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A MOLECULAR DECOMPOSITION FOR $H^p(\mathbb{Z}^n)$

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Abstract. In this work, for the range $\frac{n-1}{n} < p \leqslant 1$, we give a molecular reconstruction theorem for $H^p(\mathbb{Z}^n)$. As an application of this result and the atomic decomposition developed by S. Boza and M. Carro in [Proc. R. Soc. Edinb., 132 A (1) (2002), 25–43], we prove that the discrete Riesz potential I_α defined on $\mathbb{Z}^n$ is a bounded operator $H^p(\mathbb{Z}^n) \to H^q(\mathbb{Z}^n)$ for $\frac{n-1}{n} < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, where $0 < \alpha < n$.

1. Introduction

S. Boza and M. Carro in [4] (see also [2]) introduced the discrete Hardy spaces on $\mathbb{Z}^n$. Following the work [4] to define the discrete Hardy spaces $H^p(\mathbb{Z}^n)$, $0 < p \leqslant 1$, we consider the discrete Poisson kernel on $\mathbb{Z}^n$, which is defined by

$$P_t^d(j) = C_n \frac{t}{(t^2 + |j|^2)^{(n+1)/2}}, \ \ t > 0, \ \ j \in \mathbb{Z}^n \setminus \{\mathbf{0}\}, \ \ P_t^d(\mathbf{0}) = 0,$$

where C_n is a normalized constant depending on the dimension. For a sequence $b = \{b(j)\}_{j \in \mathbb{Z}^n}$, let

$$\|b\|_{\ell^p(\mathbb{Z})} = \begin{cases} \left(\sum\limits_{j \in \mathbb{Z}^n} |b(j)|^p \right)^{1/p}, & 0 < p < \infty \\ \sup\limits_{j \in \mathbb{Z}^n} |b(j)|, & p = \infty \end{cases}.$$

A sequence $b = \{b(j)\}_{j \in \mathbb{Z}^n}$ is said to belong to $\ell^p(\mathbb{Z}^n)$, $0 < p \leqslant \infty$, if $\|b\|_{\ell^p(\mathbb{Z}^n)} < \infty$.

Then, for $0 < p \leqslant 1$, we define

$$H^p(\mathbb{Z}^n) = \left\{ b \in \ell^p(\mathbb{Z}^n) : \sup_{t>0} |(P_t^d *_{\mathbb{Z}^n} b)| \in \ell^p(\mathbb{Z}^n) \right\}, \tag{1}$$

equipped with the quasi-norm given by

$$\|b\|_{H^p(\mathbb{Z}^n)} := \|b\|_{\ell^p(\mathbb{Z}^n)} + \| \sup_{t>0} |(P_t^d *_{\mathbb{Z}^n} b)| \|_{\ell^p(\mathbb{Z}^n)}. \tag{2}$$

They also gave a variety of distinct approaches, based on differing definitions, all leading to the same notion of Hardy spaces $H^p(\mathbb{Z}^n)$, we will recall these definitions on the full range $0 < p \leqslant \infty$ in Section 3.

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On $\mathbb{Z}$ (see [2]), the discrete Hardy spaces $H^p(\mathbb{Z})$, $0<p<\infty$, can be defined to be the space of all sequences $b=\{b(j)\}\in\ell^p(\mathbb{Z})$ such that $\|b\|_{H^p(\mathbb{Z})}:=\|b\|_{\ell^p(\mathbb{Z})}+\|Hb\|_{\ell^p(\mathbb{Z})}<\infty$, where H is the discrete Hilbert transform given by

$$(Hb)(j)=\sum_{i\neq j}\frac{b(i)}{j-i}.$$

For $0<\alpha<n$ and a sequence $b=\{b(i)\}_{i\in\mathbb{Z}^n}$, we consider the discrete Riesz potential defined by

$$(I_\alpha b)(j)=\sum_{i\in\mathbb{Z}^n\setminus\{j\}}\frac{b(i)}{|i-j|^{n-\alpha}},\quad j\in\mathbb{Z}^n. \tag{3}$$

In [6] (cf. also [7], pp. 288), G. H. Hardy, J. E. Littlewood and G. Pólya proved the following inequality

$$\left|\sum_{i\neq j}\sum|i-j|^{-\lambda}b(i)c(j)\right|\leqslant C\|b\|_{\ell^p(\mathbb{Z})}\|c\|_{\ell^q(\mathbb{Z})} \tag{4}$$

for

$$p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}>1,\ \lambda=2-\frac{1}{p}-\frac{1}{q}\ \text{(so that } 0<\lambda<1).$$

Taking $\lambda=1-\alpha$ in (4), it follows that the operator I_α is bounded from $\ell^p(\mathbb{Z})$ into $\ell^q(\mathbb{Z})$ for $1<p<\alpha^{-1}$ and $\frac{1}{q}=\frac{1}{p}-\alpha$.

