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AN INTERPOLATION PROCESS ON THE ROOTS OF LAGUERRE POLYNOMIALS WITH AN ADDITIONAL CONDITION

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Abstract. This paper is devoted to studying a Pál-type interpolation problem on the zeros of Laguerre polynomials of degree n and its derivative of degree $n-1$. Here, we work on an interpolation on the polynomials with an additional condition on the zeros of Laguerre polynomials. The mixed type interpolation problem is studied in a unified way. The aim of this paper is to consider a special problem of mixed type $(0;0,1)$-interpolation on the zeros of Laguerre polynomials. In this paper we prove the regularity of the problem and determine explicit formulae of the interpolation. Under certain conditions, we obtain an estimate over the nonnegative real number line.

1. Introduction

In 2004, Lénárd [4] investigated the Pál type interpolation problem on the nodes of Laguerre abscissas. In 1975, Pál [7] demonstrated that there is no distinct polynomial of degree $\leqslant 2n-2$ when function values are dictated on one set of n points and derivatives values on another set of $n-1$ points, but note that there is a unique polynomial having the degree $\leqslant 2n-1$ when function value is defined at one more point that does not belong to the previous collection of n points.

Many authors [5, 6, 9–11, 16], and [8] have studied about interpolation problems when the function values and its sequential derivatives are specified at the provided set of points. When using lacunary interpolation, non-consecutive derivatives are used in the interpolation procedure to retrieve the data. Pál has also introduced a modification of the Hermite-Fejér interpolation.

In the modification, the function values and the first derivatives are prescribed on two inter scaled systems of nodal points $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$, that is $-\infty < x_1 < x_1^* < x_2 < \cdots < x_{n-1} < x_{n-1}^* < x_n < +\infty$, where

$$w_n(x) = (x-x_1)(x-x_2)\dots(x-x_n) \tag{1}$$

and $w_n'(x) = n(x-x_1^*)\dots(x-x_{n-1}^*)$. He proved that for any given system of real numbers $\{u_k\}_{k=1}^{n}$ and $\{u_k'\}_{k=1}^{n-1}$, there exists a polynomial $Q_{2n-1}(x)$ of minimal degree $(2n-1)$ satisfying the following interpolational properties:

$$Q_{2n-1}(x_k) = u_k \qquad (k = 1, 2, \dots, n),$$
$$Q_{2n-1}'(x_k^*) = u_k' \qquad (k = 1, 2, \dots, n-1).$$

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This determined interpolational polynomial is not unique; hence, for the uniqueness of the polynomial an additional condition is required. Introducing the additional condition $Q_{2n-1}(x_0) = 0$ at an additional knot $x_0 \neq x_k$ $(k = 1, 2, \ldots, n)$ Pál proved the uniqueness of the polynomial and gave an explicit formula for it.

In 1992, Xie [15] presented a new explicit formula of Pál-type interpolation on the interval $[-1, 1]$ with the additional knot x_n^*, where x_n^* is equal to one of the nodal points x_k $(k = 1, \ldots, n)$. Earlier, in 1985, Eneduanya [1] investigated the special case for the Legendre polynomial. For uniqueness, Eneduanya also used the additional nodal point $x_n^* = -1$. Szili [13] investigated the Pál-type interpolation on the Hermite-polynomials with the additional point $x_0 = 0$. Joó and Szabó [3] gave a common generalization of the classical Fejér interpolation [2] and Pál interpolation. Szili [14] investigated the integrated Legendre polynomial's roots in the inverse Pál interpolational problem.

The Pál type interpolation problem, on the nodes of Laguerre abscissas, was also studied by Lénárd [4]. The function values in Pál type interpolation are specified at the zeros of $w_n(x)$, whereas the derivative values are specified at the zeros of $w_n'(x)$.

Laguerre polynomial $L_n^{(k)}(x)$ $(k > -1)$ has n unique real zeros in $(0, \infty)$ and the zeros of $L_n^{(k)}(x)$ and $L_n^{(k)'}(x)$ can be found from the inter scaled system of nodal points $0 \leqslant \xi_0 < \xi_1^* < \xi_1 < \ldots < \xi_{n-1} < \xi_n^* < \xi_n < \infty$.

