Journal of Classical Analysis
Volume 25, Number 2 (2025), 173–183 doi:10.7153/jca-2025-25-11

QUASIPERIODICITY OF SOME TRANSCENDENTAL ENTIRE FUNCTIONS

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Abstract. The main aim of this article is to study the ψ-plus quasiperiodicity of complex differential polynomials, complex difference polynomials of certain types. Some new results about ψ-plus quasiperiodicity are obtained.

1. Introduction and results

We first assume that the reader is familiar with the basic results and notations of the Nevanlinna theory and difference Nevanlinna theory with one complex variable, which can be found in [1, 2, 5, 18, 19]. In the past ten years, there were lots of research focusing on the periodicity of entire function, readers can refer to [4, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

In 2024, Liu et al. [8] presented two definitions of quasiperiodic functions below which describe the φ-time quasiperiodicity and ψ-plus quasiperiodicity for meromorphic functions, where $\varphi(z)$ and $\psi(z)$ are two polynomials such that $\varphi(z) \not\equiv 0,1$ and $\psi(z) \not\equiv 0$.

DEFINITION 1. If $f(z)$ satisfies $f(z+c) = \varphi(z)f(z)$, $\varphi(z)$ is a polynomial and $\varphi(z) \not\equiv 0,1$, then $f(z)$ is a φ-time quasiperiodic function.

If $\varphi(z)$ reduces to a constant q and $(q \neq 0,1)$, then $f(z)$ is sometimes called geometric quasiperiodic function. Actually, if $c=1$ and the polynomial $\varphi(z)$ has the following presentation

$$\varphi(z) := \lambda \prod_{k=1}^{n} (z-\alpha_k),$$

where $\lambda \neq 0$, $\alpha_k (k=1,2,\cdots,n)$ are complex numbers, then the φ-time quasiperiodic functions $f(z)$ can be written as

$$f(z) = g(z)\lambda^z \prod_{k=1}^{n} \Gamma(z-\alpha_k),$$

Mathematics subject classification (2020): 30D35, 39A45.
Keywords and phrases: Differential polynomials, complex difference polynomials, transcendental entire function, Nevanlinna theory, quasiperiodicity, ψ-plus quasiperiodic function.
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where $g(z)$ is an arbitrary periodic function with period 1 and $\Gamma(z)$ is the gamma function, see [3, pp. 115–116]. We also know that any non-constant rational function cannot be a φ-time quasiperiodic function. The transcendental φ-time quasiperiodic function $f(z)$ are of order $\rho(f) \geqslant 1$.

DEFINITION 2. If $f(z)$ satisfies $f(z+c) = f(z) + \psi(z)$, $\psi(z)$ is a polynomial and $\psi(z) \not\equiv 0$, then $f(z)$ is a ψ-plus quasiperiodic function or quasiperiodic function mod $\psi(z)$.

If $\psi(z)$ is a non-zero constant, then $f(z)$ is also sometimes called arithmetic quasiperiodic function. The ψ-plus quasiperiodic function can be written as $f(z) = \pi(z) + \varsigma(z)$, where $\pi(z)$ is a periodic function with period c and $\varsigma(z)$ is a polynomial that satisfies $\varsigma(z+c) - \varsigma(z) = \psi(z)$. Thus, any non-constant polynomials are ψ-plus quasiperiodic function. For transcendental ψ-plus quasiperiodic function $f(z)$, we also see that $\rho(f) \geqslant 1$ for the reason that $\rho(\pi) \geqslant 1$ whenever $\pi(z)$ is a non-constant periodic function with period c. Let us recall the Yang's Conjecture in [15].

YANG'S CONJECTURE. Let f be a transcendental entire function and k be a positive integer. If $ff^{(k)}$ is a periodic function, then f is also a periodic function.

In 2020, Liu et al. [7] promoted the concept of Generalized Yang's Conjecture.