Later, Y. Kanjin and M. Satake in [8], based on some results of [2], furnished a molecular decomposition for $H^p(\mathbb{Z})$ analogous to the ones given by M. Taibleson and G. Weiss in [14] for the Hardy spaces $H^p(\mathbb{R}^n)$. With this framework, they obtain the Marcinkiewicz multiplier theorem on $H^p(\mathbb{Z})$ and proved the $H^p(\mathbb{Z})\to H^q(\mathbb{Z})$ boundedness of discrete Riesz potential I_α, for $n=1$, $0<p<\alpha^{-1}$ and $\frac{1}{q}=\frac{1}{p}-\alpha$.

In [11], we studied the following fractional series operator on $\mathbb{Z}$,

$$(T_{\alpha,\beta}b)(j)=\sum_{i\neq\pm j}\frac{b(i)}{|i-j|^{\alpha}|i+j|^{\beta}},\quad (j\in\mathbb{Z}),$$

where $\alpha,\beta>0$, $\alpha+\beta=1-\gamma$ and $0\leqslant\gamma<1$. For $0\leqslant\gamma<1$, $0<p<\frac{1}{\gamma}$ and $\frac{1}{q}=\frac{1}{p}-\gamma$, we obtained, by using the atomic decomposition of $H^p(\mathbb{Z})$ developed in [2], the $H^p(\mathbb{Z})\to\ell^q(\mathbb{Z})$ boundedness of the operator $T_{\alpha,\beta}$. Moreover, for $\alpha=\beta=\frac{1-\gamma}{2}$ we show that there exists $\varepsilon\in\left(0,\frac{1}{3}\right)$ such that for every $0\leqslant\gamma<\varepsilon$ the operator $T_{\frac{1-\gamma}{2},\frac{1-\gamma}{2}}$ is not bounded from $H^p(\mathbb{Z})$ into $H^q(\mathbb{Z})$ for $0<p\leqslant\frac{1}{1+\gamma}$ and $\frac{1}{q}=\frac{1}{p}-\gamma$.

In [13], for $0<\lambda<1$, E. Stein and S. Wainger introduce the fractional operator I_λ, defined for sequences b on $\mathbb{Z}$ by

$$(I_\lambda b)(n)=\sum_{m=1}^{\infty}\frac{b(n-m^2)}{m^\lambda},$$

and the operator J_λ, defined for sequences b on $\mathbb{Z}^2$ by

$$(J_\lambda b)(n_1,n_2) = \sum_{m=1}^{\infty} \frac{b(n_1-m,n_2-m^2)}{m^\lambda},$$

and establish certain $\ell^p \to \ell^q$ estimates for them. D. Oberlin extends the results of Stein and Wainger in [9]. The $H^p \to \ell^q$ and $H^p \to H^q$ estimates for these operators are not known.

The purpose of this article is to continue the study about the behavior of discrete Riesz potential I_α on discrete Hardy spaces $H^p(\mathbb{Z}^n)$, with $n \geqslant 1$, began by the author in [12]. By means of the atomic characterization of $H^p(\mathbb{Z}^n)$ developed in [4] (see Theorem 2 below) and the boundedness of discrete fractional maximal operator, we proved in [12] that I_α is a bounded operator $H^p(\mathbb{Z}^n) \to \ell^q(\mathbb{Z}^n)$ for $0 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

The main aim of this work is to prove the following result.

THEOREM 4. *Let $0 < \alpha < n$ and let I_α be the discrete Riesz potential given by* (3). *If $\frac{n-1}{n} < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then there exists a positive constant C such that*

$$\|I_\alpha b\|_{H^q(\mathbb{Z}^n)} \leqslant C\|b\|_{H^p(\mathbb{Z}^n)} \tag{5}$$

for all $b \in H^p(\mathbb{Z}^n)$.

To prove this result, as in [8], we furnish a molecular decomposition for the elements of $H^p(\mathbb{Z}^n)$, on the range $\frac{n-1}{n} < p \leqslant 1$. More precisely, our strategy is to prove that the discrete Riesz potential I_α maps discrete atoms into discrete molecules, analogous to what is done in the Euclidean setting (see Proposition 2 below).

The paper is organized as follows. Section 2 begins with the preliminaries. Others characterizations of Hardy spaces on $\mathbb{Z}^n$ and its atomic decomposition are established in Section 3. In Section 4, we present the concept of molecule in $\mathbb{Z}^n$ and prove some of its basic properties. In Section 5, we obtain the molecular decomposition for $H^p(\mathbb{Z}^n)$, $\frac{n-1}{n} < p \leqslant 1$, and give a criterion for the $H^p(\mathbb{Z}^n) \to H^q(\mathbb{Z}^n)$ boundedness of certain linear operators, when $\frac{n-1}{n} < p \leqslant q \leqslant 1$. Finally, in Section 6 we prove the Theorem 4.

2. Preliminaries

Throughout this paper, C will denote a positive real constant not necessarily the same at each occurrence. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For every $A \subset \mathbb{Z}^n$, we denote by $\#A$ the cardinality of the set A. Given a real number $s \geqslant 0$, we write $\lfloor s \rfloor$ for the integer part of s. As usual $\mathscr{S}(\mathbb{R}^n)$ denotes the set of Schwartz functions on $\mathbb{R}^n$. If β is the multiindex $\beta = (\beta_1,\ldots,\beta_n) \in \mathbb{N}_0^n$, then $[\beta] := \beta_1 + \ldots + \beta_n$ and $j^\beta := j_1^{\beta_1} \cdots j_n^{\beta_n}$ for every $j = (j_1,\ldots,j_n) \in \mathbb{Z}^n$.

