$. Therefore, $x_m = y_m$ for almost all $m \in \mathbb{N}$. $\square$

COROLLARY 1. *Every statistically convergent sequence in an S-metric space has a convergent subsequence.*

Proof. Let $\{x_n\}$ be a statistically convergent sequence. So, there is a convergent sequence $\{y_n\}$ such that $x_n = y_n$ almost everywhere, i.e., $\delta(A) = 0$, where $A = \{m \in \mathbb{N} : y_m \neq x_m\}$. Let us enumerate the set $\mathbb{N} \setminus A$ by $\{n_1 < n_2 < \ldots\}$. Therefore, $y_{n_k} = x_{n_k}$ $\forall\ k = 1,2,3\ldots$ and since $\{y_{n_k}\}$ is convergent, $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n\}$. $\square$

DEFINITION 10. Let (X,S) be an S-metric space. If every statistically Cauchy sequence is statistically convergent, then (X,S) is called statistically complete.

THEOREM 5. *If $\{x_n\}$ is Cauchy then it is statistically Cauchy in an S-metric space.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X,S). Let $\varepsilon > 0$. Then there exists a natural number k such that $S(x_n,x_n,x_m) < \varepsilon$ for every $n,m \geqslant k$. So, in particular $S(x_n,x_n,x_k) < \varepsilon \ \forall n \geqslant k$. So, the set $B(\varepsilon) = \{n \in \mathbb{N} : S(x_n,x_n,x_k) \geqslant \varepsilon\} \subset \{1,2,3,\ldots(k-1)\} = Q$(say). Since Q is finite, so $\delta(Q) = 0$. Hence $\delta(B(\varepsilon)) = 0$. Therefore, $\{x_n\}$ is statistically Cauchy. $\square$

THEOREM 6. *Every statistically complete S-metric space is complete.*

Proof. Let (X,S) be a statistically complete S-metric space. Suppose that $\{x_n\}$ is a Cauchy sequence in (X,S), then it is statistically Cauchy sequence in (X,S). Since (X,S) is statistically complete, so $\{x_n\}$ is statistically convergent. By the Corollary 1, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to a point $x \in X$. Since $\{x_n\}$ is Cauchy, for $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \frac{\varepsilon}{3}$ for $n,m \geqslant N_1$. On the other hands, $\{x_{n_k}\}$ converges to x, so there exists k_0 such that $S(x_{n_k},x_{n_k},x) < \frac{\varepsilon}{3}$ $\forall \ k \geqslant k_0$. So, in particular $S(x_{n_{k_0}},x_{n_{k_0}},x) < \frac{\varepsilon}{3}$ and let $N = max\{N_1, n_{k_0}\}$. Then for $n \geqslant N$, we have

$$\begin{aligned} S(x_n,x_n,x) &\leqslant S(x_n,x_n,x_{n_{k_0}}) + S(x_n,x_n,x_{n_{k_0}}) + S(x,x,x_{n_{k_0}}) \\ &= S(x_n,x_n,x_{n_{k_0}}) + S(x_n,x_n,x_{n_{k_0}}) + S(x_{n_{k_0}},x_{n_{k_0}},x) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence $lim_{n\to\infty} x_n = x$. So, (X,S) is a complete S-metric space. $\square$

DEFINITION 11. (cf. [5]) A sequence $\{x_n\}$ in an S-metric space (X,S) is said to be statistically bounded if for any fixed $u \in X$ there exists a positive real number B such that

$$\delta(\{n \in N : S(x_n,x_n,u) \geqslant B\}) = 0.$$

THEOREM 7. *Every statistically Cauchy sequence is statistically bounded in an S-metric space (X,S).*