Let $\{\xi_i\}_{i=0}^n$ and $\{\xi_i^*\}_{i=1}^n$ be the arbitrary two sets of nodal points in an infinite interval $[0, \infty)$ inter scaled such that

$$0 \leqslant \xi_0 < \xi_1^* < \xi_1 < \ldots < \xi_{n-1} < \xi_n^* < \xi_n < +\infty. \tag{2}$$

Then there exists a unique polynomial $R_n(x)$ having a degree less than or equal to $3n + k - 1$ that meets the interpolation requirements listed below:

$$\begin{cases} R_n(\xi_i) = \alpha_i^*, & i = 1, 2, \ldots, n \\ R_n(\xi_i^*) = \beta_i^*, & i = 1, 2, \ldots, n-1 \\ R_n'(\xi_i^*) = \gamma_i^*, & i = 1, 2, \ldots, n-1 \\ R_n^{(j)}(\xi_0) = \varphi_0^{*(j)}, & j = 0, 1, \ldots, k \end{cases} \tag{3}$$

$$R_n(0) = 0, \tag{4}$$

where $\{\alpha_i^*\}_{i=1}^n$, $\{\beta_i^*\}_{i=1}^{n-1}$, $\{\gamma_i^*\}_{i=1}^{n-1}$, and $\{\varphi_0^{*(j)}\}_{j=0}^k$, are arbitrary real numbers. The object is to consider the problem of explicit representation and estimation of the sequence $\{R_n(x)\}$ of polynomials of degree $\leqslant 3n + k - 1$.

2. Preliminaries

We have used some well known results of the Laguerre polynomial $L_n^{(k)}(x)$ which are as follows:

The differential equation of the Laguerre polynomial is given by

$$xD^2L_n^{(k)}(x) + (1 + k - x)DL_n^{(k)}(x) + nL_n^{(k)}(x) = 0, \tag{5}$$

where n is a positive integer and $k > -1$.

For the zeros of $L_n^{(k)}(x)$, we have the results from Srivastava [10]:

$$2\sqrt{x_j} = \frac{1}{\sqrt{n}}[j\pi + \mathrm{O}(1)] \tag{6}$$

$$|L_n^{(k)'}(x_j)| \sim j^{-k-\frac{3}{2}} n^{k+1}, \quad (0 < x_j \leqslant \Omega,\ n = 1,2,3,\ldots) \tag{7}$$

$$|L_n^{(k)}(x)| = \begin{cases} x^{-\frac{k}{2}-\frac{1}{4}} \mathrm{O}(n^{\frac{k}{2}-\frac{1}{4}}), & cn^{-1} \leqslant x \leqslant \Omega \\ \mathrm{O}(n^k), & 0 \leqslant x \leqslant cn^{-1} \end{cases} \tag{8}$$

$$\mathrm{O}(l_j(x)) = \mathrm{O}(l_j^*(x)) = 1, \tag{9}$$

$$|x - x_k| \sim \frac{k^2}{n}. \tag{10}$$

Now we also have some properties of fundamental polynomials of the Lagrange interpolation which are given in Szegő [12] as:

$$l_j(x) = \frac{L_n^{(k)}(x)}{L_n^{(k)\prime}(x_j)\,(x - x_j)}, \tag{11}$$

$$l_j^*(x) = \frac{L_n^{(k)'}(x)}{L_n^{(k)''}(y_j)(x - y_j)}. \tag{12}$$

Here the degree of the polynomial $l_j(x)$ is $n-1$ and the degree of the polynomial $l_j^*(x)$ is $n-2$.