The following results will be useful in the study of the discrete molecules presented in Section 4 and in the obtaining of the $H^p(\mathbb{Z}^n) \to H^q(\mathbb{Z}^n)$ boundedness for the discrete Riesz potential which will be established in Section 6.

In the sequel, for $j=(j_1,\dots,j_n)\in\mathbb{Z}^n$ we put $|j|_\infty=\max\{|j_k|:k=1,\dots,n\}$ and $|j|=(j_1^2+\cdots+j_n^2)^{1/2}$.

Our first result generalizes [12, Lemma 2.2], so we will give the main steps of its proof.

LEMMA 1. *If $\varepsilon>0$ and $N\in\mathbb{N}$, then*

$$\sum_{|j|_\infty\geqslant N}\frac{1}{|j|^{n+\varepsilon}}\leqslant 2^n n^{n+\varepsilon}\left(2+\frac{2^{\frac{\varepsilon}{n}}n}{\varepsilon}\right)^n N^{-\varepsilon}. \tag{6}$$

Proof. For every $j=(j_1,\dots,j_n)\in\mathbb{Z}^n$ such that $|j|_\infty\geqslant N$, by Multinomial Theorem, we obtain that

$$(|j_1|+|j_2|+\cdots+|j_n|)^n\geqslant\max\{N,|j_1|\}\cdot\max\{N,|j_2|\}\cdots\max\{N,|j_n|\}.$$

So, by applying the $\frac{n+\varepsilon}{n}$-power on this inequality, we get

$$(|j_1|+|j_2|+\cdots+|j_n|)^{n+\varepsilon}\geqslant\max\{N^{1+\frac{\varepsilon}{n}},|j_1|^{1+\frac{\varepsilon}{n}}\}\cdot\max\{N^{1+\frac{\varepsilon}{n}},|j_2|^{1+\frac{\varepsilon}{n}}\} \tag{7}$$

$$\cdots\max\{N^{1+\frac{\varepsilon}{n}},|j_n|^{1+\frac{\varepsilon}{n}}\},$$

for all $j=(j_1,\dots,j_n)\in\mathbb{Z}^n$ such that $|j|_\infty\geqslant N$.

Now, by (7) and following the proof of Lemma 2.2 in [12], for every $L\geqslant N$ we have

$$\sum_{N\leqslant|j|_\infty\leqslant L}\frac{1}{|j|^{n+\varepsilon}}\leqslant 2^n n^{n+\varepsilon}\left(N^{-\frac{\varepsilon}{n}}+\sum_{j_1=N}^{L}\frac{1}{j_1^{1+\frac{\varepsilon}{n}}}\right)^n.$$

Finally, letting L tend to infinity, we obtain

$$\sum_{|j|_\infty\geqslant N}\frac{1}{|j|^{n+\varepsilon}}\leqslant 2^n n^{n+\varepsilon}\left(2+\frac{2^{\frac{\varepsilon}{n}}n}{\varepsilon}\right)^n N^{-\varepsilon}. \quad\square$$

LEMMA 2. ([5, Example 3.2.10]) *Let $0<\alpha<n$ and $\Phi\in\mathscr{S}(\mathbb{R}^n)$ such that $\widehat{\Phi}(\xi)=1$ if $|\xi|\leqslant1$ and $\widehat{\Phi}(\xi)=0$ if $|\xi|>2$. For $x\in\mathbb{R}^n$ and $R>0$, we put*

$$\mu_{\alpha,R}(x)=\sum_{j\in\mathbb{Z}^n\setminus\{\mathbf{0}\}}|j|^{\alpha-n}\widehat{\Phi}(j/R)\,e^{2\pi i(j\cdot x)}.$$

Then,

$$\lim_{R\to\infty}|\mu_{\alpha,R}(x)|\leqslant C|x|^{-\alpha},\ \text{ for all } x\in[-1/2,1/2)^n\setminus\{\mathbf{0}\},$$

where C is independent of x.

Given $R>0$, we consider the set E_R of slowly increasing $C^\infty(\mathbb{R}^n)$ functions f with $\operatorname{supp}\widehat{f}\subset[-R,R]^n$. The elements of E_R are called functions of exponential type R. For this class of functions we have the following result.

LEMMA 3. ([1, Lemma 3]) *Let* $0 < p < \infty$ *and* $0 < R < \frac{1}{2}$, *then there exists a positive constant* C *such that*

$$\sum_{j\in\mathbb{Z}^n} |f(j)|^p \leqslant C \int_{\mathbb{R}^n} |f(x)|^p dx,$$

for every function f *of exponential type* R.

3. Hardy spaces on $\mathbb{Z}^n$

We briefly recall others characterizations of the spaces $H^p(\mathbb{Z}^n)$ given in [4]. By [4, Theorem 2.7] the discrete Poisson kernel P_t^d in (1) and (2) can be substituted by a discrete kernel Φ_t^d, where $\Phi \in \mathscr{S}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \Phi = 1$, $\Phi_t^d(j) = t^{-n}\Phi(j/t)$ if $j \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and $\Phi_t^d(\mathbf{0}) = 0$. Moreover, the respective H^p quasi-norms are equivalent.

