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WEIERSTRASS DIVISION POINTS AND THE ETA FUNCTION LAW

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Abstract. New proofs of the celebrated transformation law for the Dedekind eta function come by inspecting pairs of related Weierstrass $\wp$-functions at appropriate division points.

1. Introduction

The eta function of Dedekind, a key player in the theory of numbers and elsewhere, assigns to each point τ in the upper half-plane $\mathscr{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ the value

$$\eta(\tau) = e^{\pi \mathrm{i} \tau/12} \prod_{j=1}^{\infty} (1 - e^{2\pi \mathrm{i} \tau j}).$$

With $q = e^{2\pi \mathrm{i} \tau}$ this is often abbreviated to

$$\eta(\tau) = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j)$$

but merely as a convenience; $\eta(\tau)$ is fundamentally a holomorphic function of τ, not a branched function of q. The transformation law to which we refer in our title states that

$$\eta(-1/\tau) = \eta(\tau)\sqrt{\tau/\mathrm{i}}$$

for each τ in the upper half-plane, the holomorphic square-root (on the right half-plane) being singled out by the condition $\sqrt{1} = 1$. Several proofs of this law are presented in [1]. Perhaps the most well known proof of this transformation law is due to Siegel [8]; his one-page proof applies the residue theorem to a suitable rhombic contour and a suitable sequence of trigonometric functions. Some proofs are based on the Poisson summation formula: [9] contains one such proof, using the triple product formula and theta functions of Jacobi; [6] contains variant proofs, using instead the pentagonal number theorem of Euler. Other proofs rest on transformation properties of Eisenstein series: see [7] and [5] for such a proof involving the exceptional (or 'forbidden') Eisenstein series E_2; see [3] for a related proof, involving instead the Ramanujan-Eisenstein series Q and R, which satisfy $Q^3 - R^2 = 1728\,\eta^{24}$. A proof using the Weierstrass zeta function and simplifying an idea due to Petersson is presented in [4]. In short, the eta function transformation law has already been provided with a number of proofs.

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Here, we add an infinite number of proofs to the foregoing partial list. The setting for our proofs is the theory of elliptic functions; to this extent, our proofs are related to the aforementioned proofs that make use of Eisenstein series, theta functions or zeta functions, but the details are very different. To describe the context of our proofs briefly, fix any odd integer N greater than unity: let p be any Weierstrass $\wp$-function and multiply one of its periods by N to obtain a second $\wp$-function P; our Nth proof proceeds simply by comparing the first derivative of p at its Nth division points with the first derivative of P at its Nth division points.

2. The Mth proof of the transformation law

As in the introduction, let $N = 2M+1$ be an odd integer greater than one. Let $p = \wp(-;\omega,\omega')$ be the Weierstrass $\wp$-function having $(2\omega, 2\omega')$ as a fundamental pair of periods. As is quite customary, we take the period ratio $\tau = \omega'/\omega$ to lie in the upper half-plane $\mathscr{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and write $q = \exp\{2\pi i \tau\}$ for the corresponding nome. Let $P = \wp(-;\omega, N\omega')$ be the Weierstrass $\wp$-function that results when a period of p (here the 'second') is multiplied by N; presented this way, P has nome q^N corresponding to the period ratio $N\tau$. We are interested in evaluating the first derivatives of p and P at their Nth division points: specifically, we evaluate p' at the multiples of $\alpha := 2\omega/N$ and evaluate P' at the multiples of $A' := 2\omega'$.