The tool which is use to measure the rate of convergence is the modulus of continuity of the function f. For a function $f : \mathbb{R} \to \mathbb{R}$, $\omega(f,\delta)$ denotes the special form of modulus of continuity given by:

$$\omega(f,\delta) = \sup_{0 \leqslant t \leqslant \delta} \| W(x+t) f(x+t) - W(x) f(x) \| + \| \tau(\delta x) W(x) f(x) \|,$$

where

$$\tau(x) = \begin{cases} |x|, & \text{if } |x| \leqslant 1 \\ 1, & \text{if } |x| > 1 \end{cases}$$

and $\| \, . \, \|$ denotes the sup-norm in $C(R)$. If $f \in C(R)$ and

$$\lim_{|x| \longrightarrow \infty} W(x) f(x) = 0,$$

then

$$\lim_{\delta \longrightarrow 0} \omega(f,\delta) = 0.$$

3. Explicit representation of interpolatory polynomial

Let $2n+1$ points in $(0, \infty)$ be given by (2). Then, for the prescribed numbers $\{\alpha_i^*\}_{i=0}^{n}$, $\{\beta_i^*\}_{i=1}^{n-1}$, $\{\gamma_i^*\}_{i=1}^{n-1}$, there exists a unique polynomial $\{R_n(x)\}$ of degree $\leqslant 3n+k-1$ satisfying the conditions (3) and (4).

The polynomial $R_n(x)$ is explicitly given by:

$$R_n(x) = \sum_{j=0}^{n} \alpha_j^* U_j(x) + \sum_{j=1}^{n-1} \beta_j^* V_j(x) + \sum_{j=1}^{n-1} \gamma_j^* W_j(x) + \sum_{j=0}^{k} \varphi_0^{*(j)} Z_j(x) \tag{13}$$

where $\{U_j(x)\}_{j=0}^{n}$, $\{V_j(x)\}_{j=1}^{n-1}$, $\{W_j(x)\}_{j=1}^{n-1}$, and $\{Z_j(x)\}_{j=0}^{k}$ are the polynomials having the degree $\leqslant 3n+k-1$. These polynomials are unique and satisfy the following conditions: for $j = 0,1,2,\ldots,n$

$$\begin{cases} U_j(x_i) = \delta_{ij}, & (i = 1,2,\ldots,n) \\ U_j(y_i) = 0, & (i = 1,2,\ldots,n-1) \\ U_j'(y_i) = 0, & (i = 1,2,\ldots,n-1) \\ U_j^{(l)}(0) = 0, & (l = 0,1,\ldots,k) \end{cases} \tag{14}$$

for $j = 1,2,\ldots,n-1$

$$\begin{cases} V_j(x_i) = 0, & (i = 1,2,\ldots,n) \\ V_j(y_i) = \delta_{ij}, & (i = 1,2,\ldots,n-1) \\ V_j'(y_i) = 0, & (i = 1,2,\ldots,n-1) \\ V_j^{(l)}(0) = 0, & (l = 0,1,\ldots,k) \end{cases} \tag{15}$$

for $j = 1,2,\ldots,n-1$

$$\begin{cases} W_j(x_i) = 0, & (i = 1,2,\ldots,n) \\ W_j(y_i) = 0, & (i = 1,2,\ldots,n-1) \\ W_j'(y_i) = \delta_{ij}, & (i = 1,2,\ldots,n-1) \\ W_j^{(l)}(0) = 0, & (l = 0,1,\ldots,k) \end{cases} \tag{16}$$

for $l = 0,1,\ldots,k$

$$\begin{cases} Z_k(x_i) = 0, & (i = 1,2,\ldots,n) \\ Z_k(y_i) = 0, & (i = 1,2,\ldots,n-1) \\ Z_k'(y_i) = 0, & (i = 1,2,\ldots,n-1) \\ Z_k^{(l)}(0) = \delta_{ij}, & (l = 0,1,\ldots,k). \end{cases} \tag{17}$$

Here δ_{ij} is a Kronecker delta,

$$\delta_{ij} = \begin{cases} 1, \; i = j \\ 0, \; i \neq j \end{cases}. \tag{18}$$

The explicit form of the $U_j(x), V_j(x), W_j(x)$ and $Z_k(x)$ are given in the following lemma.

LEMMA 1. *The fundamental polynomial* $\{U_j(x)\}_{j=0}^{n}$ *satisfying the interpolatory condition* (14) *is given by:*

$$U_0(x) = \frac{L_n^{(k)}(x)\left[L_n^{(k)'}(x)\right]^2}{L_n^{(k)}(0)\left[L_n^{(k)'}(0)\right]^2} \tag{19}$$

and for $j = 1, 2 \ldots, n$

$$U_j(x) = \frac{x^{k+2} l_j(x)\left[L_n^{(k)'}(x)\right]^2}{x_j^{k+2}\left[L_n^{(k)'}(x_j)\right]^2} \tag{20}$$

where $l_j(x)$ *is given by* (11).