GENERALIZED YANG'S CONJECTURE. Let $f(z)$ be a transcendental entire function and n, k be positive integers. If $f^n f^{(k)}$ is a periodic function, then f is also a periodic function.

In 2024, Liu et al. [8] proposed Generalized Yang's Conjecture for quasiperiodicity.

φ-TIME QUASIPERIODIC VERSION OF GENERALIZED YANG'S CONJECTURE. Let $f(z)$ be a transcendental entire function and n, k be positive integers. If $f^n f^{(k)}$ is a φ-time quasiperiodic function, then f is also a φ-time quasiperiodic function.

ψ-PLUS QUASIPERIODIC VERSION OF GENERALIZED YANG'S CONJECTURE. Let $f(z)$ be a transcendental entire function and n, k be positive integers. If $f^n f^{(k)}$ is a ψ-plus quasiperiodic function, then f is also a ψ-plus quasiperiodic function. They mainly considered the ψ-plus quasiperiodicity and obtained

THEOREM 1. [8] *Let f be a transcendental entire function with $\rho_2(f) < 1$ and n, k be positive integers.*

(i) If $n \geqslant 3$, then $f^n(z)f^{(k)}(z)$ cannot be a ψ-plus quasiperiodic function with period c.

(ii) If $n \leqslant 2$, then $f^n(z)f'(z)$ cannot be a ψ-plus quasiperiodic function with period c.

The following theorem can be seen as another version of Theorem 1 where we allow n to have negative integer values.

THEOREM 2. [8] *Let f be a transcendental entire function with $\rho_2(f) < 1$ and n, k be positive integers.*

(i) If $n \geqslant 4$, then $\frac{f^{(k)}}{f^n}$ cannot be a ψ-plus quasiperiodic function with period c.

(ii) If $n \geqslant 2$*, then* $\frac{f'}{f^n}$ *cannot be a* ψ*-plus quasiperiodic function with period* c*.*

Based on Generalized Yang's Conjecture, Liu et al. also considered the periodicity of $f^n(f^m-1)f^{(k)}$.

THEOREM 3. [16] *Let* f *be a transcendental entire function and* n*,* m *and* k *be positive integers. If* $f^n(f^m-1)f^{(k)}$ *is a periodic function with period* c*, and one of following conditions is satisfied:*

(i) $f(z) = e^{h(z)}$*, where* $h(z)$ *is an entire function;*

(ii) $f(z)$ *has a non-zero Picard exceptional value and* $f(z)$ *is of finite order;*

(iii) $f^n(f^m-1)f^{(k+1)}$ *is a periodic function with period* c*; then* $f(z)$ *is also a periodic function.*

Inspired by Theorem 3, can we consider the ψ-plus quasiperiodicity about $f^n(f^m-1)f^{(k)}$? In this paper, our results are listed as follows.

THEOREM 4. *Let* f *be a transcendental entire function with* $\rho_2(f) < 1$ *and* n*,* m*,* k *be positive integers.*

(i) If $n \geqslant 3$*, then* $f^n(f^m-1)f^{(k)}$ *cannot be a* ψ*-plus quasiperiodic function with period* c*.*

(ii) If $n \geqslant \frac{m}{2}+1$*, then* $f^n(f^m-1)f'$ *cannot be a* ψ*-plus quasiperiodic function with period* c*.*

The following theorem can be seen as another version of Theorem 4 where we allow n to have negative integer values.

THEOREM 5. *Let* f *be a transcendental entire function with* $\rho_2(f) < 1$ *and* n*,* m*,* k *be positive integers. If* $n+m \geqslant 4$*, then* $\frac{f^{(k)}}{f^n(f^m-1)}$ *cannot be a* ψ*-plus quasiperiodic function with period* c*.*

Liu et al. also provided a result on the Delay Yang's Conjecture for quasiperiodicity.

