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BOUNDEDNESS OF HIGHER ORDER COMMUTATORS OF FRACTIONAL VARIABLE ROUGH HARDY OPERATORS ON GRAND VARIABLE HERZ SPACES

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Abstract. In this paper, we obtain the boundedness of higher order commutators of fractional variable rough Hardy operators on grand variable Herz spaces $\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)}$.

1. Introduction

In the last two decades it was evident that classical function spaces are no longer appropriate for studying a number of modern problems arising in many mathematical models of applied sciences. It thus became necessary to introduce and study new function spaces. Such spaces are: variable exponent Lebesgue and Sobolev spaces, grand function spaces, Morrey-type spaces, amalgam spaces, Herz spaces, their hybrid variants, etc (see e.g., the monographs [3], [5], [15], [16] and references therein dedicated to new function spaces). Morrey spaces describe local regularity more precisely than Lebesgue spaces. As a result, one can use Morrey spaces widely not only in Harmonic Analysis but also in the theory of PDEs. We refer to the recent monographs [28] for Morrey-type spaces and applications.

The classical Hardy operator

$$Hg(x) := \frac{1}{x}\int_0^x g(t)dt,$$

and its dual

$$H^*g(x) := \int_x^\infty \frac{g(t)}{t}dt, \ \ x>0,$$

play an important role in the study of number of problems of harmonic analysis and differential equations (see, e.g., [19]).

First generalizations of Herz spaces with variable exponent were given in [11] (see also [1]), where the authors studied the boundedness of sublinear operators in these spaces. The variable exponent Herz-Morrey (VEHM briefly) space is the generalization of the Herz spaces with variable exponent. The VEHM space was initially

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defined by the author in [12]. The boundednes for fractional Hardy operator on Herz-Morrey spaces with variable exponent was proved in [41]. It should be emphasized that continual Herz spaces with variable exponents were defined in [27], where the authors explored the mapping properties of sublinear operators in these spaces.

Grand Morrey spaces introduced in [22] and took considerable amount of attention of researchers, author proved the boundedness of class of integral operators in newly defined grand Morrey spaces (see also [25] for further generalizations). Grand variable exponent Lebesgue spaces were spaces were introduced in [14]. Recently, the same authors studied the boundedness of integral operators in grand variable exponent Morrey spaces (see [13]).

Grand variable exponent Herz-Morrey (GVEHM briefly) spaces were introduced in [30], where the boundedness of Riesz potential operator in these spaces were also proved. Then boundedness of Marcinkiewicz integral operator of variable order in GVEHM spaces was studied in [36] (see also [20], [40], [34], [39], [32] for other boundedness results in these spaces). These spaces were introduced in the spirit of [23] and [24], where mapping properties of sublinear operators were studied as well (see also [31], [35] for more results in these spaces). Grand weighted Herz spaces and grand weighted Herz-Morrey spaces were introduced and studied in [29, 37] respectively. Boundedness of other operators were then studied in subsequent works such as [2, 33, 36]. Motivated by the above results, in this article we investigate the boundedness of higher order commutators of fractional variable Hardy operators (FVHOs briefly) with rough kernels from GVEMS $\dot{K}^{\alpha(\cdot),u),\theta}_{p(\cdot)}(\mathbb{R}^n)$ to another one $\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)}(\mathbb{R}^n)$ with a certain weight function, where $q(\cdot)$ is the Sobolev variable exponent of $p(\cdot)$.

2. Preliminaries

This section is dedicated to some necessary definitions and important lemmas to prove our main results.

2.1. Function spaces with variable exponent

For this section we refer to [3–5, 8, 9, 18].

DEFINITION 1. Let H be a measurable subset of $\mathbb{R}^n$ and let $p\colon H \to [1,\infty)$ is a measurable function. We suppose that

$$1 \leqslant p_-(H) \leqslant p(x) \leqslant p_+(H) < \infty, \tag{1}$$

where $p_- := \operatorname{ess}\inf_{x\in H} p(x), \quad p_+ := \operatorname{ess}\sup_{x\in H} p(x).$

(a) Lebesgue space with variable exponent $L^{p(\cdot)}(H)$ is the class of functions of H such that

$$L^{p(\cdot)}(H) = \left\{ f : \int_H \left(\frac{|f(y)|}{\gamma} \right)^{p(y)} dy < \infty, \text{ where } \gamma \text{ is a constant} \right\}.$$

Norm in $L^{p(\cdot)}(H)$ is defined as follows

$$\|f\|_{L^{p(\cdot)}(H)} = \inf\left\{\gamma > 0 : \int_H \left(\frac{|f(y)|}{\gamma}\right)^{p(y)} dy \leqslant 1\right\}.$$

(b) The space $L^{p(\cdot)}_{\mathrm{loc}}(H)$ is the following class of functions:

$$L^{p(\cdot)}_{\mathrm{loc}}(H) := \left\{f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset H\right\}.$$

Let us recall the well-known log-Hölder continuity condition (or Dini-Lipschitz condition) for $p : H \mapsto (0,\infty)$: there is a positive constant C such that for all $x,y \in H$ with $|x-y| \leqslant \frac{1}{2}$,

$$|p(x) - p(y)| \leqslant \frac{C}{-\ln|x-y|}. \tag{2}$$

Further, we say that $p(\cdot)$ satisfies the decay condition if there exists $p_\infty := p(\infty) = \lim\limits_{|x|\to\infty} p(x)$, and there is a positive constant $C_\infty > 0$ such that

$$|p(h) - p_\infty| \leqslant \frac{C_\infty}{\ln(e+|h|)}. \tag{3}$$

We will need also the log Hölder continuity condition at 0 for $p(\cdot)$: there are constants $C_0 > 0$ such that for all $|h| \leqslant \frac{1}{2}$,

$$|p(h) - p(0)| \leqslant \frac{C_0}{\ln|h|}. \tag{4}$$

The best possible constant C in 2 (resp. C_∞ in 3) is called log-Hölder continuity or log-Dini-Lipschitz constant (resp. decay constant) for the exponent $p(\cdot)$.

The following definitions and notations will be used throughout the manuscript:

(i) Let $f \in L^1_{\mathrm{loc}}(H)$ be a locally integrable function, then the Hardy-Littlewood maximal operator M is defined as

$$(Mf)(y) := \sup_{s>0} s^{-n} \int\limits_{B(y,s)} |f(y)|dy, \quad (y \in H),$$

where

$$B(y,s) := \{x \in H : |y-x| < s\}.$$

(ii) The set $\mathfrak{P}(H)$ is consists of all measurable functions $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

(iii) $\mathfrak{P}^{\log} = \mathfrak{P}^{\log}(H)$ consists of all functions $q(\cdot) \in \mathfrak{P}(H)$ satisfying (1) and (2).

(iv) $\mathfrak{P}_\infty(H)$ (resp. $\mathfrak{P}_{0,\infty}(H)$) is the subset of $\mathfrak{P}(H)$ consisting of functions which satisfy condition (3) (resp. both conditions (3) and (4)).