Proof. Let $\varepsilon > 0$ and let $\{x_n\}$ be a statistically Cauchy sequence in (X,S). Then there exists $N \in \mathbb{N}$ such that $\delta(B) = 0$, where $B = \{n \in N : S(x_n,x_n,x_N) \geqslant \frac{\varepsilon}{2}\}$. Let $k \in B^c$, then $S(x_k,x_k,x_N) < \frac{\varepsilon}{2}$. Let $u \in X$ be fixed, then for the same $\varepsilon > 0$, we have

$$\begin{aligned} S(x_k,x_k,u) &\leqslant S(x_k,x_k,x_N) + S(x_k,x_k,x_N) + S(u,u,x_N) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + S(x_N,x_N,u) \\ &= \varepsilon + S(x_N,x_N,u) \\ &= a \text{ (say)}. \end{aligned}$$

So, $k \in \{n \in N : S(x_n, x_n, u) < a\}$. Therefore, $B^c \subset \{n \in N : S(x_n, x_n, u) < a\}$. Since $\delta(B) = 0$, $\delta(B^c) = 1$. Hence $\delta(\{n \in N : S(x_n, x_n, u) < a\}) = 1$. This implies that $\delta(\{n \in N : S(x_n, x_n, u) \geqslant a\}) = 0$. Therefore, the sequence $\{x_n\}$ is statistically bounded in X. □

COROLLARY 2. *Every statistically convergent sequence is statistically bounded in an S-metric space.*

4. Rough statistical convergence in an S-metric space

DEFINITION 12. A sequence $\{x_n\}$ in an S-metric space (X,S) is said to be rough statistically convergent or r-statistically convergent (or, in short r-st convergent) to x, if for every $\varepsilon > 0$ and for some roughness degree r, $\delta(\{n \in \mathbb{N} : S(x_n, x_n, x) \geqslant r + \varepsilon\}) = 0$.

For $r = 0$, the rough statistical convergence becomes the statistical convergence in any S-metric space (X,S). If a sequence $\{x_n\}$ is rough statistically convergent to x, then we denote it by the notation $x_n \xrightarrow{r\text{-}st} x$ and x is said to be a rough statistical limit point (or r-st limit point) of $\{x_n\}$. The set of all r-st limit points of a sequence $\{x_n\}$ is said to be the r-st limit set. We denote it by $st\text{-}LIM^r x_n$, i.e., $st\text{-}LIM^r x_n = \left\{x \in X : x_n \xrightarrow{r\text{-}st} x\right\}$. For the degree of roughness $r > 0$, the r-st limit may not be unique.

THEOREM 8. *Every rough convergent sequence in an S-metric space (X,S) is rough statistically convergent.*

Proof. Let a sequence $\{\xi_n\}$ be rough convergent to ξ in (X,S). Let $\varepsilon > 0$ be arbitrary. Then for $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $S(\xi_n, \xi_n, \xi) < r + \varepsilon$ $\forall n \geqslant m$. Then the set $A = \{n \in \mathbb{N} : S(\xi_n, \xi_n, \xi) \geqslant r + \varepsilon\} \subset \{1, 2, 3, \ldots (m-1)\}$. Since A is a finite set, so $\delta(A) = 0$. Hence $\{\xi_n\}$ is rough statistically convergent in (X,S). □

REMARK 2. The converse of the above theorem may not be true. The following example shows that a rough statistically convergent sequence may not be rough convergent in an S-metric space.

EXAMPLE 2. Let $X = \mathbb{R}^2$ and $||.||$ be the Euclidean norm on X, then $S(x,y,z) = ||x - z|| + ||y - z||$, $\forall x, y, z \in X$ is an S-metric on X. Let a sequence $\{\xi_n\}$ be defined by

$$\xi_n = \begin{cases} (p,p), & \text{if } n = p^2 \text{ for } p \in \mathbb{N}, \\ (0,0), & \text{if } n \neq p^2 \text{ and } n \text{ is even}, \\ (1,1), & \text{if } n \neq p^2 \text{ and } n \text{ is odd}. \end{cases}$$

Let $\varepsilon > 0$ be arbitrary and consider the point $\xi_1 = (1,1)$. Then $A_1 = \{n \in \mathbb{N} : S(\xi_n, \xi_n, \xi_1) \geqslant 2\sqrt{2} + \varepsilon\} \subset \{1^2, 2^2, 3^2, 4^2, 5^2 \ldots\} = M$ (say).