In order to evaluate the derivative p' of $p = \wp(-;\omega,\omega')$ at its Nth division points, we avail ourselves of the following standard expression

$$p'(z) = -\sigma(2z)/\sigma^4(z). \tag{1}$$

Here, the Weierstrass sigma function $\sigma = \sigma(-;\omega,\omega')$ may be expressed by the factorization

$$\sigma(z) = \frac{2\omega}{\pi} \exp\left\{\frac{\zeta(\omega)z^2}{2\omega}\right\} \sin\left(\frac{\pi z}{2\omega}\right) \prod_{j=1}^{\infty} \left\{\frac{(1-q^j e^-(z))(1-q^j e^+(z))}{(1-q^j)^2}\right\} \tag{2}$$

where $\zeta(-;\omega,\omega') = \zeta = \sigma'/\sigma$ is the corresponding Weierstrass zeta function and where

$$e^{\pm}(z) = \exp\{\pm \pi i z/\omega\}.$$

As a standard reference for these Weierstrass functions, we cite Chapter XX in the classic text [10] by Whittaker and Watson: in particular, see page 448 for the sigma function expansion (2) and page 459 for the sigma function duplication formula (1). In order to evaluate P' at its Nth division points, we use the corresponding expression

$$P'(z) = -\Sigma(2z)/\Sigma^4(z) \tag{3}$$

where now

$$\Sigma(z) = \frac{2\omega}{\pi} \exp\left\{\frac{Z(\omega)z^2}{2\omega}\right\} \sin\left(\frac{\pi z}{2\omega}\right) \prod_{j=1}^{\infty} \left\{\frac{(1-Q^j e^-(z))(1-Q^j e^+(z))}{(1-Q^j)^2}\right\} \tag{4}$$

with $Z = \Sigma'/\Sigma$ the corresponding zeta function and with $Q = q^N$ abbreviating the appropriate nome for P.

THEOREM 1. *If* $1 \leqslant m \leqslant M$ *then*

$$\mathrm{i}\,\frac{\omega^3}{\pi^3}\,P'(mA') = q^m \prod_{j=1}^{\infty}\Big\{(1-Q^j)^6\,\frac{(1-Q^jq^{-2m})(1-Q^jq^{2m-N})}{(1-Q^jq^{-m})^4(1-Q^jq^{m-N})^4}\Big\}.$$

Proof. We evaluate the expression (3) for $P'(z)$ at $z = mA'$; each of the Sigma functions in (4) has four factors, which we consider in turn. The $2\omega/\pi$ coefficients together amount to $\pi^3/8\omega^3$ while the exponential factors cancel. For the trigonometric factors, we require

$$\sin\frac{\pi mA'}{2\omega} = \sin m\pi\tau = \frac{e^{2m\pi\mathrm{i}\tau}-1}{2\mathrm{i}e^{m\pi\mathrm{i}\tau}}$$

whence

$$\sin^4\frac{\pi mA'}{2\omega} = \frac{(q^m-1)^4}{16q^{2m}}$$

and similarly

$$\sin\frac{\pi m2A'}{2\omega} = \sin 2m\pi\tau = \frac{q^{2m}-1}{2\mathrm{i}q^m};$$

the trigonometric factors thus contribute

$$\frac{q^{2m}-1}{2\mathrm{i}q^m}\,\frac{16q^{2m}}{(q^m-1)^4} = 8\,\mathrm{i}\,q^m\,\frac{1-q^{2m}}{(1-q^m)^4}.$$

For the infinite products we use

$$e^{\pm}(mA') = e^{\pm\pi\mathrm{i}m2\omega'/\omega} = q^{\pm m} \text{ and } e^{\pm}(2mA') = q^{\pm 2m}$$

to see that their contribution is

$$\prod_{j=1}^{\infty}\Big\{(1-Q^j)^6\,\frac{(1-Q^jq^{-2m})(1-Q^jq^{+2m})}{(1-Q^jq^{-m})^4(1-Q^jq^{+m})^4}\Big\};$$

here, recall that $Q = q^N$ to see that we may write

$$\prod_{j=1}^{\infty}\Big\{\frac{(1-Q^jq^{+2m})}{(1-Q^jq^{+m})^4}\Big\} = \frac{(1-q^m)^4}{1-q^{2m}}\prod_{j=1}^{\infty}\Big\{\frac{(1-Q^jq^{2m-N})}{(1-Q^jq^{m-N})^4}\Big\}.$$

Finally, assemble the contributions of all four types of factor to conclude the proof. □

We now multiply the results of this Theorem as m runs through $1,\dots,M$. The outcome is pleasing: it yields a generalization of the Borwein cubic theta function c from the case $N = 3$.