Now, for $s = 1,\dots,n$, we introduce the discrete Riesz kernels K_s^d on $\mathbb{Z}^n$, that is

$$K_s^d(j) = \frac{j_s}{|j|^{n+1}}, \quad \text{for } s = 1,\dots,n \text{ and } j = (j_1,\dots,j_n) \in \mathbb{Z}^n \setminus \{\mathbf{0}\},$$

and $K_s^d(\mathbf{0}) = 0$. The discrete Riesz transforms, R_s^d, applied to a sequence $b = \{b(j)\}_{j\in\mathbb{Z}^n}$ are the convolution operators

$$(R_s^d b)(m) = (K_s^d *_{\mathbb{Z}^n} b)(m) = \sum_{j\neq m} b(j)\frac{m_s - j_s}{|m-j|^{n+1}}, \quad s = 1,\dots,n.$$

Then, for $0 < p < \infty$, one defines

$$H^p_{\text{Riesz}}(\mathbb{Z}^n) = \left\{ b \in \ell^p(\mathbb{Z}^n) : R_s^d b \in \ell^p(\mathbb{Z}^n),\, s = 1,\dots,n \right\},$$

equipped with the quasi-norm given by

$$\|b\|_{H^p_{\text{Riesz}}(\mathbb{Z}^n)} := \|b\|_{\ell^p(\mathbb{Z}^n)} + \sum_{s=1}^{n} \|R_s^d b\|_{\ell^p(\mathbb{Z}^n)}. \tag{8}$$

On the range $\frac{n-1}{n} < p \leqslant 1$, Boza and Carro in [4] obtained the following characterization.

THEOREM 1. ([4, Theorem 2.6]) *Let* $\frac{n-1}{n} < p \leqslant 1$, *then*

$$H^p_{Riesz}(\mathbb{Z}^n) = H^p(\mathbb{Z}^n),$$

with equivalent H^p *quasi-norms.*

For $1 < p < \infty$, we define $H^p(\mathbb{Z}^n) := H^p_{\text{Riesz}}(\mathbb{Z}^n) = \ell^p(\mathbb{Z}^n)$ (this last identity follows from [1, Proposition 12]) and put $H^\infty(\mathbb{Z}^n) := \ell^\infty(\mathbb{Z}^n)$. In [3], S. Boza and M. Carro established the connection between the boundedness of convolution operators on $H^p(\mathbb{R}^n)$ and some related operators on $H^p(\mathbb{Z}^n)$.

In [4], the authors also gave an atomic characterization of $H^p(\mathbb{Z}^n)$ for $0 < p \leqslant 1$. Before establishing this result we give the definition of (p, p_0, L)-atom in $H^p(\mathbb{Z}^n)$ and a supporting result about atoms.

DEFINITION 1. Let $0 < p \leqslant 1 < p_0 \leqslant \infty$, $d_p := \lfloor n(p^{-1}-1) \rfloor$ ($\lfloor s \rfloor$ indicates the integer part of $s \geqslant 0$) and $L \geqslant d_p$, with $L \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We say that a sequence $a = \{a(j)\}_{j \in \mathbb{Z}^n}$ is an (p, p_0, L)-atom centered at a discrete cube $Q \subset \mathbb{Z}^n$ if the following three conditions hold:

(a1) $\operatorname{supp} a \subset Q$,

(a2) $\|a\|_{\ell^{p_0}(\mathbb{Z}^n)} \leqslant (\#Q)^{1/p_0 - 1/p}$,

(a3) $\sum_{j \in Q} j^\beta a(j) = 0$, for all multi-index $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ with $[\beta] = \beta_1 + \cdots + \beta_n \leqslant L$, where $j^\beta := j_1^{\beta_1} \cdots j_n^{\beta_n}$.

REMARK 1. It is easy to check that every (p, ∞, L)-atom is an (p, p_0, L)-atom for each $1 < p_0 < \infty$.

LEMMA 4. *Let $0 < p \leqslant 1 < p_0 < \infty$ and $L \geqslant d_p$. Given an (p, p_0, L)-atom $a = \{a(j)\}_{j \in \mathbb{Z}^n}$, we put*

$$\widehat{a}(x) = \sum_{j \in \mathbb{Z}^n} a(j) e^{2\pi i (j \cdot x)}, \ \ x \in \mathbb{R}^n.$$

Then

$$|\widehat{a}(x)| \leqslant (2\pi)^{L+1} |x|^{L+1} \sum_{j \in \mathbb{Z}^n} |a(j)| |j|^{L+1} e^{2\pi \sqrt{n} |j|}, \ \ \textit{for } |x| \leqslant \sqrt{n}.$$

Proof. From the identity $e^{2\pi i (j \cdot x)} = \sum_{k=0}^{\infty} \frac{(2\pi i (j \cdot x))^k}{k!}$, we have that

$$e^{2\pi i (j \cdot x)} - \sum_{k=0}^{L} \frac{(2\pi i (j \cdot x))^k}{k!} = \sum_{k=L+1}^{\infty} \frac{(2\pi i (j \cdot x))^k}{k!}.$$