(v)

$$\chi_l := \chi_{F_l},\ F_l := B_l \setminus B_{l-1},$$

$$B_l := B(0,2^l) = \{x \in \mathbb{R}^n : |x| < 2^l\}, \quad l \in \mathbb{Z}.$$

We denote by $I^{\Upsilon(x)}$ (or $I^{\Upsilon(\cdot)}$) the Riesz potential operator with variable parameter $\Upsilon(x)$ defined on $\mathbb{R}^n$ and given by the formula:

$$I^{\Upsilon(\cdot)}f(x) = \int_{\mathbb{R}^n} \frac{f(s)}{|x-s|^{n-\Upsilon(x)}} ds, \quad 0 < \Upsilon(\cdot) < n. \tag{5}$$

Riesz potentials play important role if harmonic analysis and PDEs, in particular in the theory of Sobolev embeddings (see, e.g., [21]). We assume that order of Riesz potential operator $\Upsilon(x)$ is not continuous, but rather that it is a measurable function in $\mathbb{R}^n$ satisfying the following conditions:

(1) $\Upsilon_0 := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} \Upsilon(x) > 0$,

(2) $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)\Upsilon(x) < n$,

(3) $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(\infty)\Upsilon(x) < n$.

The following proposition is the one of the main requirement to prove our main results. The next proposition deals with weighted Sobolev inequality in variable Lebesgue space proved in [17].

PROPOSITION 2.1. *Suppose that* $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n) \cap \mathfrak{B}^{\log}(\mathbb{R}^n) \cap \mathfrak{B}_\infty(\mathbb{R}^n)$*, and assume*

$$1 < p(\infty) \leqslant p(x) \leqslant p_+ < \infty.$$

Let $\Upsilon(x)$ *satisfies the above conditions* (1), (2) *and* (3). *Then we have following weighted Sobolev-type estimate for the fractional operator* $I^{\Upsilon(z)}$,

$$\|(1+|\cdot|)^{-\lambda(\cdot)} I^{\Upsilon(\cdot)}(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $q(\cdot)$ *is the variable Sobolev exponent defined by*

$$\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\Upsilon(\cdot)}{n},$$

and

$$\lambda(\cdot) = C_\infty \Upsilon(\cdot)\left(1 - \frac{\Upsilon(\cdot)}{n}\right) \leqslant \frac{n}{4} C_\infty, \tag{6}$$

C_∞ *is being the decay constant for* $p(\cdot)$ *(see* (3)*).*

The following remark follows from the Dini-Lipschitz condition for variable parameter $\Upsilon(\cdot)$:

REMARK 2.2.

(i) If $\Upsilon(x)$ is satisfying the condition (3): $|\Upsilon(x)-\Upsilon_\infty| \leqslant \frac{C_\infty}{\ln(e+|x|)}$ for $x \in \mathbb{R}^n$. Then $(1+|\cdot|)^{-\lambda(\cdot)}$ is equivalent to the weight $(1+|\cdot|)^{-\lambda_\infty}$.

(ii) One can replace the variable parameter $\Upsilon(x)$ of Riesz potential operator $I^{\Upsilon(z)}$ by $\Upsilon(y)$ in the case of potentials over bounded domain, where y is the integration variable, since the Dini-Lipschitz condition implies that

$$C_1|x-y|^{n-\Upsilon(y)}| \leqslant |x-y|^{n-\Upsilon(x)} \leqslant C_2|x-y|^{n-\Upsilon(y)}.$$

2.2. Variable exponent Herz spaces

Classical Herz spaces were introduced in [7]. Later these spaces attracted a considerable interest of researcher from various viewpoint (see, e.g., [6]).

In this section we will recall definitions of the classical and variable exponent Herz spaces.

DEFINITION 2. [7] Let $u,v \in [1,\infty)$, $a \in \mathbb{R}$, the norms of classical versions of homogeneous and non-homogeneous Herz spaces are given below,

$$\|g\|_{K^a_{u,v}(\mathbb{R}^n)} := \|g\|_{L^u(B(0,1))} + \left\{ \sum_{l\in\mathbb{N}} 2^{lav} \left(\int\limits_{F_{l-1,l}} |g(y)|^u dy \right)^{\frac{v}{u}} \right\}^{\frac{1}{v}}, \tag{7}$$

$$\|g\|_{\dot{K}^a_{u,v}(\mathbb{R}^n)} := \left\{ \sum_{l\in\mathbb{Z}} 2^{lav} \left(\int\limits_{F_{l-1,l}} |g(y)|^u dy \right)^{\frac{v}{u}} \right\}^{\frac{1}{v}}, \tag{8}$$

respectively, where $F_{t,\tau} := B(0,2^\tau)\backslash B(0,2^t)$.

Herz spaces with variable exponents have been recently introduced in [11] (see also [1]).

DEFINITION 3. Let $u \in [1,\infty)$, $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ and $a \in \mathbb{R}$. The homogeneous version of Herz space $\dot{K}^{a,u}_{v(\cdot)}(\mathbb{R}^n)$ can be defined as

$$\dot{K}^{a,u}_{v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{a,u}_{v(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{9}$$

where

$$\|g\|_{\dot{K}^{a,u}_{v(\cdot)}(\mathbb{R}^n)} = \left(\sum_{l\in\mathbb{Z}} \|2^{la} g \chi_l\|^u_{L^{v(\cdot)}} \right)^{\frac{1}{u}}.$$

DEFINITION 4. Let $u \in [1,\infty)$, $a \in \mathbb{R}$ and $v(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. The non-homogeneous Herz space $K^{a,u}_{v(\cdot)}(\mathbb{R}^n)$ can be defined as

$$K^{a,u}_{v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{K^{a,u}_{v(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{10}$$

where

$$\|g\|_{K^{a,u}_{v(\cdot)}(\mathbb{R}^n)} = \|g\|_{L^{v(\cdot)}(B(0,1))} + \left(\sum_{k\in\mathbb{Z}} \|2^{ka} g\chi_k\|^u_{L^{v(\cdot)}} \right)^{\frac{1}{u}}.$$

Now we define higher order commutators of rough variable Hardy operators.

Let $\mathbb{S}^{n-1}$ is denoting the unit sphere in $\mathbb{R}^n$ with the normalized Lebesgue measure. $\Phi \in L^r(\mathbb{S}^{n-1})$ is a function of degree zero which is a homogeneous such that

$$\int\limits_{\mathbb{S}^{n-1}} \Phi(z') d\Phi(z') = 0, \tag{11}$$

where $z' = z/|z|$ and z is not zero.

Let f is a locally integrable function on $\mathbb{R}^n$, $b \in BMO(\mathbb{R}^n)$, $0 \leqslant \Upsilon(z) < n$ and $m > 0$. Now the higher order commutators of fractional variable Hardy operators with rough kernels can be defined as

$$\mathscr{H}^m_b f(z) := \frac{1}{|z|^{n-\Upsilon(z)}} \int_{|x|<|z|} \Phi(z-x)[b(z)-b(x)]^m f(x)dx,$$

$$\mathscr{H}^{m*}_b f(z) := \int_{|x|\geqslant|z|} \frac{\Phi(z-x)[b(z)-b(x)]^m f(x)}{|x|^{n-\Upsilon(z)}} dx, \quad z \in \mathbb{R}^n \setminus 0.$$

DEFINITION 5. Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $u \in [1,\infty)$, $q : \mathbb{R}^n \to [1,\infty)$, $\theta > 0$. A grand variable Herz (GVH briefly) space $\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)}$ is defined by

$$\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)} = \left\{ g \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)}} < \infty \right\},$$

where

$$\|g\|_{\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)}} = \sup_{\varepsilon>0} \left(\varepsilon^\theta \sum_{k\in\mathbb{Z}} \|2^{k\alpha(\cdot)} g\chi_k\|^{u(1+\varepsilon)}_{L^{q(\cdot)}} \right)^{\frac{1}{u(1+\varepsilon)}}.$$

The next proposition is the generalization of variable exponents Herz spaces in [1]. We omit the proof of the proposition since is essentially similar to the proof given in [1] and with slight modification we can obtain following result in grand variable Herz spaces.