THEOREM 6. [8] *Let* $f(z)$ *be a transcendental entire function with* $\rho_2(f) < 1$*.*

(i) If $n \geqslant 2$ *and* $f^n(z)f(z+\eta)$ *is a* φ*-time quasiperiodic function with period* c*, then* $f(z+c) = G(z)f(z)$*, where* $G(z)$ *is a rational function satisfying* $G^n(z)G(z+\eta) = \varphi(z)$*.*

(ii) If $n \geqslant 4$*, then* $f^n(z)f(z+\eta)$ *is not* ψ*-plus quasiperiodic function with period* c*.*

In Theorem 6, if we replace $f(z+\eta)$ by the difference operator $\Delta_\eta f(z)$, we can obtain the following theorem.

THEOREM 7. *Let* $f(z)$ *be a transcendental entire function with* $\rho_2(f) < 1$*. If* $n \geqslant 5$*, then* $f^n(z)\Delta_\eta f(z)$ *cannot be a* ψ*-plus quasiperiodic function with period* c*.*

Liu et al. also provided a result on the Delay-Differential Yang's Conjecture for quasiperiodicity.

THEOREM 8. [8] *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$. If $n \geqslant 2$, then $[f^n(z)f(z+c)]^{(k)}$ cannot be a ψ-plus quasiperiodic function with period η.*

In Theorem 8, if we replace $f(z+c)$ by the difference operator $\Delta_c f(z)$, we can obtain the following theorem.

THEOREM 9. *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$ and n, k be positive integers. If $n \geqslant 3$, then $[f^n(z)\Delta_c f(z)]^{(k)}$ cannot be a ψ-plus quasiperiodic function with period η.*

2. Some lemmas

To prove our theorems, we require the following lemmas.

LEMMA 1. [[18], Theorem 1.12] *Let $f(z)$ be a meromorphic function in the complex plane and $P(f) = a_0 f^n + a_1 f^{n-1} + \cdots + a_n$, where $a_0(\not\equiv 0)$, $a_1, \cdots, a_n$ are small functions of $f(z)$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

LEMMA 2. [10] *Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$. If $n \geqslant 1$, then*

$$T(r, f^n(z)(f(z+c) - f(z))) \geqslant nT(r, f) + S(r, f).$$

LEMMA 3. [6] *Let $f(z)$ be a transcendental meromorphic function with $\rho_2(f) < 1$. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f),$$

and

$$N(r, f(z+c)) = N(r, f) + S(r, f),$$
$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r, f).$$

LEMMA 4. [[18], Theorem 1.24] *Let $f(z)$ be a transcendental meromorphic function and k be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leqslant T(r, f^{(k)}) - T(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f),$$

and

$$N\left(r, \frac{1}{f^{(k)}}\right) \leqslant N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

LEMMA 5. [18] *Suppose $f(z)$ is a transcendental meromorphic function in the complex plane and $\alpha_1(z), \alpha_2(z), \cdots, \alpha_q(z)$ are q distinct small functions of $f(z)$. Then*

$$(q-1)T(r, f) \leqslant N(r, f) + \sum_{j=1}^{q} N\left(r, \frac{1}{f - a_j}\right) + S(r, f).$$

3. Proof of Theorem 4

Proof. (i). If $f^n(f^m-1)f^{(k)}$ is a ψ-plus quasiperiodic function with period c, we can assume that

$$f^n(z)(f^m(z)-1)f^{(k)}(z) = f^n(z+c)(f^m(z+c)-1)f^{(k)}(z+c)+\psi(z),$$

where $\psi(z)$ is a non-zero polynomial, that is,

$$f^n(z)(f^m(z)-1)f^{(k)}(z)-\psi(z) = f^n(z+c)(f^m(z+c)-1)f^{(k)}(z+c).$$

By shifting the equation above forward, we have

$$f^n(z)(f^m(z)-1)f^{(k)}(z)-\psi_1(z) = f^n(z+2c)(f^m(z+2c)-1)f^{(k)}(z+2c),$$

where $\psi_1(z)=\psi(z)+\psi(z+c)$. Furthermore, by shifting q times, we obtain

$$f^n(z)(f^m(z)-1)f^{(k)}(z)-\psi_{q-1}(z) = f^n(z+qc)(f^m(z+qc)-1)f^{(k)}(z+qc),$$

where $\psi_{q-1}(z)=\psi(z)+\psi(z+c)+\cdots+\psi(z+(q-1)c)$ and q is a positive integer.