Again, consider the point $\xi_2=(0,0)$. Then $A_2=\{n\in\mathbb{N}:S(\xi_n,\xi_n,\xi_2)\geqslant 2\sqrt{2}+\varepsilon\}\subset M$.

Since $\delta(M)=0$, we have that $\delta(A_1)=\delta(A_2)=0$.

So, $\{\xi_n\}$ is rough statistically convergent to $(1,1)$ and $(0,0)$ of roughness degree $2\sqrt{2}$.

But

$$S(\xi_n,\xi_n,\xi_2)=2||\xi_n-\xi_2||=\begin{cases}2\sqrt{2}p, & \text{if } n=p^2,\\ 0, & \text{if } n\neq p^2 \text{ and } n \text{ is even},\\ 2\sqrt{2}, & \text{if } n\neq p^2 \text{ and } n \text{ is odd}.\end{cases}$$

So, when $n=p^2$, there does not exist any positive integer n_0 such that the condition $S(\xi_n,\xi_n,\xi_2)<r+\varepsilon$ for all $n\geqslant n_0$ holds. Hence $\{\xi_n\}$ is not rough convergent to $(0,0)$ of any roughness degree $r>0$. Similarly, it can be shown that $\{\xi_n\}$ is not rough convergent to $(1,1)$.

DEFINITION 13. [21] The diameter of a set A in an S-metric space (X,S) is defined by

$$diam(A)=sup\ \{S(x,x,y):x,y\in A\}.$$

THEOREM 9. *Let $\{x_n\}$ be a r-statistically convergent sequence in an S-metric space (X,S). Then the diameter of st-$LIM^r x_n$ is not greater than $3r$, i.e., dim(st-$LIM^r x_n$) $\leqslant 3r$.*

Proof. If possible, suppose that $diam(st\text{-}LIM^r x_n)>3r$. Then there exist elements $\xi,\eta\in st\text{-}LIM^r x_n$ such that $S(\xi,\xi,\eta)>3r$. Take $\varepsilon\in\left(0,\frac{S(\xi,\xi,\eta)}{3}-r\right)$. Since $\xi,\eta\in st\text{-}LIM^r x_n$, $\delta(A_1)=0$ and $\delta(A_2)=0$, where $A_1=\{n\in\mathbb{N}:S(x_n,x_n,\xi)\geqslant r+\varepsilon\}$ and $A_2=\{n\in\mathbb{N}:S(x_n,x_n,\eta)\geqslant r+\varepsilon\}$. From the property of natural density, we can write $\delta(A_1^c\cap A_2^c)=1$. Now, for all $n\in A_1^c\cap A_2^c$

$$\begin{aligned}S(\xi,\xi,\eta)&\leqslant S(\xi,\xi,x_n)+S(\xi,\xi,x_n)+S(\eta,\eta,x_n)\\&=S(x_n,x_n,\xi)+S(x_n,x_n,\xi)+S(x_n,x_n,\eta)\ \text{(by Lemma 1)}\\&<(r+\varepsilon)+(r+\varepsilon)+(r+\varepsilon)\\&=3(r+\varepsilon)\\&=3r+3\varepsilon\\&=3r+3\left\{\frac{S(\xi,\xi,\eta)}{3}-r\right\}\\&=3r+S(\xi,\xi,\eta)-3r\\&=S(\xi,\xi,\eta),\ \text{which is a contradiction.}\end{aligned}$$

Hence the diameter of st-$LIM^r x_n$ is not greater than $3r$, i.e., $diam(st\text{-}LIM^r x_n)\leqslant 3r$. $\square$

THEOREM 10. *If a sequence $\{x_n\}$ statistically convergent to x' in an S-metric space (X,S), then st-LIM$^r x_n = B_S[x',r]$.*

Proof. Let a sequence $\{x_n\}$ be statistically convergent to x'. Let $\varepsilon > 0$. Then $\delta(A) = 0$, where $A = \{n \in \mathbb{N} : S(x_n,x_n,x') \geqslant \frac{\varepsilon}{3}\}$.