PROPOSITION 2.3. *Let* α, p, q *are as defined in definition* 2.7*, then*

$$\begin{aligned}\|f\|_{\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)}(\mathbb{R}^n)} &= \sup_{\psi>0}\left(\varepsilon^{\theta}\sum_{k\in\mathbb{Z}}\|2^{k\alpha(\cdot)}f\chi_k\|^{u(1+\varepsilon)}_{L^{q(\cdot)}}\right)^{\frac{1}{u(1+\varepsilon)}}\\ &\approx \sup_{\psi>0}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)u(1+\varepsilon)}\|f\chi_k\|^{u(1+\varepsilon)}_{L^{q(\cdot)}}\right)^{\frac{1}{u(1+\varepsilon)}}\\ &\quad + \sup_{\psi>0}\left(\varepsilon^{\theta}\sum_{k=0}^{\infty}2^{k\alpha_\infty u(1+\varepsilon)}\|f\chi_k\|^{u(1+\varepsilon)}_{L^{q(\cdot)}}\right)^{\frac{1}{u(1+\varepsilon)}}.\end{aligned}$$

DEFINITION 6. (BMO space) A BMO function is a locally integrable function u whose mean oscillation given by $\frac{1}{|Q|}\int_Q |u(y)-u_Q|dy$ is bounded, i.e.,

$$\|u\|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |u(y)-u_Q|dy < \infty.$$

2.3. Basic Lemmas

LEMMA 2.4. [27] *Let* $D>1$ *be a constant and let* $p(\cdot)\in\mathfrak{P}_{0,\infty}(\mathbb{R}^n)$*. Then there are positive constants* c_0 *and* c_∞

$$\frac{1}{c_0}s^{\frac{n}{p(0)}} \leqslant \|\chi_{B(0,Ds)\setminus B(0,s)}\|_{p(\cdot)} \leqslant t_0 s^{\frac{n}{p(0)}}, \ \textit{for}\ \ 0<s\leqslant 1 \tag{12}$$

and

$$\frac{1}{c_\infty}s^{\frac{n}{p_\infty}} \leqslant \|\chi_{B(0,Ds)\setminus B(0,s)}\|_{p(\cdot)} \leqslant c_\infty s^{\frac{n}{p_\infty}}, \ \textit{for}\ \ s\geqslant 1, \tag{13}$$

respectively, where the constants $c_0 \geqslant 1$ *and* $c_\infty \geqslant 1$ *depend on* D *but do not depend on* s*.*

Let $\frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1$. Then the Hölder inequality in in the variable exponent case has the form:

$$\int f(x)g(x)dx \leqslant \kappa\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}},$$

where $\kappa = \frac{1}{p_-}+\frac{1}{(p')_-}$.

The next statement is the generalized Hölder inequality for variable exponent Lebesgue spaces (see [3], [15]):

LEMMA 2.5. *Let* H *be a measurable subset of* $\mathbb{R}^n$ *and let* $p(\cdot)$ *be an exponent such that* $1\leqslant p_-(H)\leqslant p_+(H)\leqslant\infty$*. Then*

$$\|fg\|_{L^{r(\cdot)}(H)} \leqslant 2^{1/r_-}\|f\|_{L^{p(\cdot)}(H)}\|g\|_{L^{q(\cdot)}(H)}$$

holds, where $f\in L^{p(\cdot)}(H)$*,* $g\in L^{q(\cdot)}(H)$ *and* $\frac{1}{r(\cdot)}=\frac{1}{p(\cdot)}+\frac{1}{q(\cdot)}$*.*

LEMMA 2.6. ([10]) *Let k be a positive integers. Then there is a positive constant C such that for all $b \in BMO(\mathbb{R}^n)$ and all $m, n \in \mathbb{Z}$ for $n > m$,*

$$C^{-1}\|b\|_{BMO(\mathbb{R}^n)}^k \leqslant \sup_B \frac{1}{\|\chi_B\|_{L^{q(\cdot)}}} \|(b-b_B)^k \chi_B\|_{L^{q(\cdot)}} \leqslant C\|b\|_{BMO(\mathbb{R}^n)}^k,$$

where the supremum is taken over all balls B;

$$\|(b-b_{B_m})^k \chi_{B_n}\|_{L^{q(\cdot)}} \leqslant C(n-m)^k \|b\|_{BMO(\mathbb{R}^n)}^k ||\chi_{B_n}||_{L^{q(\cdot)}}. \tag{14}$$

LEMMA 2.7. ([38]) *If $a > 0$, $s \in [1,\infty]$, $0 < d \leqslant s$ and $-m+(m-1)ds < u < \infty$, then*

$$\left(\int\limits_{|x_2| \leqslant a|x_1|} |x_2|^u |\Phi(x_1-x_2)|^d dx_2 \right)^{1/d} \leqslant |x_1|^{(u+m)/d} \|\Phi\|_{L^s(\mathbb{S}^{m-1})}.$$

3. Main results and their proofs

Now we formulate and prove the boundedness of the higher order commutators of fractional variable Hardy operators with rough kernels on grand variable Herz spaces under the log-Hölder continuity condition (log-condition and decay condition at infinity) on exponent of spaces. As it can bee seen that grand variable Herz spaces is the generalization of Herz spaces with variable exponent, our main results hold for Herz spaces for variable exponent. Our main results: can be stated as follows

THEOREM 3.1. *Let $0 < v \leqslant 1$, $\alpha(\cdot), q(\cdot) \in \mathfrak{B}_{0,\infty}(\mathbb{R}^n)$ with $1 < q^- \leqslant q^+ < \infty$, $1 \leqslant u < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let Φ be homogeneous of degree zero and $\Phi \in L^s(\mathbb{S}^{n-1})$, $s > q'^-$. Let α be such that :*

(i) $-\frac{n}{q_1(0)} - v - \frac{n}{s} < \alpha(0) < \frac{n}{q_1'(0)} - v - \frac{n}{s}$

(ii) $-\frac{n}{q_{1\infty}} - v - \frac{n}{s} < \alpha_\infty < \frac{n}{q_{1\infty}'} - v - \frac{n}{s}$.