THEOREM 2.

$$\mathrm{i}^M\,\frac{\omega^{3M}}{\pi^{3M}}\prod_{m=1}^{M}P'(mA') = q^{M(M+1)/2}\Big\{\prod_{j=1}^{\infty}\frac{(1-q^{Nj})^N}{1-q^j}\Big\}^3.$$

Proof. The product on the left-hand side is clear, as is the factor $q^{M(M+1)/2}$ on the right-hand side; for the balance on the right-hand side we argue as follows. The various factors $1-Q^j$ of course contribute

$$\prod_{j=1}^{\infty}(1-Q^j)^{6M} = \prod_{j=1}^{\infty}(1-q^{Nj})^{6M}.$$

To handle the remaining factors on the right-hand side, let us abbreviate $1-Q^j q^{-k}$ to $[k]$; in the jth factor, we then have

$$\prod_{m=1}^{M}\frac{[2m][N-2m]}{[m]^4[N-m]^4} = \frac{1}{[2M]^3[2M-1]^3\cdots[1]^3} = \left\{\frac{[0]}{[2M][2M-1]\cdots[1][0]}\right\}^3.$$

Here, taking the product as j runs over the positive integers, the numerators yield

$$\prod_{j=1}^{\infty}(1-Q^j)^3 = \prod_{j=1}^{\infty}(1-q^{Nj})^3$$

while the denominators yield the cube of

$$\prod_{j=1}^{\infty}\{(1-q^{Nj-2M})\cdots(1-q^{Nj-1})(1-q^{Nj})\} = \prod_{j=1}^{\infty}(1-q^j).$$

Finally, assemble the pieces and recall that $6M+3=3N$ to conclude the proof. □

We complement the foregoing evaluations by performing the corresponding evaluations of p'. As the calculations follow similar lines to those that have gone before, we may be brief. For these calculations, we abbreviate the 'first' Nth root of unity by ε: thus

$$\varepsilon := \exp\{2\pi \mathrm{i}/N\}.$$

THEOREM 3. *If* $1 \leqslant m \leqslant M$ *then*

$$-8\frac{\omega^3}{\pi^3}p'(m\alpha) = \frac{\sin(2m\pi/N)}{\sin^4(m\pi/N)}\prod_{j=1}^{\infty}\left\{(1-q^j)^6\frac{(1-q^j\varepsilon^{-2m})(1-q^j\varepsilon^{2m-N})}{(1-q^j\varepsilon^{-m})^4(1-q^j\varepsilon^{m-N})^4}\right\}.$$

Proof. Along Theorem 1 lines, calculating $p'(m\alpha)$ by tracking the four types of factor in each sigma function (2) that apears in (1). Again, the $2\omega/\pi$ coefficients contribute $\pi^3/8\omega^3$ and the exponentials cancel. The trigonometric factors contribute $\sin^4(\pi m\alpha/2\omega) = \sin^4(m\pi/N)$ in the denominator and $\sin(2m\pi/N)$ in the numerator. Finally, the contribution of the infinite products is found by noting that $e^-(m\alpha) = \varepsilon^{-m}$ and $e^+(m\alpha) = \varepsilon^m = \varepsilon^{m-N}$. □

The result of multiplying these evaluations as m runs through $1,\ldots,M$ is also pleasing: it leads to a generalization of the Borwein cubic theta function b from the case $N=3$.