Let $y \in B_S[x',r] = \{y \in X : S(y,y,x') \leqslant r\}$. Then using the triangular properties of S-metric space, we have for $n \in A^c$

$$\begin{aligned} S(x_n,x_n,y) &\leqslant S(x_n,x_n,x') + S(x_n,x_n,x') + S(y,y,x') \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + r \\ &< r + \varepsilon. \end{aligned}$$

So, $\{n \in \mathbb{N} : S(x_n,x_n,y) \geqslant r+\varepsilon\} \subset A$. Therefore, $\delta(\{n \in \mathbb{N} : S(x_n,x_n,y) \geqslant r+\varepsilon\}) = 0$. So, $y \in$ st-LIM$^r x_n$. Hence

$$B_S[x',r] \subset st\text{-}LIM^r x_n. \tag{1}$$

Again, let $y \in$ st-LIM$^r x_n$ and let $\varepsilon > 0$ be arbitrary. Then $\delta(B) = 0$, where $B = \{n \in \mathbb{N} : S(x_n,x_n,y) \geqslant r + \frac{\varepsilon}{3}\}$. Now, let $n \in A^c \cap B^c$

$$\begin{aligned} S(x',x',y) &\leqslant S(x',x',x_n) + S(x',x',x_n) + S(y,y,x_n) \\ &= S(x_n,x_n,x') + S(x_n,x_n,x') + S(x_n,x_n,y) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + r + \frac{\varepsilon}{3} \\ &= (r+\varepsilon). \end{aligned}$$

So, $S(y,y,x') < r+\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $S(y,y,x') \leqslant r$. So, $y \in B_S[x',r]$. Hence

$$st\text{-}LIM^r x_n \subset B_S[x',r]. \tag{2}$$

From (1) and (2), we get st-LIM$^r x_n = B_S[x',r]$. □

THEOREM 11. *Let $\{x_n\}$ be a r-statistically convergent sequence in an S-metric space (X,S) and let $\{\xi_n\}$ be a convergent sequence in st-LIM$^r x_n$ converging to ξ. Then ξ must belongs to st-LIM$^r x_n$.*

Proof. If st-LIM$^r x_n = \phi$, then there is nothing to prove. Let us consider the case where st-LIM$^r x_n \neq \phi$. Let $\{\xi_n\}$ be a sequence in st-LIM$^r x_n$ such that $\{\xi_n\}$ is convergent to ξ. Let $\varepsilon > 0$ be arbitrary. So, for $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that $S(\xi_n,\xi_n,\xi) < \frac{\varepsilon}{3}$ $\forall n \geqslant n_1$. Now, let us choose an $n_0 \in \mathbb{N}$ such that $n_0 > n_1$. Then we can write $S(\xi_{n_0},\xi_{n_0},\xi) < \frac{\varepsilon}{3}$. Since $\{\xi_n\} \subset$ st-LIM$^r x_n$, so we have $\xi_{n_0} \in$ st-LIM$^r x_n$ and $\delta(\{n \in \mathbb{N} : S(x_n,x_n,\xi_{n_0}) \geqslant r + \frac{\varepsilon}{3}\}) = 0$. Now, we show that

$$\{n \in \mathbb{N} : S(x_n,x_n,\xi) < r+\varepsilon\} \supseteq \left\{n \in \mathbb{N} : S(x_n,x_n,\xi_{n_0}) < r + \frac{\varepsilon}{3}\right\}. \tag{3}$$

Let $k \in \{n \in \mathbb{N} : S(x_n,x_n,\xi_{n_0}) < r + \frac{\varepsilon}{3}\}$.