Then

$$\|(1+|x|)^{-\lambda(x)} \mathscr{H}_b^m(g)\|_{\dot{K}_{q_2(\cdot)}^{\alpha(\cdot),u),\theta}(\mathbb{R}^n)} \leqslant C\|b\|_{BMO(\mathbb{R}^n)}^m \|g\|_{\dot{K}_{q_1(\cdot)}^{\alpha(\cdot),u),\theta}(\mathbb{R}^n)}.$$

Proof. Let $f \in \dot{K}_{q_2(\cdot)}^{\alpha(\cdot),u),\theta}(\mathbb{R}^n)$, and $f(z) = \sum_{j=-\infty}^{\infty} f(z)\chi_j(z) = \sum_{j=-\infty}^{\infty} f_j(z)$, by using Hölder's inequality we have

$$\begin{aligned}
&|\mathscr{H}_b^m(f)(z)\chi_k(z)| \\
&\leqslant \frac{1}{|z|^{n-\Upsilon(\cdot)}} \int_{F_j} |f(x)||\Phi(z-x)|[b(z)-b(x)]^m dx \chi_k(z) \\
&\leqslant \frac{1}{|z|^{n-\Upsilon(\cdot)}} \left(|b(z)-b_{B_j}|^m \int_{F_j} |\Phi(z-x)||f(x)|dx \right. \\
&\qquad \left. + \int_{F_j} |\Phi(z-x)||b(x)-b_{B_j}|^m |f(x)|dx \right) \chi_k(z)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{|z|^{n-\Upsilon(\cdot)}}\Bigg(|b(z)-b_{B_j}|^m \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}} \|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_1'(\cdot)}} \\
&\quad + \sum_{j=-\infty}^{k} \|\Phi(z-\cdot)(b-b_{B_j})^m \chi_j(\cdot)\|_{L^{q_1'(\cdot)}} \|f_j\|_{L^{q_1(\cdot)}}\Bigg)\chi_k(z) \\
&\leqslant \frac{1}{|z|^{n-\Upsilon(\cdot)}} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}} \big(|b(z)-b_{B_j}|^m \|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_1'(\cdot)}} \\
&\quad + \|\Phi(z-\cdot)(b-b_{B_j})^m \chi_j(\cdot)\|_{L^{q_1'(\cdot)}}\big)\chi_k(z) \\
&\leqslant C|z|^{-n} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}} \big(|b(z)-b_{B_j}|^m \|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_1'(\cdot)}} \\
&\quad + \|\Phi(z-\cdot)(b-b_{B_j})^m \chi_j(\cdot)\|_{L^{q_1'(\cdot)}}\big)|z|^{\Upsilon(z)}\big)\chi_k(z).
\end{aligned}
$$

It is known, see e.g. [41] that

$$
\begin{aligned}
I^{\Upsilon(\cdot)}((b(z)-b_{B_j})^m\chi_k)(z) &\geqslant I^{\Upsilon(\cdot)}((b(z)-b_{B_j})^m\chi_k)(z).(\chi_{B_k})(z) \\
&= \left(\int_{F_k} \frac{|b(z)-b_{B_j}|^m}{|z-z_2|^{n-\Upsilon(z)}} dz_2\right)\chi_{B_k}(z) \\
&\geqslant C|b(z)-b_{B_j}|^m |z|^{\Upsilon(z)} \chi_{B_k}(z) \\
&\geqslant C|b(z)-b_{B_j}|^m |z|^{\Upsilon(z)} \chi_k(z).
\end{aligned}
$$

As we know that $s > q'^{-}$, we define $q_1(\cdot)$ by the relation $\frac{1}{q_1'(x)} = \frac{1}{q_1(x)} + \frac{1}{s}$. By Lemma 2.7 and generalized Hölder's inequality we find that

$$
\begin{aligned}
\|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_1'(\cdot)}} &\leqslant 2\|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)} \|\chi_j(\cdot)\|_{L^{q_1(\cdot)}} \\
&\leqslant 2^{-jv+2}\left(\int_{2^{j-1}<|x|<2^j} |\Phi(z-x)|^s |x|^{sv} dx\right)^{1/s} \|\chi_j\|_{L^{q_1(\cdot)}} \\
&\leqslant 2^{-jv+2} 2^{k(v+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \| \|\chi_j\|_{L^{q_1(\cdot)}}.
\end{aligned}
$$

Similarly, by Lemma 2.6 we get

$$
\begin{aligned}
&\|(b-b_{B_j})^m(\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_1'(\cdot)}} \\
&\qquad \leqslant \|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)} \|(b-b_{B_j})^m \chi_j(\cdot)\|_{L^{q_1(\cdot)}} \\
&\qquad \leqslant C\|b\|^m_{BMO(\mathbb{R}^n)} \|\chi_j\|_{L^{q_1(\cdot)}} \|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)} \\
&\qquad \leqslant C\|b\|^m_{BMO(\mathbb{R}^n)} 2^{-jv} 2^{k(v+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \| \|\chi_j\|_{L^{q_1(\cdot)}}.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\|\chi_k(1+|z|)^{-\lambda(z)}\mathscr{H}_b^m(f_j)\|_{L^{q_2(\cdot)}}\\
&\leqslant C\sum_{j=-\infty}^{k}|z|^{-n}\|f_j\|_{L^{q_1(\cdot)}}\Big(\|(1+|z|)^{-\lambda(z)}|z|^{\Upsilon(z)}(b-b_{B_j})^m\chi_k\|_{L^{q_2(\cdot)}}\|\Phi(z-\cdot)\chi_j(\cdot)\|_{q_1'(\cdot)}\\
&\quad+\|\Phi(z-\cdot)(b-b_{B_j})^m\chi_j(\cdot)\|_{L^{q_1'(\cdot)}}\|(1+|z|)^{-\lambda(z)}I^{\Upsilon(\cdot)}(\chi_{B_k})\|_{L^{q_2(\cdot)}}\Big)\\
&\leqslant C\sum_{j=-\infty}^{k}|z|^{-n}\|f_j\|_{L^{q_1(\cdot)}}\\
&\quad\times\Big(\|(1+|z|)^{-\lambda(z)}I^{\Upsilon(\cdot)}((b-b_{B_j})^m\chi_k)\|_{L^{q_2(\cdot)}}2^{-jv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_j\|_{L^{q_1(\cdot)}}\\
&\quad+\|\|b\|_{BMO(\mathbb{R}^n)}^m2^{-jv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_j\|_{L^{q_1(\cdot)}}\|(1+|z|)^{-\lambda(z)}I^{\Upsilon(\cdot)}(\chi_k)\|_{L^{q_2(\cdot)}}\Big)\\
&\leqslant C\sum_{j=-\infty}^{k}|z|^{-n}\|f_j\|_{L^{q_1(\cdot)}}\big((k-j)^m\|b\|_{BMO(\mathbb{R}^n)}^m2^{-jv}2^{k(v+\frac{n}{s})}\|\chi_k\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1(\cdot)}}\\
&\quad+\|b\|_{BMO(\mathbb{R}^n)}^m2^{-jv}2^{k(v+\frac{n}{s})}\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q_1(\cdot)}}\big)\\
&\leqslant C\sum_{j=-\infty}^{k}|z|^{-n}2^{-jv}2^{k(v+\frac{n}{s})}(k-j)^m\|b\|_{BMO(\mathbb{R}^n)}^m\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}.
\end{aligned}
$$