Using Lemma 1 and the first main theorem of Nevanlinna theory, we have

$$\begin{aligned}
(n+m+1)T(r,f) &= T(r,f^{n+1}(f^m-1))+S(r,f)\\
&= T\left(r,f^{n+1}(f^m-1)\frac{f^{(k)}}{f^{(k)}}\right)+S(r,f)\\
&\leqslant T(r,f^n(f^m-1)f^{(k)})+T\left(r,\frac{f}{f^{(k)}}\right)+S(r,f)\\
&\leqslant T(r,f^n(f^m-1)f^{(k)})+T\left(r,\frac{f^{(k)}}{f}\right)+S(r,f)\\
&\leqslant T(r,f^n(f^m-1)f^{(k)})+N\left(r,\frac{f^{(k)}}{f}\right)+S(r,f)\\
&\leqslant T(r,f^n(f^m-1)f^{(k)})+T(r,f)+S(r,f),
\end{aligned}$$

thus, we have

$$T(r,f^n(f^m-1)f^{(k)}) \geqslant (n+m)T(r,f)+S(r,f). \tag{1}$$

We will affirm that $\psi_s(z)\equiv 0(s=0,1,2,\cdots,q-1)$, where $\psi_0(z)=\psi(z)$. Otherwise, if $\psi_s(z)$ are mutually distinct, using Lemma 3, Lemma 4 and Lemma 5, we obtain

$$\begin{aligned}
\overline{N}\left(r,\frac{1}{f^{(k)}(z+sc)}\right) &\leqslant N\left(r,\frac{1}{f^{(k)}(z+sc)}\right)\\
&\leqslant N\left(r,\frac{1}{f(z+sc)}\right)+k\overline{N}(r,f(z+sc))+S(r,f)\\
&\leqslant T(r,f)+S(r,f),
\end{aligned}$$

then

$$
\begin{aligned}
&(q-1)T(r,f^n(f^m-1)f^{(k)})\\
\leqslant\ & N(r,f^n(f^m-1)f^{(k)})+\sum_{s=0}^{q-1}\overline{N}\left(r,\frac{1}{f^n(f^m-1)f^{(k)}-\psi_s(z)}\right)+S(r,f^n(f^m-1)f^{(k)})\\
\leqslant\ & \sum_{s=1}^{q}\overline{N}\left(r,\frac{1}{f^n(z+sc)(f^m(z+sc)-1)f^{(k)}(z+sc)}\right)+S(r,f^n(f^m-1)f^{(k)})\\
\leqslant\ & \sum_{s=1}^{q}\left[\overline{N}\left(r,\frac{1}{f^n(z+sc)}\right)+\overline{N}\left(r,\frac{1}{f^m(z+sc)-1}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}(z+sc)}\right)\right]\\
&+S(r,f^n(f^m-1)f^{(k)})\\
\leqslant\ & \sum_{s=1}^{q}[(m+2)T(r,f(z+sc))]+S(r,f^n(f^m-1)f^{(k)})\\
\leqslant\ & q(m+2)T(r,f)+S(r,f).
\end{aligned}
$$

Combining (1) and (2), we have $(q-1)(n+m)\leqslant q(m+2)$, however if $n\geqslant 3$, by taking a large enough q, we get a contradiction.

Hence, at least two of the functions $\psi_s(z)$ are identically the same, thus, $\psi(z)\equiv 0$ for the reason that $\psi(z)$ is a polynomial.