Proof. For $j = 1, 2 \ldots, n$, let

$$U_j^*(x) = u_1 x^{k+2} l_j(x)[L_n^{(k)'}(x)]^2 \tag{21}$$

be a polynomial of degree $\leqslant 3n+k-1$. We can easily check that $U_j^*(x)$ satisfies the equation (14) provided

$$u_1 = \frac{1}{x_j^{k+2}\left[L_n^{(k)'}(x_j)\right]^2}. \tag{22}$$

Thus,

$$U_j^*(x) \equiv U_j(x), \tag{23}$$

which completes the proof of the lemma. □

LEMMA 2. *The fundamental polynomial* $\{V_j(x)\}_{j=1}^{n-1}$ *satisfying the interpolatory condition* (15) *is given by: for* $j = 1, 2 \ldots, n-1$

$$V_j(x) = \frac{x^{k+2}[l_j^*(x)]^2 L_n^{(k)}(x)}{y_j^{k+2} L_n^{(k)}(y_j)}\left[1 + \frac{(k+y_j)}{y_j}(x - y_j)\right] \tag{24}$$

where $l_j^*(x)$ *is given by* (12).

Proof. For $j = 1, 2 \ldots, n-1$, let

$$V_j^*(x) = V_1 x^{k+2}[l_j^*(x)]^2 L_n^{(k)}(x) + V_2 x^{k+2} l_j^*(x) L_n^{(k)}(x) L_n^{(k)'}(x) \tag{25}$$

be a polynomial of degree $\leqslant 3n+k-1$. We can easily check that $V_j^*(x)$ satisfies the equation (15) provided

$$V_1 = \frac{1}{y_j^{k+2} L_n^{(k)}(y_j)}. \tag{26}$$

Also for $j = 1,2,\ldots,n-1$, the condition $V_j^{*'}(y_i) = 0$ provides

$$V_2 = \frac{(k+y_j)}{y_j^{k+2} L_n^{(k)}(y_j) L_n^{(k)''}(y_j)}. \tag{27}$$

Thus,

$$V_j^*(x) \equiv V_j(x) \tag{28}$$

which completes the proof of the lemma. □

LEMMA 3. *The fundamental polynomial* $\{W_j(x)\}_{j=1}^{n-1}$ *satisfying the interpolatory condition* (16) *is given by: for* $j = 1,2,\ldots,n-1$

$$W_j(x) = \frac{x^{k+2} l_j^*(x) L_n^{(k)}(x) L_n^{(k)'}(x)}{y_j^{k+2} L_n^{(k)}(y_j) L_n^{(k)''}(y_j)} \tag{29}$$

where $l_j^*(x)$ *is given by* (12).

Proof. For $j = 1,2,\ldots,n-1$, let

$$W_j^*(x) = w_1 x^{k+2} l_j^*(x) L_n^{(k)}(x) L_n^{(k)'}(x) \tag{30}$$

be a polynomial of degree $\leqslant 3n+k-1$. We can easily check that $W_j^*(x)$ satisfies the equation (16) provided

$$w_1 = \frac{1}{y_j^{k+2} L_n^{(k)}(y_j) L_n^{(k)''}(y_j)}. \tag{31}$$

Thus,

$$W_j^*(x) \equiv W_j(x) \tag{32}$$

it completes the lemma's proof. □

LEMMA 4. *The interpolatory condition* (17) *is satisfied by the basic polynomial* $\{Z_j(x)\}_{j=0}^{k}$*, which is provided by: for* $j = 0,1,2,\ldots,k-1$

$$\begin{aligned} Z_j(x) = a_j(x) x^{j+1} [L_n^{(k)}(x)]^2 L_n^{(k)'}(x) + x^{k+1} L_n^{(k)}(x) L_n^{(k)'}(x) \\ \left[z_j^* - \frac{L_n^{(k)}(x) a_j(x) + b_j(x) L_n^{(k)'}(x)}{x^{k-j}} \right] \end{aligned} \tag{33}$$

and

$$Z_k(x) = \frac{1}{k! L_n^{(k)}(0) [L_n^{(k)'}(0)]^2} x^{k+1} L_n^{(k)}(x) [L_n^{(k)'}(x)]^2 \tag{34}$$

where the degree of the polynomial $a_j(x)$ *is at most* $(k-j-1)$ *and the degree of the polynomial* $b_j(x)$ *is at most* $(k-j)$.