$$
\begin{aligned}
&\|(1+|z|)^{-\lambda(z)}\mathscr{H}_b^m(f)\|_{\dot K_{q_2(\cdot)}^{\alpha(\cdot),u),\theta}(\mathbb{R}^n)}\\
&=\sup_{\theta>0}\left(\varepsilon^\theta\sum_{k\in\mathbb{Z}}\|2^{k\alpha(\cdot)}\chi_k(1+|z|)^{-\lambda(z)}\mathscr{H}_b^m(f)\|_{L^{q_2(\cdot)}}^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&\leqslant\sup_{\theta>0}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)u(1+\varepsilon)}\left(\sum_{j=-\infty}^{k}|z|^{-n}2^{-jv}2^{k(v+\frac{n}{s})}(k-j)^m\right.\right.\\
&\qquad\left.\left.\times\|b\|_{BMO(\mathbb{R}^n)}^m\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\right)^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&\qquad+\sup_{\theta>0}\left(\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\alpha_\infty u(1+\varepsilon)}\left(\sum_{j=-\infty}^{k}|z|^{-n}2^{-jv}2^{k(v+\frac{n}{s})}(k-j)^m\right.\right.\\
&\qquad\left.\left.\times\|b\|_{BMO(\mathbb{R}^n)}^m\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\right)^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&=:E_1+E_2.
\end{aligned}
$$

Now we will find estimate for E_1 by using the fact $z\in F_k$ and $j\leqslant k$.

Applying above-derived results to E_1 to get

$$\begin{aligned}E_1 &\leqslant \sup_{\theta>0}\Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)u(1+\varepsilon)}\Bigg(\sum_{j=-\infty}^{k} 2^{-kn}2^{-jv}2^{k(v+\frac{n}{s})}(k-j)^m \\ &\quad \times \|b\|^m_{BMO(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}}\|\chi_j\|_{L^{q_1(\cdot)}}\|\chi_k\|_{L^{q_1(\cdot)}}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\ &\leqslant C\sup_{\theta>0}\Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{ka(0)u(1+\varepsilon)} \\ &\quad \times \Bigg(\sum_{j=-\infty}^{k}(k-j)^m\|b\|^m_{BMO(\mathbb{R}^n)}2^{(j-k)(\frac{n}{q_1'(0)}-v-\frac{n}{s})}\|f_j\|_{L^{q_1(\cdot)}}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}.\end{aligned}$$

Let $v_1 := \frac{n}{q_1'(0)} - \alpha(0) - v - \frac{n}{s} > 0$. Applying Hölder's inequality, Fubini's theorem for series and the fact that $2^{-u(1+\varepsilon)} < 2^{-u}$ we obtain

$$\begin{aligned}E_1 &\leqslant C\sup_{\theta>0}\Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1}\Bigg(\sum_{j=-\infty}^{k} 2^{\alpha(0)j}\|f_j\|_{L^{q_1(\cdot)}}(k-j)^m\|b\|^m_{BMO(\mathbb{R}^n)}2^{v_1(j-k)}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\ &\leqslant C\|b\|_{BMO(\mathbb{R}^n)}\sup_{\theta>0}\Bigg[\varepsilon^\theta \sum_{k=-\infty}^{-1}\Bigg(\sum_{j=-\infty}^{k} 2^{\alpha(0)u(1+\varepsilon)j}\|f_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}}2^{v_1u(1+\varepsilon)(j-k)/2}\Bigg) \\ &\quad \times \Bigg(\sum_{j=-\infty}^{k}(k-j)^{mu(1+\varepsilon)/2}2^{v_1(u(1+\varepsilon))'(j-k)/2}\Bigg)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Bigg]^{\frac{1}{u(1+\varepsilon)}} \\ &\leqslant C\|b\|_{BMO(\mathbb{R}^n)}\sup_{\theta>0}\Bigg[\varepsilon^\theta \sum_{k=-\infty}^{-1}\sum_{j=-\infty}^{k} 2^{\alpha(0)u(1+\varepsilon)j}\|f_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}}2^{v_1u(1+\varepsilon)(j-k)/2}\Bigg]^{\frac{1}{u(1+\varepsilon)}} \\ &\leqslant C\|b\|_{BMO(\mathbb{R}^n)}\sup_{\theta>0}\Bigg(\varepsilon^\theta \sum_{j=-\infty}^{-1} 2^{\alpha(0)u(1+\varepsilon)j}\|f_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}}\sum_{k=j}^{-1}2^{v_1u(1+\varepsilon)(j-k)/2}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\ &\leqslant C\|b\|_{BMO(\mathbb{R}^n)}\sup_{\theta>0}\Bigg(\varepsilon^\theta \sum_{j=-\infty}^{-1} 2^{\alpha(0)u(1+\varepsilon)j}\|f_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\ &= C\|b\|_{BMO(\mathbb{R}^n)}\sup_{\theta>0}\Bigg(\varepsilon^\theta \sum_{j\in\mathbb{Z}} 2^{\alpha(\cdot)u(1+\varepsilon)j}\|f_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\ &\leqslant C\|b\|_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{\alpha(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Now for E_2 using Minkowski's inequality we have

$$
\begin{aligned}
E_2 &\leqslant \sup_{\theta>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty u(1+\varepsilon)}\left(\sum_{j=-\infty}^{k}\|\chi_k(1+|z|)^{-\lambda(z)}\mathscr{H}_b^m(f_j)\|_{L^{q_2(\cdot)}}\right)^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&\leqslant \sup_{\theta>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty u(1+\varepsilon)}\left(\sum_{j=-\infty}^{-1}\|\chi_k(1+|z|)^{-\lambda(z)}\mathscr{H}_b^m(f_j)\|_{L^{q_2(\cdot)}}\right)^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&\quad+\sup_{\theta>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty u(1+\varepsilon)}\left(\sum_{j=0}^{k}\|\chi_k(1+|z|)^{-\lambda(z)}\mathscr{H}_b^m(f_j)\|_{L^{q_2(\cdot)}}\right)^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&:= A_1 + A_2.
\end{aligned}
$$

The estimate for A_2 follows in a similar manner to E_1 with $q_1'(0)$ replaced by $q_{1\infty}'$, and by using the fact that $w_1 := \frac{n}{q_{1\infty}'} - a_\infty - v - \frac{n}{s} > 0$. For A_1, using Lemma 2.4 we get

$$
\begin{aligned}
A_1 &\leqslant C\sup_{\theta>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{ka_\infty u(1+\varepsilon)}\left(\sum_{j=-\infty}^{-1}\|\chi_k(1+|z|)^{-\lambda(z)}\mathscr{H}_b^m(f_j)\|_{L^{q_2(\cdot)}}\right)^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&\leqslant C\|b\|_{BMO(\mathbb{R}^n)}^m \sup_{\theta>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{ka_\infty u(1+\varepsilon)}\right.\\
&\quad\times\left.\left(\sum_{j=-\infty}^{-1}(k-j)^m 2^{-kn}2^{k(v+\frac{n}{s}+\frac{n}{q_{1\infty}})}2^{-jv+\frac{jn}{q_1(0)}}\|f_j\|_{L^{q_1(\cdot)}}\right)^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&\leqslant C\|b\|_{BMO(\mathbb{R}^n)}^m \sup_{\theta>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty} 2^{k(a_\infty+v+\frac{n}{s}-\frac{n}{q_{1\infty}'})u(1+\varepsilon)}\right.\\
&\quad\times\left.\left(\sum_{j=-\infty}^{-1}(k-j)^m 2^{-jv+\frac{jn}{q_1(0)}}\|f_j\|_{L^{q_1(\cdot)}}\right)^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&\leqslant C\|b\|_{BMO(\mathbb{R}^n)}^m \sup_{\theta>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty}\left(\sum_{j=-\infty}^{-1}(k-j)^m 2^{-jv+\frac{jn}{q_1(0)}}\|f_j\|_{L^{q_1(\cdot)}}\right)^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&\leqslant C\|b\|_{BMO(\mathbb{R}^n)}^m\\
&\quad\times\sup_{\theta>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty}\left(\sum_{j=-\infty}^{-1}2^{j\alpha(0)}(k-j)^m 2^{j\left(\frac{n}{q_1(0)}-\frac{n}{s}-v-\alpha(0)\right)}\|f_j\|_{L^{q_1(\cdot)}}\right)^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}.
\end{aligned}
$$