So,

$$\left| e^{2\pi i (j \cdot x)} - \sum_{k=0}^{L} \frac{(2\pi i (j \cdot x))^k}{k!} \right| \leqslant (2\pi)^{L+1} |j|^{L+1} |x|^{L+1} e^{2\pi |j| |x|}. \tag{9}$$

Now, by the moment condition of the atom $a(\cdot)$, we have that

$$\widehat{a}(x) = \sum_{j \in \mathbb{Z}^n} a(j) \left(e^{2\pi i (j \cdot x)} - \sum_{k=0}^{L} \frac{(2\pi i (j \cdot x))^k}{k!} \right). \tag{10}$$

Finally, (10) and (9) lead to

$$|\widehat{a}(x)| \leqslant (2\pi)^{L+1} |x|^{L+1} \sum_{j \in \mathbb{Z}^n} |a(j)| |j|^{L+1} e^{2\pi \sqrt{n} |j|},$$

for all $|x| \leqslant \sqrt{n}$. □

The atomic decomposition for members in $H^p(\mathbb{Z}^n)$, $0 < p \leqslant 1$, developed in [4] is as follows:

THEOREM 2. ([4, Theorem 3.7]) *Let $0 < p \leqslant 1$, $L \geqslant d_p$ and $b \in H^p(\mathbb{Z}^n)$. Then there exist a sequence of (p,∞,L)-atoms $\{a_k\}_{k=0}^{+\infty}$, a sequence of scalars $\{\lambda_k\}_{k=0}^{+\infty}$ and a positive constant C, which depends only on p and n, with $\sum_{k=0}^{+\infty} |\lambda_k|^p \leqslant C\|b\|_{H^p(\mathbb{Z}^n)}^p$ such that $b = \sum_{k=0}^{+\infty} \lambda_k a_k$, where the series converges in $H^p(\mathbb{Z}^n)$.*

4. Molecules in $\mathbb{Z}^n$

In this section we present a type of molecules in $\mathbb{Z}^n$, which are analogous to the continuous version introduced by Taibleson and Weiss in [14], and will prove that these discrete molecules belong to $H^p(\mathbb{Z}^n)$.

DEFINITION 2. Let $0 < p \leqslant 1 < p_0 \leqslant \infty$, $r > \frac{1}{p} - \frac{1}{p_0}$ and $d_p = \lfloor n(p^{-1}-1) \rfloor$. A sequence $M = \{M(j)\}_{j\in\mathbb{Z}^n}$ is called a (p,p_0,r,d_p)-molecule centered at $m_0 \in \mathbb{Z}^n$ if M satisfies the following conditions:

(m1) $\mathscr{N}_{p,p_0,r}(M) := \|M\|_{\ell^{p_0}}^{1-\theta} \||\cdot - m_0|^{nr} M\|_{\ell^{p_0}}^{\theta} < \infty$, where $\theta = (1/p - 1/p_0)/r$,

(m2) $\sum_{j\in\mathbb{Z}^n} j^\beta M(j) = 0$, for all multi-index $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ with $[\beta] \leqslant d_p$.

We call $\mathscr{N}_{p,p_0,r}(M)$ the molecule norm of M, which is also denoted by $\mathscr{N}(M)$.

REMARK 2. Let $0 < p \leqslant 1 < p_0 \leqslant \infty$, $L \geqslant d_p$ and $r > \frac{1}{p} - \frac{1}{p_0}$. If $a(\cdot)$ is an (p,p_0,L)-atom centered at a cube $Q \subset \mathbb{Z}^n$, then $a(\cdot)$ is a (p,p_0,r,d_p)-molecule centered at each $m_0 \in Q$ with $\mathscr{N}(a) \leqslant C$, where C is independent of the atom $a(\cdot)$ and m_0.

LEMMA 5. *Let $0 < p \leqslant 1 < p_0 \leqslant \infty$, $r > \frac{1}{p} - \frac{1}{p_0}$. If M is a (p,p_0,r,d_p)-molecule centered at $m_0 \in \mathbb{Z}^n$, then*

$$\|M\|_{\ell^p(\mathbb{Z}^n)} \leqslant C\mathscr{N}(M),$$

where C depends only on n, p, p_0 and r.

Proof. Without loss of generality we assume $m_0 = \mathbf{0}$. From the moment condition (m2) of the molecule M, it follows that

$$|M(\mathbf{0})| \leqslant \sum_{j\neq\mathbf{0}} |M(j)|.$$

Then, for $0 < p \leqslant 1$

$$\sum_{j\in\mathbb{Z}^n} |M(j)|^p \leqslant 2\sum_{j\neq\mathbf{0}} |M(j)|^p.$$

By Hölder inequality, we have

$$\sum_{j\neq\mathbf{0}} |M(j)|^p \leqslant \left(\sum_{j\neq\mathbf{0}} |j|^{-nrp(p_0/p)'}\right)^{1/(p_0/p)'} \left(\sum_{j\neq\mathbf{0}} |j|^{nrp_0} |M(j)|^{p_0}\right)^{p/p_0}.$$