Proof. Let us suppose that $Z_j(x)$, for some fixed $j \in \{0,1,\ldots,k-1\}$, be

$$Z_j^*(x) = a_j^*(x)x^{j+1}[L_n^{(k)}(x)]^2 L_n^{(k)'}(x) + x^{k+1}L_n^{(k)}(x)L_n^{(k)'}(x)b_n^*(x) \tag{35}$$

where the degree of the polynomial $a_j^*(x)$ is $(k-j-1)$ and the degree of the polynomial $q_n^*(x)$ is n. Also note that for $(l=0,1,\ldots,j-1)$, $Z_j^{(l)}(0)=0$. We know that $L_n^{(k)}(x_i)=0$ and $L_n^{(k)'}(y_i)=0$. Therefore, $Z_j(x_i)=0$ and $Z_j(y_i)=0$ for $i=1,2,\ldots,n$. The coefficient of the polynomial $a_j^*(x)$, for $(l=j,\ldots,k-1)$, is calculated by:

$$Z_j^{(l)}(0) = \frac{d^l}{dx^l}\left[a_j^*(x)x^j[L_n^{(k)}(x)]^2 L_n^{(k)'}(x)\right]_{x=0} = \delta_{ij}. \tag{36}$$

Now using the condition $Z_j'(y_i)=0$ of (17), we will have

$$q_n^*(y_i) = -(y_i)^{j-k}L_n^{(k)}(y_i)a_j^*(y_i) \tag{37}$$

this will imply the value of $g_n^*(x)$ as:

$$q_n^*(x) = -\frac{L_n^{(k)}(x)a_j^*(x) + b_j^*(x)L_n^{(k)'}(x)}{x^{k-j}}, \tag{38}$$

where the degree of the polynomial $a_j^*(x)$ is at most $(k-j-1)$ and the degree of the polynomial $b_j^*(x)$ is atmost $(k-j)$. By using the equations (35) and (38), we get the polynomial $Z_j(x)$ of degree $\leqslant 3n+k-1$ satisfying the equation (17). □

Now we state our main theorem.

THEOREM 1. *Assuming that the interpolatory function* $f:\mathbb{R}\to\mathbb{R}$ *is continuous as well as differentiable such that*

$$C(m) = \{f(x) : f(x) = \mathrm{O}(x^m) \text{ as } x \longrightarrow \infty\};$$

where m *is a non negative integer,* f *is continuous function in the interval* $[0,\infty)$, *then for each* $f \in C(m)$ *and a non negative* k,

$$R_n(x) = \sum_{j=0}^{n}\alpha_j^* U_j(x) + \sum_{j=1}^{n-1}\beta_j^* V_j(x) + \sum_{j=1}^{n-1}\gamma_j^* W_j(x) + \sum_{j=0}^{k}\varphi_0^{*(j)} Z_j(x) \tag{39}$$

satisfies the relation:

$$|R_n(x)-f(x)| = \mathrm{O}(1)\omega\left(f,\frac{\log n}{\sqrt{n}}\right), \qquad \text{for } 0 \leqslant x \leqslant cn^{-1} \tag{40}$$

$$|R_n(x)-f(x)| = \mathrm{O}(1)\omega\left(f,\frac{\log n}{\sqrt{n}}\right), \qquad \text{for } cn^{-1} \leqslant x \leqslant \Omega \tag{41}$$

here ω *represents the modulus of continuity.*

Prior to proving theorem 1, first estimate the values of the following fundamental polynomials, which are listed below:

4. Estimation of the fundamental polynomials

THEOREM 2. *Let us assume the fundamental polynomial* $U_j(x)$, *for* $j=0,1,2,\ldots,n$ *is presented by:*

$$U_j(x)=\frac{x^{k+2}l_j(x)\left[L_n^{(k)'}(x)\right]^2}{x_j^{k+2}\left[L_n^{(k)'}(x_j)\right]^2}, \tag{42}$$

then we have

$$\sum_{j=0}^{n}|U_j(x)|=\mathrm{O}(1), \qquad \textit{for } 0\leqslant x\leqslant\Omega. \tag{43}$$