By applyig Hölder's inequality we have

$$
\begin{aligned}
A_1 \leqslant & C\|b\|_{BMO(\mathbb{R}^n)}^m \\
& \times \sup_{\theta>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty}\left(\sum_{j=-\infty}^{-1} 2^{j\alpha(0)}(k-j)^m 2^{j\left(\frac{n}{q_1(0)}-\frac{n}{s}-v-\alpha(0)\right)}\|f_j\|_{L^{q_1(\cdot)}}\right)^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}} \\
\leqslant & C\|b\|_{BMO(\mathbb{R}^n)}^m \sup_{\theta>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty}\left(\sum_{j=-\infty}^{-1} 2^{j\alpha(0)u(1+\varepsilon)}\|f_j\|_{L^{q_1(\cdot)}}^{u(1+\varepsilon)}\right)\right. \\
& \times\left.\left(\sum_{j=-\infty}^{-1} 2^{j\left(\frac{n}{q_1(0)}-\frac{n}{s}-v-\alpha(0)\right)(u(1+\varepsilon))'}\right)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\right)^{\frac{1}{u(1+\varepsilon)}} \\
\leqslant & C\|b\|_{BMO(\mathbb{R}^n)}^m \sup_{\theta>0}\left(\varepsilon^\theta \sum_{k=0}^{\infty}\left(\sum_{j=-\infty}^{-1} 2^{j\alpha(0)u(1+\varepsilon)}\|f_j\|_{L^{q_1(\cdot)}}^{u(1+\varepsilon)}\right)\right)^{\frac{1}{u(1+\varepsilon)}} \\
\leqslant & C\|b\|_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}_{q_1(\cdot)}^{\alpha(\cdot),u),\theta}(\mathbb{R}^n)}.
\end{aligned}
$$

Combining these estimates we conclude that

$$
\|(1+|z|)^{-\lambda(z)}\mathscr{H}_b^m(f)\|_{\dot{K}_{q_2(\cdot)}^{\alpha(\cdot),u),\theta}(\mathbb{R}^n)} \leqslant C\|b\|_{BMO(\mathbb{R}^n)}^m\|f\|_{\dot{K}_{q_1(\cdot)}^{\alpha(\cdot),u),\theta}(\mathbb{R}^n)}. \quad \square
$$

THEOREM 3.2. *Let* $0<v\leqslant 1$*,* $\alpha(\cdot), q(\cdot)\in\mathfrak{B}_{0,\infty}(\mathbb{R}^n)$ *with* $1<q^-\leqslant q^+<\infty$*,* $1\leqslant u<\infty$ *and* $b\in BMO(\mathbb{R}^n)$*. Let* Φ *be homogeneous of degree zero and* $\Phi\in L^s(\mathbb{S}^{n-1})$*,* $s>q'^-$*. Let* α *be such that :*

(i) $-\frac{n}{q_2(0)}-v-\frac{n}{s}<\alpha(0)<\frac{n}{q_2'(0)}-v-\frac{n}{s}$

(ii) $-\frac{n}{q_{2\infty}}-v-\frac{n}{s}<\alpha_\infty<\frac{n}{q_{2\infty}'}-v-\frac{n}{s}$.

Then

$$
\|(1+|x|)^{-\lambda(x)}\mathscr{H}_b^{m*}(g)\|_{\dot{K}_{q_2(\cdot)}^{\alpha(\cdot),u),\theta}(\mathbb{R}^n)} \leqslant C\|b\|_{BMO(\mathbb{R}^n)}^m\|g\|_{\dot{K}_{q_1(\cdot)}^{\alpha(\cdot),u),\theta}(\mathbb{R}^n)}.
$$

Proof. This proof is similar to that of the previous result; therefore we only give omit some details.

Let $f\in\dot{K}_{q_2(\cdot)}^{\alpha(\cdot),u),\theta}(\mathbb{R}^n)$, and $f(z)=\sum_{j=-\infty}^{\infty}f(z)\chi_j(z)=\sum_{j=-\infty}^{\infty}f_j(z)$, we have

$$
\begin{aligned}
& |(1+|z|)^{-\lambda(z)}\mathscr{H}_b^{m*}(f)(z).\chi_k(z)| \\
& \leqslant \sum_{j=k+1}^{\infty}\int_{F_j}\frac{1}{|z|^{n-\Upsilon(\cdot)}}|\Phi(z-x)||f(x)|[b(z)-b(x)]^m dx.(1+|z|)^{-\lambda(z)}\chi_k(z) \\
& \leqslant \left(|b(z)-b_{B_j}|^m\sum_{j=k+1}^{\infty}\int_{F_j}|z|^{\Upsilon(x)-n}(1+|z|)^{-\lambda(z)}|\Phi(z-x)||f(x)|dx\right. \\
& \quad\left.+\sum_{j=k+1}^{\infty}\int_{F_j}|z|^{\Upsilon(x)-n}(1+|z|)^{-\lambda(z)}|b(x)-b_{B_j}|^m|\Phi(z-x)||f(x)|dx\right).\chi_k(z)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \Bigg(|b(z)-b_{B_j}|^m \sum_{j=k+1}^{\infty} \|f_j\|_{L^{q_1(\cdot)}} \||z|^{\Upsilon(\cdot)-n}(1+|z|)^{-\lambda(z)}\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_1'(\cdot)}} \\
&\quad + \sum_{j=k+1}^{\infty} \||z|^{\Upsilon(\cdot)-n}(1+|z|)^{-\lambda(z)}\Phi(z-\cdot)(b-b_{B_j})^m\chi_j(\cdot)\|_{L^{q_1'(\cdot)}} \|f_j\|_{L^{q_1(\cdot)}} \Bigg) \cdot \chi_k(z) \\
&\leqslant \sum_{j=k+1}^{\infty} \|f_j\|_{L^{q_1(\cdot)}} \Big(|b(z)-b_{B_j}|^m \||z|^{\Upsilon(\cdot)-n}(1+|z|)^{-\lambda(z)}\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_1'(\cdot)}} \\
&\quad + \||z|^{\Upsilon(\cdot)-n}(1+|z|)^{-\lambda(z)}(b-b_{B_j})^m\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_1'(\cdot)}} \Big) \cdot \chi_k(z).
\end{aligned}
$$