(ii) Assume that $f^n(f^m-1)f'$ is a ψ-plus quasiperiodic function with period c, then

$$f^n(z)(f^m(z)-1)f'(z)=f^n(z+c)(f^m(z+c)-1)f'(z+c)+\psi(z).$$

Hence we have

$$\frac{f^{n+m+1}(z)}{n+m+1}-\frac{f^{n+1}(z)}{n+1}-\frac{f^{n+m+1}(z+c)}{n+m+1}+\frac{f^{n+1}(z+c)}{n+1}=\omega(z),$$

where $\omega'(z)=\psi(z)$.

In this case, we have

$$
\begin{aligned}
&f^{n+1}(z)\left[f^m(z)-\frac{n+m+1}{n+1}\right]\\
=\ & f^{n+1}(z+c)\left[f^m(z+c)-\frac{n+m+1}{n+1}\right]+(n+m+1)\omega(z).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&f^{n+1}(z+c)\left[f^m(z+c)-\frac{n+m+1}{n+1}\right]\\
=\ & f^{n+1}(z+2c)\left[f^m(z+2c)-\frac{n+m+1}{n+1}\right]+(n+m+1)\omega(z+c),
\end{aligned}
$$

we also have

$$
\begin{aligned}
&f^{n+1}(z)\left[f^m(z)-\frac{n+m+1}{n+1}\right]\\
=\ & f^{n+1}(z+2c)\left[f^m(z+2c)-\frac{n+m+1}{n+1}\right]+(n+m+1)\big[\omega(z)+\omega(z+c)\big].
\end{aligned}
$$

By Lemma 3, Lemma 4 and Lemma 5, we obtain

$$\begin{aligned}
&2T\left(r, f^{n+1}\left(f^m - \frac{n+m+1}{n+1}\right)\right)\\
&\leqslant \overline{N}\left(r, f^{n+1}\left(f^m - \frac{n+m+1}{n+1}\right)\right) + \overline{N}\left(r, \frac{1}{f^{n+1}(f^m - \frac{n+m+1}{n+1})}\right)\\
&+\overline{N}\left(r, \frac{1}{f^{n+1}(f^m - \frac{n+m+1}{n+1}) - (n+m+1)\omega(z)}\right)\\
&+\overline{N}\left(r, \frac{1}{f^{n+1}(f^m - \frac{n+m+1}{n+1}) - (n+m+1)(\omega(z)+\omega(z+c))}\right) + S(r,f)\\
&\leqslant \overline{N}\left(r, \frac{1}{f^{n+1}(f^m - \frac{n+m+1}{n+1})}\right) + \overline{N}\left(r, \frac{1}{f^{n+1}(z+c)(f^m(z+c) - \frac{n+m+1}{n+1})}\right)\\
&+\overline{N}\left(r, \frac{1}{f^{n+1}(z+2c)(f^m(z+2c) - \frac{n+m+1}{n+1})}\right) + S(r,f)\\
&\leqslant 3(m+1)T(r,f) + S(r,f).
\end{aligned}$$

Hence, we have $n \leqslant \frac{m+1}{2}$, which contradicts with $n \geqslant \frac{m}{2} + 1$. $\square$

4. Proof of Theorem 5

Proof. If $\frac{f^{(k)}}{f^n(f^m-1)}$ is a ψ-plus quasiperiodic function with period c, we obtain

$$\frac{f^{(k)}(z)}{f^n(z)(f^m(z)-1)} + \zeta_{q-1}(z) = \frac{f^{(k)}(z+qc)}{f^n(z+qc)(f^m(z+qc)-1)},$$