Proof. From the polynomial $U_j(x)$ we have

$$\sum_{j=0}^{n}\left|U_j(x)\right|\leqslant\sum_{j=0}^{n}\frac{\left|x^{k+2}\right|\left|l_j(x)\right|\left[L_n^{(k)'}(x)\right]^2}{\left|x_j^{k+2}\right|\left[L_n^{(k)'}(x_j)\right]^2}, \tag{44}$$

As $U_j(x)$ is independent of $L_n^{(k)}(x)$, and by using the equations (6), (7), and (9), we get the desired result.

$$\sum_{j=0}^{n}|U_j(x)|=\mathrm{O}(1), \quad \text{for } 0\leqslant x\leqslant\Omega. \quad \square$$

THEOREM 3. *Let us assume the fundamental polynomial* $V_j(x)$, *for* $j=1,2,\ldots,$ $n-1$ *is presented by:*

$$V_j(x)=\frac{x^{k+2}[l_j^*(x)]^2L_n^{(k)}(x)}{y_j^{k+2}L_n^{(k)}(y_j)}\left[1+\frac{(k+y_j)}{y_j}(x-y_j)\right] \tag{45}$$

then we have

$$\sum_{j=1}^{n-1}|V_j(x)|=\mathrm{O}(n^{-1}), \qquad \textit{for } 0\leqslant x\leqslant\Omega. \tag{46}$$

Proof. Rewrite the polynomial as:

$$V_j(x)=\frac{x^{k+2}[l_j^*(x)]^2L_n^{(k)}(x)}{y_j^{k+2}L_n^{(k)}(y_j)}+\frac{(k+y_j)}{y_j}\frac{x^{k+2}[l_j^*(x)]^2L_n^{(k)}(x)}{y_j^{k+2}L_n^{(k)}(y_j)}(x-y_j) \tag{47}$$

then we have

$$\sum_{j=1}^{n-1}\left|V_j(x)\right| \leqslant \sum_{j=1}^{n-1}\frac{\left|x^{k+2}\right|\left[l_j^*(x)\right]^2\left|L_n^{(k)}(x)\right|}{\left|y_j^{k+2}\right|\left|L_n^{(k)}(y_j)\right|} + \sum_{j=1}^{n-1}\frac{\left|(k+y_j)\right|\left|x^{k+2}\right|\left[l_j^*(x)\right]^2\left|L_n^{(k)}(x)\right|\left|(x-y_j)\right|}{\left|y_j\right|\left|y_j^{k+2}\right|\left|L_n^{(k)}(y_j)\right|} \tag{48}$$

by using the equations (6), (8), (9), and (10) we get the desired result.

$$\sum_{j=1}^{n}|V_j(x)| = \mathrm{O}(n^{-1}), \quad \text{for } 0 \leqslant x \leqslant \Omega. \quad \square$$

THEOREM 4. *Let us suppose the fundamental polynomial* $W_j(x)$*, for* $j = 1,2,\ldots,$ $n-1$*, is presented by:*

$$W_j(x) = \frac{x^{k+2} l_j^*(x) L_n^{(k)}(x) L_n^{(k)'}(x)}{y_j^{k+2} L_n^{(k)}(y_j) L_n^{(k)''}(y_j)} \tag{49}$$

then we have

$$\sum_{j=1}^{n-1}|W_j(x)| = \mathrm{O}(n^{-1}), \qquad \textit{for } 0 \leqslant x \leqslant cn^{-1} \tag{50}$$

$$\sum_{j=1}^{n-1}|W_j(x)| = \mathrm{O}(1), \qquad \textit{for } cn^{-1} \leqslant x \leqslant \Omega. \tag{51}$$