We note that

$$
\begin{aligned}
I^{\Upsilon(\cdot)}((b(z)-b_{B_j})^m\chi_k)(z) &\geqslant I^{\Upsilon(\cdot)}(b(z)-b_{B_j})^m(\chi_k)(z).(\chi_{B_k})(z) \\
&= \int_{B_k} \frac{|b(z)-b_{B_j}|^m}{|z-z_2|^{n-\Upsilon(z)}} dz_2 . \chi_{B_k}(z) \\
&\geqslant C|b(z)-b_{B_j}|^m |z|^{\Upsilon(z)} . \chi_{B_k}(z) \\
&\geqslant C|b(z)-b_{B_j}|^m |z|^{\Upsilon(z)} . \chi_k(z).
\end{aligned}
$$

Consequently, we find that

$$
\begin{aligned}
&\|\chi_k(1+|z|)^{-\lambda(z)}\mathscr{H}_b^{m^*}(f\chi_j)\|_{L^{q_2(\cdot)}} \\
&\leqslant C \sum_{j=k+1}^{\infty} \|f_j\|_{L^{q_1(\cdot)}} \Big(\|(b-b_{B_j})^m\chi_k\|_{L^{q_2(\cdot)}} \||z|^{\Upsilon(\cdot)-n}(1+|z|)^{-\lambda(z)}\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_1'(\cdot)}} \\
&\quad + \||z|^{\Upsilon(\cdot)-n}(1+|z|)^{-\lambda(z)}(b-b_{B_j})^m\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_1'(\cdot)}} \|\chi_k\|_{L^{q_2(\cdot)}} \Big) \\
&\leqslant C \sum_{j=k+1}^{\infty} |z|^{-n} \|f_j\|_{L^{q_1(\cdot)}} \big((k-j)^m \|b\|^m_{BMO(\mathbb{R}^n)} \|\chi_k\|_{L^{q_2(\cdot)}} \|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_2'(\cdot)}} \\
&\quad + \|b\|^m_{BMO(\mathbb{R}^n)} \|\Phi(z-\cdot)\chi_j(\cdot)\|_{q_2'(\cdot)} \|\chi_k\|_{L^{q_2(\cdot)}} \big) \\
&\leqslant C \sum_{j=k+1}^{\infty} |z|^{-n}(k-j)^m \|b\|^m_{BMO(\mathbb{R}^n)} \|f_j\|_{L^{q_1(\cdot)}} \|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_2'(\cdot)}} \|\chi_k\|_{L^{q_2(\cdot)}}.
\end{aligned}
$$

By Lemma 2.7 we define $q_2(\cdot)$ by the relation $\frac{1}{q_2'(x)} = \frac{1}{q_2(x)} + \frac{1}{s}$ then by using generalized Hölder's inequality we have

$$
\begin{aligned}
\|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_2'(\cdot)}} &\leqslant \|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)} \|\chi_j(\cdot)\|_{L^{q_2(\cdot)}} \\
&\leqslant 2^{-jv} \left(\int_{2^{j-1}<|x|<2^j} |\Phi(z-x)|^s |x|^{sv} dx \right)^{1/s} \|\chi_j\|_{L^{q_2(\cdot)}} \\
&\leqslant 2^{-jv} 2^{k(v+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \| \|\chi_j\|_{L^{q_2(\cdot)}}.
\end{aligned}
$$

Similarly, by Lemma 2.6 we have

$$\begin{aligned}
&\|(b-b_{B_j})^m(\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_2'(\cdot)}}\\
&\quad\leqslant \|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)}\|(b-b_{B_j})^m\chi_j(\cdot)\|_{L^{q_2(\cdot)}}\\
&\quad\leqslant C\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_j\|_{L^{q_2(\cdot)}}\|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^s(\mathbb{R}^n)}\\
&\quad\leqslant C\|b\|^m_{BMO(\mathbb{R}^n)}2^{-jv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_j\|_{L^{q_2(\cdot)}}.
\end{aligned}$$

$$\begin{aligned}
&\|(1+|z|)^{-\lambda(z)}\mathscr{H}_b^{m*}(f)\|_{\dot K^{\alpha(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)}\\
&=\sup_{\theta>0}\left(\varepsilon^\theta\sum_{k\in\mathbb{Z}}\|2^{k\alpha(\cdot)}\chi_k(1+|z|)^{-\lambda(z)}\mathscr{H}_b^{m*}(f)\|^{u(1+\varepsilon)}_{L^{q_2(\cdot)}}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&\leqslant\sup_{\theta>0}\left[\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)u(1+\varepsilon)}\left(\sum_{j=k+1}^{\infty}|z|^{-n}(k-j)^m\right.\right.\\
&\quad\left.\left.\times\|b\|^m_{BMO(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}}\|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_2'(\cdot)}}\|\chi_k\|_{L^{q_2(\cdot)}}\right)^{u(1+\varepsilon)}\right]^{\frac{1}{u(1+\varepsilon)}}\\
&\quad+\sup_{\theta>0}\left[\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\alpha_\infty u(1+\varepsilon)}\left(\sum_{j=k+1}^{\infty}|z|^{-n}(k-j)^m\right.\right.\\
&\quad\left.\left.\times\|b\|^m_{BMO(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}}\|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_2'(\cdot)}}\|\chi_k\|_{L^{q_2(\cdot)}}\right)^{u(1+\varepsilon)}\right]^{\frac{1}{u(1+\varepsilon)}}\\
&=:E_1+E_2.
\end{aligned}$$

Now we estimate of E_2. Since $k\in\mathbb{Z}$ and $j\geqslant k+1$ with $z\in F_k$, we get

$$\begin{aligned}
E_2&\leqslant\sup_{\theta>0}\left[\varepsilon^\theta\sum_{k=0}^{\infty}2^{k\alpha(\cdot)u(1+\varepsilon)}\left(\sum_{j=k+1}^{\infty}|z|^{-n}(k-j)^m\right.\right.\\
&\quad\left.\left.\times\|b\|^m_{BMO(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}}\|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_2'(\cdot)}}\|\chi_k\|_{L^{q_2(\cdot)}}\right)^{u(1+\varepsilon)}\right]^{\frac{1}{u(1+\varepsilon)}}\\
&\leqslant C\|b\|_{BMO)(\mathbb{R}^n)}\sup_{\theta>0}\left(\varepsilon^\theta\sum_{k=0}^{\infty}\left(\sum_{j=k+1}^{\infty}2^{a_\infty j}\|f_j\|_{L^{q_1(\cdot)}}(k-j)^m2^{d(k-j)}\right)^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}},
\end{aligned}$$

where $d := \frac{n}{q_{2\infty}} + a_\infty + v + \frac{n}{s} > 0$. Applying Hölder's theorem for series and the obvious estimate $2^{-u(1+\varepsilon)} < 2^{-u}$ we find that