Then $S(x_k,x_k,\xi_{n_0}) < r+\frac{\varepsilon}{3}$. So, we can write

$$\begin{aligned} S(\xi,\xi,x_k) &\leqslant S(\xi,\xi,\xi_{n_0})+S(\xi,\xi,\xi_{n_0})+S(x_k,x_k,\xi_{n_0}) \\ &= S(\xi_{n_0},\xi_{n_0},\xi)+S(\xi_{n_0},\xi_{n_0},\xi)+S(x_k,x_k,\xi_{n_0}) \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+r+\frac{\varepsilon}{3} \\ &= r+\varepsilon. \end{aligned}$$

Hence $k \in \{n \in \mathbb{N} : S(\xi,\xi,x_n) < r+\varepsilon\} = \{n \in \mathbb{N} : S(x_n,x_n,\xi) < r+\varepsilon\}$. Therefore, (3) holds. Since the set on the right-hand side of (3) has natural density 1, so the natural density of the set on the left-hand side of (3) is equal to 1. So, $\delta(\{n \in \mathbb{N} : S(x_n,x_n,\xi) \geqslant r+\varepsilon\}) = 0$. Hence $\xi \in st\text{-}LIM^r x_n$. Therefore, the theorem follows. □

THEOREM 12. *Every S-metric space (X,S) is first countable.*

Proof. Let $\tau(S)$ be the topology generated by the base v, where $v = \{B_S(x,\varepsilon) : x \in X, \varepsilon > 0\}$. Let us consider $u = \{B_S(x,\frac{1}{p}) : x \in X, p \in \mathbb{N}\}$. Then u is a basis for $\tau(S)$. For let, $A \in \tau(S)$ and $x \in A$ be arbitrary element. Then, since v is a basis for $\tau(S)$, there exists $\varepsilon > 0$ such that $x \in B_S(x,\varepsilon) \subset A$. Choose $p \in \mathbb{N}$ so that $\frac{1}{p} < \varepsilon$. Then $B_S(x,\frac{1}{p}) \subset B_S(x,\varepsilon)$. Thus $x \in B_S(x,\frac{1}{p}) \subset B_S(x,\varepsilon) \subset A$. So, u forms a basis for $\tau(S)$. Since u is countable, (X,S) is first countable. □

COROLLARY 3. *Let $\{x_n\}$ be a r-statistically convergent sequence in an S-metric space (X,S). Then $st\text{-}LIM^r x_n$ is a closed set for any roughness degree $r \geqslant 0$.*

Proof. Since the S-metric space (X,S) is first countable, the result follows directly from Theorem 11. □

REMARK 3. In [22], it is proved that a sequence in a first countable space is statistically convergent to x if and only if it is S^*-convergent to x, where a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a topological space X is said to be S^*-convergent to $x \in X$ if there is $A \subset \mathbb{N}$ with $\delta(A) = 1$ such that $lim_{m\to\infty, m\in A} x_m = x$. So, in view of Theorem 12, if $\{x_n\}$ is statistically convergent to x, it is S^*-convergent. So, there is $A \subset \mathbb{N}$ with $\delta(A) = 1$ such that $lim_{m\to\infty, m\in A} x_m = x$. Let $\{y_n\}$ be a sequence such that

$$y_m = \begin{cases} x_m & \text{if } m \in A, \\ x & \text{if } m \in \mathbb{N}\setminus A. \end{cases}$$

Then $\{y_n\}$ is a convergent sequence and the set $\{m \in \mathbb{N} : y_m \neq x_m\} \subset A^c$. Since $\delta(A) = 1$, $\delta(A^c) = 0$. So, $\delta(\{m \in \mathbb{N} : y_m \neq x_m\}) = 0$. So, $x_n = y_n$ for almost all $n \in \mathbb{N}$. This is the conclusion of Theorem 4.

THEOREM 13. *Let (X,S) be an S-metric space. Then a sequence $\{x_n\}$ is statistically bounded in (X,S) if and only if there exists a non-negative real number r such that $st\text{-}LIM^r x_n \neq \phi$.*

Proof. Let $p \in X$ be a fixed element in X. Since the sequence $\{x_n\}$ is statistically bounded in (X,S), so there exists a positive real number M such that $\delta(\{n \in \mathbb{N} : S(x_n,x_n,p) \geqslant M\}) = 0$. Let $\varepsilon > 0$ be arbitrary and $r = M$. Then $M < r+\varepsilon$. So, we have the inclusion

$$\{n \in \mathbb{N} : S(x_n,x_n,p) < r+\varepsilon\} \supset \{n \in \mathbb{N} : S(x_n,x_n,p) < M\}. \tag{4}$$