We observe that $rp(p_0/p)'-1>0$. Then, by applying Lemma 1 with $N=1$ and $\varepsilon=n(rp(p_0/p)'-1)$, we obtain

$$\|M\|_{\ell^p(\mathbb{Z}^n)} \leqslant C\||\cdot|^{nr}M\|_{\ell^{p_0}(\mathbb{Z}^n)}. \tag{11}$$

We put $\sigma=\left(\|M\|_{\ell^{p_0}}^{-1}\||\cdot|^{nr}M\|_{\ell^{p_0}}\right)^{1/nr}$. If $\sigma\leqslant 2$, then $\||\cdot|^{nr}M\|_{\ell^{p_0}}\leqslant 2^{nr}\|M\|_{\ell^{p_0}}$. This and (11) lead to

$$\|M\|_{\ell^p}\leqslant C\|M\|_{\ell^{p_0}}^{1-\theta}\||\cdot|^{nr}M\|_{\ell^{p_0}}^{\theta}=C\mathcal{N}(M),$$

where $\theta=(1/p-1/p_0)/r$. Now, we assume $\sigma>2$, and put

$$\sum_{j\in\mathbb{Z}^n}|M(j)|^p=\sum_{|j|_\infty<\lfloor\sigma\rfloor}|M(j)|^p+\sum_{|j|_\infty\geqslant\lfloor\sigma\rfloor}|M(j)|^p=J_1+J_2.$$

For J_1, by Hölder inequality, we have

$$J_1\leqslant\|M\|_{\ell^{p_0}}^p\left(\sum_{|j|_\infty<\lfloor\sigma\rfloor}1\right)^{1/(p_0/p)'}\leqslant C\|M\|_{\ell^{p_0}}^p\sigma^{n\left(1-\frac{p}{p_0}\right)}=C\mathcal{N}(M)^p.$$

For J_2, by Hölder inequality and Lemma 1 with $\varepsilon=n(rp(p_0/p)'-1)$ and $N=\lfloor\sigma\rfloor$, we obtain

$$J_2\leqslant\||\cdot|^{nr}M\|_{\ell^{p_0}}^p\left(\sum_{|j|_\infty\geqslant\lfloor\sigma\rfloor}|j|^{-nrp(p_0/p)'}\right)^{1/(p_0/p)'}$$

$$\leqslant C\||\cdot|^{nr}M\|_{\ell^{p_0}}^p\lfloor\sigma\rfloor^{-n\left(rp-\frac{1}{(p_0/p)'}\right)},$$

since $\sigma>2$, it follows that

$$\leqslant C\||\cdot|^{nr}M\|_{\ell^{p_0}}^p(\sigma-1)^{-n\left(rp-\frac{1}{(p_0/p)'}\right)}$$

$$\leqslant C\||\cdot|^{nr}M\|_{\ell^{p_0}}^p\sigma^{-n\left(rp-\frac{1}{(p_0/p)'}\right)}=C\mathcal{N}(M)^p.$$

This concludes the proof. □

LEMMA 6. *Let $0<p\leqslant 1<p_0\leqslant\infty$, $r>\frac{1}{p}-\frac{1}{p_0}$ and let M be a (p,p_0,r,d_p)-molecule centered at $m_0\in\mathbb{Z}^n$. For $\Phi\in\mathcal{S}(\mathbb{R}^n)$, we put*

$$\widetilde{M}(x)=\sum_{j\in\mathbb{Z}^n}M(j)\Phi(x-j),\ \ x\in\mathbb{R}^n.$$

Then, $\widetilde{M}^d$ is a (p,p_0,r,d_p)-molecule centered at m_0 with

$$\mathcal{N}(\widetilde{M}^d)\leqslant C\mathcal{N}(M),$$

where C does not depends on M, and $\widetilde{M}^d$ is the restriction of $\widetilde{M}$ to $\mathbb{Z}^n$.

Proof. The proof is similar to the one given in [8, Lemma 2]. □

PROPOSITION 1. *Let* $n \geqslant 2$, $\frac{n-1}{n} < p \leqslant 1 < p_0 \leqslant \infty$, *and* $r > \frac{1}{p} - \frac{1}{p_0}$. *If* M *is a* $(p, p_0, r, 0)$*-molecule, then* $M \in H^p(\mathbb{Z}^n)$ *with*

$$\|M\|_{H^p(\mathbb{Z}^n)} \leqslant C\mathscr{N}(M),$$

where C *is independent of* M.

Proof. For $n \geqslant 2$, $\frac{n-1}{n} < p \leqslant 1$ we have $d_p = 0$, by Lemma 5, (8) and Theorem 1, it is enough to show

$$\sum_{s=1}^{n} \|R_s^d M\|_{\ell^p(\mathbb{Z}^n)} \leqslant C\mathscr{N}(M), \tag{12}$$

for discrete $(p, p_0, r, 0)$-molecules M. From the invariance by translations of the operator R_s^d, we can assume that the molecules are centered at $\mathbf{0}$. Then, we fix a real number $0 < R < \frac{1}{2}$ and let Φ be a radial function of $\mathscr{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{\Phi} \subset [-R, R]^n$ with $\widehat{\Phi}(\xi) \equiv 1$ on the ball $B(\mathbf{0}, R/2)$. Given a discrete $(p, p_0, r, 0)$-molecule M, by Lemma 5, we have that $M \in \ell^p(\mathbb{Z}^n) \subset \ell^1(\mathbb{Z}^n)$, and so

$$\widetilde{M}(x) = \sum_{j \in \mathbb{Z}^n} M(j)\Phi(x - j) \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap E_R \subset L^2(\mathbb{R}^n).$$