$$\begin{aligned}
E_2 \leqslant & C\|b\|_{BMO)(\mathbb{R}^n)} \sup_{\theta>0}\Bigg[\varepsilon^\theta \sum_{k=0}^{\infty}\Bigg(\sum_{j=k+1}^{\infty} 2^{a_\infty u(1+\varepsilon)j}\|f_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}} 2^{du(1+\varepsilon)(k-j)/2}\Bigg) \\
& \times \Bigg(\sum_{j=k+1}^{\infty}(k-j)^{m(u(1+\varepsilon))'/2} 2^{d(u(1+\varepsilon))'(k-j)/2}\Bigg)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Bigg]^{\frac{1}{u(1+\varepsilon)}} \\
\leqslant & C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\theta>0}\Bigg[\varepsilon^\theta \sum_{k=0}^{\infty}\sum_{j=k+1}^{\infty} 2^{a_\infty u(1+\varepsilon)j}\|f_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}} 2^{du(1+\varepsilon)(k-j)/2}\Bigg]^{\frac{1}{u(1+\varepsilon)}} \\
\leqslant & C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\theta>0}\Bigg(\varepsilon^\theta \sum_{j=0}^{\infty} 2^{a_\infty u(1+\varepsilon)j}\|f_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}} \sum_{k=0}^{j-1} 2^{du(1+\varepsilon)(k-j)/2}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
< & C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\theta>0}\Bigg(\varepsilon^\theta \sum_{j\in\mathbb{Z}} 2^{a_\infty u(1+\varepsilon)j}\|f_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}} \sum_{k=-\infty}^{j-1} 2^{du(1+\varepsilon)(k-j)/2}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
= & C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\theta>0}\Bigg(\varepsilon^\theta \sum_{j\in\mathbb{Z}} 2^{\alpha(\cdot)u(1+\varepsilon)j}\|f_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
\leqslant & C\|b\|^m_{BMO(\mathbb{R}^n)}\|f\|_{\dot{K}^{\alpha(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For E_1, by using Minkowski's inequality we get

$$\begin{aligned}
E_1 \leqslant & \sup_{\theta>0}\Bigg[\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)u(1+\varepsilon)}\Bigg(\sum_{j=k+1}^{-1}|z|^{-n}(k-j)^m \\
& \times \|b\|^m_{BMO(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}}\|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_2'(\cdot)}}\|\chi_k\|_{L^{q_2(\cdot)}}\Bigg)^{u(1+\varepsilon)}\Bigg]^{\frac{1}{u(1+\varepsilon)}} \\
& + \sup_{\theta>0}\Bigg[\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)u(1+\varepsilon)}\Bigg(\sum_{j=0}^{\infty}|z|^{-n}(k-j)^m \\
& \times \|b\|^m_{BMO(\mathbb{R}^n)}\|f_j\|_{L^{q_1(\cdot)}}\|\Phi(z-\cdot)\chi_j(\cdot)\|_{L^{q_2'(\cdot)}}\|\chi_k\|_{L^{q_2(\cdot)}}\Bigg)^{u(1+\varepsilon)}\Bigg]^{\frac{1}{u(1+\varepsilon)}} \\
:= & E_{11} + E_{12}.
\end{aligned}$$

We can easily find the estimate for E_{11} by similar way to E_2. We simply replace $q_{2\infty}$ by $q_2(0)$ and use the facet that $\frac{n}{q_2(0)} + a(0) + v + \frac{n}{s} > 0$. For E_{12}, we obtain

$$2^{-jn}2^{-jv}2^{k(v+\frac{n}{s})}\|\chi_k\|_{L^{q_2(\cdot)}}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leqslant C2^{-j(\frac{n}{q_{2\infty}}+v+\frac{n}{s})}2^{k(\frac{n}{q_2(0)}+v+\frac{n}{s})}. \tag{15}$$

$$E_{12} \leqslant C\|b\|^m_{BMO)(\mathbb{R}^n)} \sup_{\theta>0} \Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)u(1+\varepsilon)} \times \Bigg(\sum_{j=0}^{\infty} (k-j)^m \|f_j\|_{L^{q_1(\cdot)}} 2^{-j(\frac{n}{q_{2\infty}}+v+\frac{n}{s})} 2^{k(\frac{n}{q_2(0)}+v+\frac{n}{s})} \Bigg)^{u(1+\varepsilon)} \Bigg)^{\frac{1}{u(1+\varepsilon)}}$$

$$\leqslant C\|b\|^m_{BMO)(\mathbb{R}^n)} \sup_{\theta>0} \Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k(\alpha(0)+\frac{n}{q_2(0)}+v+\frac{n}{s})u(1+\varepsilon)} \times \Bigg(\sum_{j=0}^{\infty} (k-j)^m \|f_j\|_{L^{q_1(\cdot)}} 2^{-j(\frac{n}{q_{2\infty}}+v+\frac{n}{s})} \Bigg)^{u(1+\varepsilon)} \Bigg)^{\frac{1}{u(1+\varepsilon)}}$$

$$\leqslant C\|b\|^m_{BMO)(\mathbb{R}^n)} \sup_{\theta>0} \Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} \Bigg(\sum_{j=0}^{\infty} (k-j)^m \|f_j\|_{L^{q_1(\cdot)}} 2^{-j(\frac{n}{q_{2\infty}}+v+\frac{n}{s})} \Bigg)^{u(1+\varepsilon)} \Bigg)^{\frac{1}{u(1+\varepsilon)}}.$$

Let us denote $\theta_1 := \frac{n}{q_{2\infty}} + v + \frac{n}{s} - \alpha_\infty$, then

$$E_{12} \leqslant C\|b\|^m_{BMO)(\mathbb{R}^n)} \sup_{\theta>0} \Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} \Bigg(\sum_{j=0}^{\infty} 2^{j\alpha_\infty} (k-j)^m \|f_j\|_{L^{q_1(\cdot)}} 2^{-j\theta_1} \Bigg)^{u(1+\varepsilon)} \Bigg)^{\frac{1}{u(1+\varepsilon)}}$$

$$\leqslant C\|b\|^m_{BMO)(\mathbb{R}^n)} \sup_{\theta>0} \Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} \sum_{j=0}^{\infty} 2^{j\alpha_\infty u(1+\varepsilon)} \|f_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}} \left(2^{-j\theta_1}\right)^{u(1+\varepsilon)/2} \times \Bigg(\sum_{j=0}^{\infty} \left((k-j)^m 2^{-j\theta_1}\right)^{(u(1+\varepsilon))'/2} \Bigg)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}} \Bigg)^{\frac{1}{u(1+\varepsilon)}}$$

$$\leqslant C\|b\|^m_{BMO)(\mathbb{R}^n)} \sup_{\theta>0} \Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} \sum_{j=0}^{\infty} 2^{j\alpha(0)u(1+\varepsilon)} \|f_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}} \left(2^{-j\theta_1}\right)^{u(1+\varepsilon)/2} \Bigg)^{\frac{1}{u(1+\varepsilon)}}$$

$$\leqslant C\|b\|^m_{BMO)(\mathbb{R}^n)} \|f\|_{\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)}(\mathbb{R}^n)}.$$

Combining the estimates for E_1 and E_2 we conclude that

$$\|(1+|z|)^{-\lambda(z)} \mathscr{H}^{m*}_b(f)\|_{\dot{K}^{\alpha(\cdot),u),\theta}_{q_2(\cdot)}(\mathbb{R}^n)} \leqslant C\|b\|^m_{BMO(\mathbb{R}^n)} \|f\|_{\dot{K}^{\alpha(\cdot),u),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.$$

which ends the proof. $\square$

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