Hence $\{n \in \mathbb{N} : S(x_n,x_n,p) \geqslant r+\varepsilon\} \subset \{n \in \mathbb{N} : S(x_n,x_n,p) \geqslant M\}$. Since $\delta(\{n \in \mathbb{N} : S(x_n,x_n,p) \geqslant M\}) = 0$, it follows that $\delta(\{n \in \mathbb{N} : S(x_n,x_n,p) \geqslant r+\varepsilon\}) = 0$. Therefore, $p \in st\text{-}LIM^r x_n$ i.e. $st\text{-}LIM^r x_n \neq \phi$.

Conversely, let $st\text{-}LIM^r x_n \neq \phi$ and let p be a r-limit of $\{x_n\}$. Then for $\varepsilon > 0$, $\delta(\{n \in \mathbb{N} : S(x_n,x_n,p) \geqslant r+\varepsilon\}) = 0$. Let $A = \{n \in \mathbb{N} : S(x_n,x_n,p) \geqslant r+\varepsilon\}$ and $M = r+\varepsilon$. Then $\delta(A) = 0$ and if $n \in A^c$, then $S(x_n,x_n,p) < r+\varepsilon = M$. So, $n \in \{n \in \mathbb{N} : S(x_n,x_n,p) < M\}$. This implies $A^c \subset \{n \in \mathbb{N} : S(x_n,x_n,p) < M\}$. So, $\{n \in \mathbb{N} : S(x_n,x_n,p) \geqslant M\} \subset A$ and hence $\delta(\{n \in \mathbb{N} : S(x_n,x_n,p) \geqslant M\}) = 0$. Therefore, $\{x_n\}$ is statistically bounded. $\square$

THEOREM 14. *Let $\{x_{n_p}\}$ be a subsequence of $\{x_n\}$ such that $\delta(\{n_1,n_2,\dots\}) = 1$, then $st\text{-}LIM^r x_n \subseteq st\text{-}LIM^r x_{n_p}$.*

Proof. Let $\{x_{n_p}\}$ be a subsequence of $\{x_n\}$ and $x \in st\text{-}LIM^r x_n$. Let $\varepsilon > 0$ be arbitrary, then the set $A = \{n \in \mathbb{N} : S(x_n,x_n,x) \geqslant r+\varepsilon\}$ has density zero. So, $\delta(A^c) = 1$. Since the set $P = \{n_1,n_2,\dots\}$ has density 1, $A^c \cap P \neq \phi$. For if $A^c \cap P = \phi$, then $P \subset A$ and so $\delta(P) = 0$, a contradiction, since $\delta(A) = 0$. Let $n_k \in A^c \cap P$. Then $S(x_{n_k},x_{n_k},x) < r+\varepsilon$. So, $\{n_p \in P : S(x_{n_p},x_{n_p},x) \geqslant r+\varepsilon\} \subset A \cup P^c$. This implies that $\delta(\{n_p \in P : S(x_{n_p},x_{n_p},x) \geqslant r+\varepsilon\}) = 0$, since $\delta(A \cup P^c) \leqslant \delta(A) + \delta(P^c) = 0+0 = 0$. Therefore, $x \in st\text{-}LIM^r x_{n_p}$. Hence $st\text{-}LIM^r x_n \subseteq st\text{-}LIM^r x_{n_p}$. $\square$

THEOREM 15. *Let $\{\xi_n\}$ and $\{\eta_n\}$ be two sequences in an S-metric space (X,S) such that $S(\xi_n,\xi_n,\eta_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Then $\{\xi_n\}$ is r-statistically convergent to ξ if and only if $\{\eta_n\}$ is r-statistically convergent to ξ.*

Proof. Let $\{\xi_n\}$ be r-statistically convergent to ξ. Let $\varepsilon > 0$. Then for $\varepsilon > 0$, $\delta(A) = 0$, where $A = \{n \in \mathbb{N} : S(\xi_n,\xi_n,\xi) \geqslant r + \frac{\varepsilon}{3}\}$.