For $s = 1, \ldots, n$ and $x \in \mathbb{R}^n$, using Fourier's inversion theorem, we introduce the continuous Riesz transforms R_s acting on $\widetilde{M}$

$$\begin{aligned}
(R_s\widetilde{M})(x) &= \left(\xi \to -iC_n \frac{\xi_s}{|\xi|} \widehat{\widetilde{M}}(\xi)\right)^{\vee}(x) \\
&= \int_{\mathbb{R}^n} (-i)C_n \frac{\xi_s}{|\xi|} \sum_{j \in \mathbb{Z}^n} M(j)\, \widehat{\Phi}(\xi)\, e^{2\pi i (x-j)\cdot\xi} \\
&= \sum_{j \in \mathbb{Z}^n} M(j)(R_s\Phi)(x - j).
\end{aligned}$$

Proceeding as in the proof of Theorem 2.6 in [4] (see Theorem 1 above), and by Lemma 5, we obtain

$$\left\|(R_s\widetilde{M})^d - R_s^d M\right\|_{\ell^p(\mathbb{Z}^n)} \leqslant C\|M\|_{\ell^p(\mathbb{Z}^n)} \leqslant C\mathscr{N}(M), \tag{13}$$

where $(R_s\widetilde{M})^d$ is the restriction of $R_s\widetilde{M}$ to $\mathbb{Z}^n$.

On the other hand, Lemma 3 applied to the function $R_s\widetilde{M}$ of exponential type R allows us to obtain

$$\|(R_s\widetilde{M})^d\|_{\ell^p(\mathbb{Z}^n)} \leqslant C\|R_s\widetilde{M}\|_{L^p(\mathbb{R}^n)}. \tag{14}$$

Finally, according to the ideas to estimate (9) in [8, Proposition 1] and Lemma 6, we obtain that $\widetilde{M}$ is a p-molecule centered at $\mathbf{0}$ in $\mathbb{R}^n$ with

$$\widetilde{\mathscr{N}}(\widetilde{M}) \leqslant C\mathscr{N}(\widetilde{M}^d) \leqslant C\mathscr{N}(M), \tag{15}$$

where C is independent of M and $\widetilde{\mathscr{N}}(\widetilde{M})$ is the continuous molecule norm. Thus $\widetilde{M} \in H^p(\mathbb{R}^n)$. Since R_s is a bounded operator $H^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$, then (14) and (15) lead to

$$\|(R_s\widetilde{M})^d\|_{\ell^p(\mathbb{Z}^n)} \leqslant C\|R_s\widetilde{M}\|_{L^p(\mathbb{R}^n)} \leqslant C\|\widetilde{M}\|_{H^p(\mathbb{R}^n)} \leqslant \widetilde{\mathscr{N}}(\widetilde{M}) \leqslant C\mathscr{N}(M).$$

So, this inequality and (13) give (12). Therefore the proof is concluded. □

5. Molecular decomposition for $H^p(\mathbb{Z}^n)$

In this section, we establish a molecular reconstruction theorem for $H^p(\mathbb{Z}^n)$, $\frac{n-1}{n} < p \leqslant 1$. As an application of this result, we give a criterion for the $H^p(\mathbb{Z}^n) \to H^q(\mathbb{Z}^n)$ boundedness of certain linear operators, when $\frac{n-1}{n} < p \leqslant q \leqslant 1$.

THEOREM 3. *Let $n \geqslant 2$, $\frac{n-1}{n} < p \leqslant 1 < p_0 \leqslant \infty$, $r > \frac{1}{p} - \frac{1}{p_0}$. If $\{M_k\}_{k=1}^{\infty}$ is a sequence of $(p, p_0, r, 0)$-molecules in $\mathbb{Z}^n$ such that $\sum_{k=1}^{\infty} \mathscr{N}(M_k)^p < \infty$, then the series $\sum_{k=1}^{\infty} M_k$ converges to a sequence h in $H^p(\mathbb{Z}^n)$ and*

$$\|h\|_{H^p(\mathbb{Z}^n)}^p \leqslant C \sum_{k=1}^{\infty} \mathscr{N}(M_k)^p, \tag{16}$$

where C does not depend on the molecules M_k.