Proof. From the polynomial $W_j(x)$ we have

$$\left|W_j(x)\right| \leqslant \frac{\left|x^{k+2}\right| |l_j^*(x)| \left|L_n^{(k)}(x)\right| \left|L_n^{(k)'}(x)\right|}{\left|y_j^{k+2}\right| \left|L_n^{(k)}(y_j)\right| \left|L_n^{(k)''}(y_j)\right|}, \tag{52}$$

$$\sum_{j=1}^{n-1}\left|W_j(x)\right| \leqslant \sum_{j=1}^{n-1}\frac{\left|x^{k+2}\right| |l_j^*(x)| \left|L_n^{(k)}(x)\right| \left|L_n^{(k)'}(x)\right|}{\left|y_j^{k+2}\right| \left|L_n^{(k)}(y_j)\right| \left|L_n^{(k)''}(y_j)\right|}, \tag{53}$$

by using the equations (6), (7), (8), and (9), we get the desired result. $\square$

REMARK. Let $C(m) = \{f(x) : f$ is continuous in $[0,\infty), f(x) = \mathrm{O}(x^m)$ as $x \longrightarrow \infty;\}$, where $m \geqslant 0$ is an integer. Then, by Szegő [12] Theorem 14.7,

$$\lim_{n\longrightarrow\infty} |f(x) - H_n^{(\alpha)}(f,x)|_I = 0, \tag{54}$$

where $I \subset (0,\infty)$ for $\alpha \geqslant 0$, or $I \subset (0,\infty)$ for $-1 < \alpha < 0$. Also note that there is a function in $C(m)$ such that $\{H_n^{(\alpha)}(f,x)\}$ diverges for $\alpha \geqslant 0$ at $x = 0$. And for the convergence rate, we have:

$$\left|f(x) - H_n^{(\alpha)}(f,x)\right|_I = \begin{cases} \mathrm{O}\left(\omega\left(f, n^{-1-\alpha}\right)\right); \ -1 < \alpha < 0 \\ \mathrm{O}\left(\omega\left(f, \frac{\log n}{\sqrt{n}}\right)\right); \ \ \alpha \geqslant -\frac{1}{2}. \end{cases} \tag{55}$$

5. Proof of the main theorem 1

Let us suppose that $S_n(x)$ be a polynomial of degree $\leqslant 3n+k-1$ and $R_n(x)$ be given by (13). Note that $R_n(x)$ is exact for every fundamental polynomial of degree $\leqslant 3n+k-1$; therefore,

$$S_n(x) = \sum_{j=0}^{n} S_n(x_j)U_j(x) + \sum_{j=1}^{n-1} S_n(y_j)V_j(x) + \sum_{j=1}^{n-1} S_n^{'}(y_j)W_j(x) + \sum_{j=0}^{k} S_n(x_0)Z_j(x) \tag{56}$$

from equations (13) and (56) we get

$$\begin{aligned} |f(x) - R_n(x)| &\leqslant |f(x) - S_n(x)| + |S_n(x) - R_n(x)| \qquad (57) \\ &\leqslant |f(x) - S_n(x)| + \sum_{j=0}^{n} \left|f(x_j) - S_n(x_j)\right| \left|U_j(x)\right| \\ &+ \sum_{j=1}^{n-1} \left|f(y_j) - S_n(y_j)\right| \left|V_j(x)\right| \\ &+ \sum_{j=1}^{n-1} \left|f^{'}(y_j) - S_n^{'}(y_j)\right| \left|W_j(x)\right| \\ &+ \sum_{j=0}^{k} \left|f^{l}(x_0) - S_n^{(l)}(x_0)\right| \left|Z_j(x)\right|. \qquad (58) \end{aligned}$$

Thus, equation (55) and the conclusions of theorem 2, 3, and 4 complete the proof of the theorem 1.

6. Conclusions

In this paper, we have proved the existence, uniqueness, explicit representation, and order of convergence of the given interpolatory problem when the zeros $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n-1}$ are prescribed on the Laguerre polynomials $L_n^{(k)}(x)$ and its derivative $L_n^{(k)'}(x)$ respectively, with an additional condition. If $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable interpolatory function, then there exists a polynomial $R_n(x)$ having the degree $\leqslant 3n+k-1$ holding the equation (3) as well as an additional condition (4). The function values of the polynomial converge uniformly to $f(x)$ on the nonnegative real number line for the large value of n.

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