Since $S(\xi_n,\xi_n,\eta_n) \longrightarrow 0$ as $n \longrightarrow \infty$, for $\varepsilon > 0$, $\exists\, k \in \mathbb{N}$ such that $S(\xi_n,\xi_n,\eta_n) \leqslant \frac{\varepsilon}{3}$, when $n \geqslant k$.

Since $\delta(A) = 0$, $\delta(A^c) = 1$. Again, $\delta(\{1,2,\dots k\}) = 0$, so $\delta(\{1,2,\dots k\}^c) = 1$. So, $A^c \cap \{1,2,\dots k\}^c \neq \phi$. If $n \in A^c \cap \{1,2,\dots k\}^c$, then

$$\begin{aligned} S(\eta_n,\eta_n,\xi) &\leqslant S(\eta_n,\eta_n,\xi_n) + S(\eta_n,\eta_n,\xi_n) + S(\xi,\xi,\xi_n) \\ &= S(\xi_n,\xi_n,\eta_n) + S(\xi_n,\xi_n,\eta_n) + S(\xi_n,\xi_n,\xi) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left(r + \frac{\varepsilon}{3}\right) \\ &= r+\varepsilon. \end{aligned}$$

Hence $A^c \cap \{1,2,\dots k\}^c \subset \{n \in \mathbb{N} : S(\eta_n, \eta_n, \xi) < r+\varepsilon\}$. So, $\{n \in \mathbb{N} : S(\eta_n, \eta_n, \xi) \geqslant r+\varepsilon\} \subset (A^c \cap \{1,2,\dots k\}^c)^c = A \cup \{1,2,\dots k\}$. Now, since $\delta(A \cup \{1,2,\dots k\}) \leqslant \delta(A) + \delta(\{1,2,\dots k\}) = 0+0 = 0$, so $\delta(\{n \in \mathbb{N} : S(\eta_n, \eta_n, \xi) \geqslant r+\varepsilon\}) = 0$. Therefore, $\{\eta_n\}$ is r-statistically convergent to ξ.

The opposite part is similar. □

DEFINITION 14. (cf. [2]) Let (X,S) be an S-metric space. Then $c \in X$ is called a statistical cluster point of a sequence $\{x_n\}$ in (X,S) if for every $\varepsilon > 0$, $\delta(\{n \in \mathbb{N} : S(x_n, x_n, c) < \varepsilon\}) \neq 0$.

THEOREM 16. *Let $\{x_n\}$ be a sequence in an S-metric space (X,S). If c is a cluster point of $\{x_n\}$, then st-LIM$^r x_n \subset B_S[c,r]$ for some $r > 0$.*

Proof. Let $\varepsilon > 0$ and let $x \in$ *st-LIM*$^r x_n$. Then for $\varepsilon > 0$, $\delta(B_1) = 0$, where $B_1 = \{n \in \mathbb{N} : S(x_n, x_n, x) \geqslant r + \frac{\varepsilon}{3}\}$. Again, since c is a cluster point of $\{x_n\}$, for the same $\varepsilon > 0$, $\delta(B_2) \neq 0$, where $B_2 = \{n \in \mathbb{N} : S(x_n, x_n, c) < \frac{\varepsilon}{3}\}$. Now, let $k \in B_1^c \cap B_2$, then $S(x_k, x_k, x) < r + \frac{\varepsilon}{3}$ and $S(x_k, x_k, c) < \frac{\varepsilon}{3}$. Therefore, we can write

$$\begin{aligned} S(c,c,x) &\leqslant S(c,c,x_k) + S(c,c,x_k) + S(x,x,x_k) \\ &= S(x_k,x_k,c) + S(x_k,x_k,c) + S(x_k,x_k,x) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \left(r + \frac{\varepsilon}{3}\right) \\ &= r + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $S(c,c,x) \leqslant r$ and hence $S(x,x,c) \leqslant r$. Therefore, $x \in B_S[c,r]$. Hence *st-LIM*$^r x_n \subset \overline{B}_S[c,r]$ holds for some $r > 0$. □

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