where $\zeta_{q-1}(z) = \psi(z) + \psi(z+c) + \cdots + \psi(z+(q-1)c)$. Since

$$\begin{aligned}
(n-1+m)T(r,f) &= T\left(r, \frac{1}{f^{n-1}(f^m-1)}\right) + S(r,f)\\
&= T\left(r, \frac{f^{(k)}}{f^n(f^m-1)}\frac{f}{f^{(k)}}\right) + S(r,f)\\
&\leqslant T\left(r, \frac{f^{(k)}}{f^n(f^m-1)}\right) + T\left(r, \frac{f}{f^{(k)}}\right) + S(r,f)\\
&\leqslant T\left(r, \frac{f^{(k)}}{f^n(f^m-1)}\right) + T\left(r, \frac{f^{(k)}}{f}\right) + S(r,f)\\
&\leqslant T\left(r, \frac{f^{(k)}}{f^n(f^m-1)}\right) + N\left(r, \frac{f^{(k)}}{f}\right) + S(r,f)\\
&\leqslant T\left(r, \frac{f^{(k)}}{f^n(f^m-1)}\right) + T(r,f) + S(r,f),
\end{aligned}$$

thus, we have

$$T\left(r,\frac{f^{(k)}}{f^n(f^m-1)}\right)\geqslant (n-2+m)T(r,f)+S(r,f). \tag{2}$$

Using Lemma 3, Lemma 4 and Lemma 5, we have

$$\begin{aligned}
&(q-1)T\left(r,\frac{f^{(k)}}{f^n(f^m-1)}\right)\\
&\leqslant \overline{N}\left(r,\frac{f^{(k)}}{f^n(f^m-1)}\right)+\sum_{s=0}^{q-1}\overline{N}\left(r,\frac{1}{\frac{f^{(k)}}{f^n(f^m-1)}-\zeta_s(z)}\right)+S\left(r,\frac{f^{(k)}}{f^n(f^m-1)}\right)\\
&\leqslant \overline{N}\left(r,\frac{1}{f^n(f^m-1)}\right)+\sum_{s=1}^{q}\overline{N}\left(r,\frac{f^n(z+sc)(f^m(z+sc)-1)}{f^{(k)}(z+sc)}\right)+S\left(r,\frac{f^{(k)}}{f^n(f^m-1)}\right)\\
&\leqslant (m+1)T(r,f)\\
&\quad+\sum_{s=1}^{q}\left[\overline{N}(r,f^n(z+sc))+\overline{N}\left(r,\frac{1}{f^{(k)}(z+sc)}\right)+\overline{N}(r,f^m(z+sc))-1)\right]\\
&\quad+S\left(r,\frac{f^{(k)}}{f^n(f^m-1)}\right)\\
&\leqslant (m+1)T(r,f)+\sum_{s=1}^{q}T(r,f(z+sc))+S\left(r,\frac{f^{(k)}}{f^n(f^m-1)}\right)\\
&\leqslant (q+m+1)T(r,f)+S(r,f).
\end{aligned} \tag{3}$$

Combining (2) and (3), we have $(q-1)(n-2+m)\leqslant q+m+1$, however if $n+m\geqslant 4$, there exists a large enough q to get a contradiction. □

5. Proof of Theorem 7

Proof. Assume that $f^n(z)\Delta_\eta f(z)$ is a ψ-plus quasiperiodic function with period c. Then

$$f^n(z+c)[f(z+c+\eta)-f(z+c)]=f^n(z)[f(z+\eta)-f(z)]+\psi(z).$$

Since $f(z)$ is a transcendental entire function with $\rho_2(f)<1$, Lemma 2, Lemma 3 and Lemma 5 imply that