THEOREM 4.

$$(-1)^M \frac{\omega^{3M}}{\pi^{3M}} \prod_{m=1}^{M} p'(m\alpha) = \left\{ \frac{1}{\sqrt{N}} \prod_{j=1}^{\infty} \frac{(1-q^j)^N}{1-q^{Nj}} \right\}^3.$$

Proof. Along Theorem 2 lines; accordingly, we need only mention the two main departures. To evaluate the product of the trigonometric ratios, we employ the familiar identity

$$\prod_{m=1}^{M} \sin\left(\frac{m\pi}{N}\right) = \frac{\sqrt{N}}{2^M}$$

which comes by letting $z \to 1$ in the factorization

$$\frac{z^N - 1}{z - 1} = \prod_{m=1}^{2M} (z - \varepsilon^m).$$

To multiply the infinite products, we employ the related elementary factorization

$$(1-q^j)(1-q^j\varepsilon)\cdots(1-q^j\varepsilon^{2M}) = 1 - q^{Nj}. \quad \square$$

We may use our $p'(m\alpha)$ evaluations to deduce alternative $P'(mA')$ evaluations, as follows. Recall that we based P on $(2\omega, 2N\omega')$ as a fundamental pair of periods. We are free to base P on any equivalent pair; we choose the equivalent pair $(2N\omega', -2\omega)$ so that $P = \wp(-; N\omega', -\omega)$. Note that the new period ratio $-1/N\tau$ also lies in the upper half-plane; we shall write

$$q_* = \exp\{-2\pi \mathrm{i}/N\tau\}$$

for the nome that this choice confers upon P. Rather than pause to record the effect that this change of perspective has on individual first derivatives, we pass directly to their product.

THEOREM 5.

$$(-1)^M N^{3M} \frac{\omega'^{3M}}{\pi^{3M}} \prod_{m=1}^{M} P'(mA') = \left\{ \frac{1}{\sqrt{N}} \prod_{j=1}^{\infty} \frac{(1-q_*^j)^N}{1-q_*^{Nj}} \right\}^3.$$

Proof. A direct translation from the result of Theorem 4: replace ω by $N\omega'$, replace α by A' and replace q by q_*. $\square$

Now compare Theorem 2 and Theorem 5: as $\tau = \omega'/\omega$ there follows at once

$$\left\{ \frac{1}{\sqrt{N}} \prod_{j=1}^{\infty} \frac{(1-q_*^j)^N}{1-q_*^{Nj}} \right\}^3 = \mathrm{i}^M N^{3M} \tau^{3M} q^{M(M+1)/2} \left\{ \prod_{j=1}^{\infty} \frac{(1-q^{Nj})^N}{1-q^j} \right\}^3. \tag{5}$$

In order to pass to the cube root here, we argue by holomorphicity in $\tau \in \mathscr{H}$ that

$$\gamma \frac{\mathrm{i}^M}{\sqrt{N}} \prod_{j=1}^{\infty} \frac{(1-q_*^j)^N}{1-q_*^{Nj}} = N^M \tau^M q^{M(M+1)/6} \prod_{j=1}^{\infty} \frac{(1-q^{Nj})^N}{1-q^j} \tag{6}$$

where for convenience we write

$$q^{M(M+1)/6} := \exp\{M(M+1)\pi \mathrm{i}\tau/3\} \tag{7}$$

and where γ is a (constant) cube root of unity. To determine γ we need only evaluate at a single value of τ. We choose $\mathrm{i}/\sqrt{N}$: at this (unique) fixed point in $\mathscr{H}$ of the involution $\tau \mapsto -1/N\tau$, we see that $q_* = q = e^{-2\pi/\sqrt{N}}$ is real and $\tau^M = \mathrm{i}^M/\sqrt{N}^M$; as the i^M coefficients cancel, it follows that $\gamma = 1$.

To simplify the presentation of our results, it is convenient to introduce two 'new' functions. Prompted by Theorem 4 we define

$$b(q) = \prod_{j=1}^{\infty} \frac{(1-q^j)^N}{1-q^{Nj}}; \tag{8}$$

the resulting function b is holomorphic in the open unit disc. Prompted by Theorem 2 we define

$$c(q) = q^{M(M+1)/6} \prod_{j=1}^{\infty} \frac{(1-q^{Nj})^N}{1-q^j}; \tag{9}$$

the resulting function of q without the leading power of q is holomorphic in the open unit disc, but $c(q)$ itself is to be regarded as a holomorphic function of τ in $\mathscr{H}$ according to (7).

In terms of these 'new' functions, we now have the following transformation laws.