$$\begin{aligned}
&nT(r,f)\\
&\leqslant T(r,f^n(z)[f(z+\eta)-f(z)])+S(r,f)\\
&\leqslant \overline{N}(r,f^n(z)[f(z+\eta)-f(z)])+\overline{N}\left(r,\frac{1}{f^n(z)[f(z+\eta)-f(z)]}\right)\\
&\quad+\overline{N}\left(r,\frac{1}{f^n(z)[f(z+\eta)-f(z)]+\psi(z)}\right)+S(r,f)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \overline{N}\left(r, \frac{1}{f^n(z)[f(z+\eta)-f(z)]}\right) + \overline{N}\left(r, \frac{1}{f^n(z+c)[f(z+c+\eta)-f(z+\eta)]}\right) \\
&\quad + S(r,f) \\
&\leqslant T(r,f) + T(r, f(z+\eta)-f(z)) + T(r, f(z+c)) + T(r, f(z+c+\eta)-f(z+\eta)) \\
&\quad + S(r,f) \\
&\leqslant T(r,f) + T(r,f) + T(r,f) + T(r, f(z+\eta)) + S(r,f) \\
&\leqslant 4T(r,f) + S(r,f),
\end{aligned}
$$

which is a contradiction with $n \geqslant 5$. Thus, we have $\psi(z) \equiv 0$. □

6. Proof of Theorem 9

Proof. If $[f^n(z)\Delta_c f(z)]^{(k)}$ is a ψ-plus quasiperiodic function with period η, we can assume that

$$[f^n(z)(f(z+c)-f(z))]^{(k)} = [f^n(z+\eta)(f(z+c+\eta)-f(z+\eta))]^{(k)} + \psi(z).$$

Integrating the above equation, we obtain

$$f^n(z)(f(z+c)-f(z)) - p(z) = f^n(z+\eta)(f(z+c+\eta)-f(z+\eta)),$$

where $p^{(k)}(z) = \psi(z)$. Shifting the equation above forward, we have

$$f^n(z)(f(z+c)-f(z)) - p(z) - p(z+\eta) = f^n(z+2\eta)(f(z+c+2\eta)-f(z+2\eta)).$$

Furthermore, by shifting q times as above, we get

$$
\begin{aligned}
&f^n(z)(f(z+c)-f(z)) - p(z) - \cdots - p(z+(q-1)\eta) \\
&= f^n(z+q\eta)(f(z+c+q\eta)-f(z+q\eta)),
\end{aligned}
$$

where $q \in N$. From Lemma 2, we have

$$nT(r,f) \leqslant T(r, f^n(z)(f(z+c)-f(z))) + S(r,f).$$

Using Lemma 3, Lemma 4 and Lemma 5, we have

$$
\begin{aligned}
&(q-1)T(r, f^n(z)(f(z+c)-f(z))) \\
&\leqslant \overline{N}(r, f^n(z)(f(z+c)-f(z))) + \sum_{s=1}^{q} \overline{N}\left(r, \frac{1}{f^n(z)(f(z+c)-f(z)) - \sum\limits_{s=1}^{q} p(z+s\eta)}\right) \\
&\quad + S(r, f^n(z)(f(z+c)-f(z))) \\
&\leqslant \sum_{s=1}^{q} \overline{N}\left(r, \frac{1}{f^n(z+s\eta)(f(z+c+s\eta)-f(z+s\eta))}\right) + S(r, f^n(z)(f(z+c)-f(z)))
\end{aligned}
$$

$$\leqslant \sum_{s=1}^{q}[T(r,f(z+s\eta))+T(r,f(z+c+s\eta)-f(z+s\eta))]+S(r,f^n(z)(f(z+c)-f(z)))$$
$$\leqslant \sum_{s=1}^{q}[T(r,f(z+s\eta))+T(r,f(z+s\eta))]+S(r,f^n(z)(f(z+c)-f(z)))$$
$$\leqslant 2qT(r,f)+S(r,f).$$

Hence, we have $(q-1)n \leqslant 2q$, however if $n \geqslant 3$, there exists a suitable q to get a contradiction. $\square$

Acknowledgement. The authors were supported by GuangDong Basic and Applied Basic Research Foundation (No. 2021A1515010062) and Fund of Education Department of Guangdong (No. 2022ZDZX1034).

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(Received October 10, 2024)

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