THEOREM 6. *Let $\tau \in \mathscr{H}$: if $q = \exp\{2\pi\mathrm{i}\tau\}$ and $q_* = \exp\{-2\pi\mathrm{i}/N\tau\}$ then*

$$\mathrm{i}^M b(q_*) = \sqrt{N} N^M \tau^M c(q).$$

$$\mathrm{i}^M \sqrt{N} c(q_*) = \tau^M b(q).$$

Proof. The first law is a simple reformulation of (6) (where $\gamma = 1$) in terms of the 'new' functions; the second follows upon replacing τ by $-1/N\tau$. □

Our Mth proof of the transformation law for the Dedekind eta function follows easily once we express b and c in terms of η: with $q = \exp\{2\pi\mathrm{i}\tau\}$ as usual, it is immediate from (8) that

$$b(q) = \frac{\eta(\tau)^N}{\eta(N\tau)}$$

while the incorporation of the leading 'power of q' in (9) ensures that

$$c(q) = \frac{\eta(N\tau)^N}{\eta(\tau)}.$$

This being so, it follows that

$$b^N(q)\,c(q) = \eta(\tau)^{N^2-1} = \eta(\tau)^{4M(M+1)} \tag{10}$$

and

$$b(q)\,c^N(q) = \eta(N\tau)^{N^2-1} = \eta(N\tau)^{4M(M+1)}. \tag{11}$$

THEOREM 7. *If* $\tau \in \mathscr{H}$ *then*

$$\eta(-1/\tau) = \eta(\tau)\sqrt{\tau/\mathrm{i}}.$$

Proof. In (11) replace τ by $-1/N\tau$: this replaces q by q_* so that

$$\eta(-1/\tau)^{4M(M+1)} = b(q_*)\,c^N(q_*)$$

whence the transformation laws in Theorem 6 imply that

$$\eta(-1/\tau)^{4M(M+1)} = \left\{\mathrm{i}^{-M}\sqrt{N}N^M\tau^M c(q)\right\}\left\{\mathrm{i}^{-M}\tau^M b(q)/\sqrt{N}\right\}^N$$

which upon simplification using (10) yields

$$\eta(-1/\tau)^{4M(M+1)} = \tau^{2M(M+1)}\eta(\tau)^{4M(M+1)}.$$

The final step in the proof is to pass to the correct root of order $4M(M+1)$. This is effected by observing that the quotient $\sqrt{\tau/\mathrm{i}}\,\eta(\tau)/\eta(-1/\tau)$ is both a holomorphic function of $\tau \in \mathscr{H}$ and a root of unity to this order, hence constant; evaluation at the fixed point $\mathrm{i} \in \mathscr{H}$ of the involution $\tau \mapsto -1/\tau$ shows this constant root of unity to be unity itself. □

We close our account with a couple of brief remarks.

Notice that $c(q)$ is a holomorphic function of q in the open unit disc except when $M \equiv 1$ modulo 3: thus, whereas in the cubic case the function $c(q)$ is led by a cube root of q, in the quintic case ($M = 2$) it is led by q and in the septic case ($M = 3$) it is led by q^2.

The functions b and c are introduced at (8) and (9) as 'new' functions. Our choice of names for these functions is of course guided in part by the established notation when $N = 3$: in this case, b is precisely the Borwein cubic theta function of the same name, while c differs from the Borwein cubic theta function of the same name only by a numerical coefficient; see [2] and [3].

Finally, if we wish to standardize these 'new' functions so that they agree with the Borwein functions when $N = 3$ then we should rescale c and so define

$$b(\tau) = \frac{\eta(\tau)^N}{\eta(N\tau)} \text{ and } c(\tau) = N^{(M+1)/2}\frac{\eta(N\tau)^N}{\eta(\tau)}. \tag{12}$$

Once this is done, the transformation laws assume the matched forms

$$\mathrm{i}^M b(q_*) = \sqrt{N^M}\,\tau^M c(q)$$

and

$$\mathrm{i}^M c(q_*) = \sqrt{N^M}\,\tau^M b(q).$$

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