Proof. For each $L \in \mathbb{N}$, we consider $S_L = \sum_{k=1}^{L} M_k$. By proposition 1 we have that $\|S_{\widetilde{L}} - S_L\|_{H^p(\mathbb{Z}^n)}^p \leqslant \sum_{k=L}^{\widetilde{L}} \mathscr{N}(M_k)^p$ for every $L \leqslant \widetilde{L}$. Since $\sum_{k=1}^{\infty} \mathscr{N}(M_k)^p < \infty$, from the completeness of $H^p(\mathbb{Z}^n)$ with respect to the metric $d(f,g) = \|f-g\|_{H^p(\mathbb{Z}^n)}^p$, it follows that there exists $h \in H^p(\mathbb{Z}^n)$ such that $h = \sum_{k=1}^{\infty} M_k$ in $H^p(\mathbb{Z}^n)$. Finally, the inequality $\|h\|_{H^p(\mathbb{Z}^n)}^p \leqslant \|h - S_L\|_{H^p(\mathbb{Z}^n)}^p + \|S_L\|_{H^p(\mathbb{Z}^n)}^p$ gives (16). □

REMARK 3. By Remarks 1 and 2, every (p, ∞, L)-atom is a $(p, p_0, r, 0)$-molecule. Then, Theorem 2 and Theorem 3 give the molecular characterization for $H^p(\mathbb{Z}^n)$, $\frac{n-1}{n} < p \leqslant 1$.

COROLLARY 1. *Let $n \geqslant 2$, $\frac{n-1}{n} < p \leqslant q \leqslant 1 < p_0 \leqslant q_0 \leqslant \infty$, $r > \frac{1}{q} - \frac{1}{q_0}$, $L \geqslant 0$ and let T be a bounded linear operator $\ell^{p_0}(\mathbb{Z}^n) \to \ell^{q_0}(\mathbb{Z}^n)$ such that $Ta(\cdot)$ is a $(q, q_0, r, 0)$-molecule for all (p, ∞, L)-atom $a(\cdot)$. Suppose that there exists an universal positive constant C_0 such that $\mathscr{N}(Ta) \leqslant C_0$ for all (p, ∞, L)-atom $a(\cdot)$, then there exists a positive constant C such that*

$$\|Tb\|_{H^q(\mathbb{Z}^n)} \leqslant C\|b\|_{H^p(\mathbb{Z}^n)}, \tag{17}$$

for all $b \in H^p(\mathbb{Z}^n)$.

Proof. For $n \geqslant 2$ and $\frac{n-1}{n} < p \leqslant q \leqslant 1$, we have that $d_p = d_q = 0$. Given $b \in H^p(\mathbb{Z}^n)$ and $L \geqslant 0$, by Theorem 2, there exist a sequence of real numbers $\{\lambda_k\}_{k=1}^{\infty}$, a sequence of (p,∞,L)-atoms $\{a_k\}_{k=1}^{\infty}$ such that $b = \sum_{k=1}^{\infty} \lambda_k a_k$ converges in $\ell^p(\mathbb{Z}^n)$ and

$$\left(\sum_{k=1}^{\infty} |\lambda_k|^p\right)^{1/p} \leqslant C\|b\|_{H^p(\mathbb{Z}^n)}. \tag{18}$$

Since $\ell^p(\mathbb{Z}^n) \subset \ell^{p_0}(\mathbb{Z}^n)$ embed continuously, we have that $b = \sum_k \lambda_k a_k$ in $\ell^{p_0}(\mathbb{Z}^n)$. Being T a bounded operator $\ell^{p_0}(\mathbb{Z}^n) \to \ell^{q_0}(\mathbb{Z}^n)$, we obtain that

$$(Tb)(j) = \sum_{k=1}^{\infty} \lambda_k (Ta_k)(j), \text{ for all } j \in \mathbb{Z}^n. \tag{19}$$

By our hypotheses, $M_k = \lambda_k(Ta_k)$ is a $(q,q_0,r,0)$-molecule for every k and

$$\mathcal{N}(M_k) \leqslant C_0|\lambda_k|. \tag{20}$$

Now, by (20), (18) and Theorem 3, there exists $h \in H^p(\mathbb{Z}^n)$ such that

$$h(j) = \sum_{k=1}^{\infty} M_k(j) = \sum_{k=1}^{\infty} \lambda_k (Ta_k)(j),$$

for all $j \in \mathbb{Z}^n$. So, (19) gives $h = Tb$. Then, (16), (20) and (18) allow us to get (17). $\square$

6. Discrete Riesz potential

Let $0 < \alpha < n$, $0 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. In this section we obtain the $H^p(\mathbb{Z}^n) \to H^q(\mathbb{Z}^n)$ boundedness of discrete Riesz potential I_α, which is defined by

$$(I_\alpha b)(j) = \sum_{i \in \mathbb{Z}^n \setminus \{j\}} \frac{b(i)}{|i-j|^{n-\alpha}}, \quad j \in \mathbb{Z}^n. \tag{21}$$

We first prove that the operator I_α maps atoms into molecules.

PROPOSITION 2. *Let $n \geqslant 2$, $0 < \alpha < \frac{n}{n-1}$, $\frac{n-1}{n} < p \leqslant \frac{n}{n+\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{n}{n-\alpha} < q_0 < \infty$, $r > \frac{1}{q} - \frac{1}{q_0}$, $L = \lfloor n(r + \frac{1}{q_0}) + \alpha \rfloor + 1$ and let I_α be the discrete Riesz potential given by (21). Then $I_\alpha a(\cdot)$ is a $(q,q_0,r,0)$-molecule for every (p,∞,L)-atom $a(\cdot)$. Moreover, there exists an universal positive constat C_0 such that $\mathcal{N}(I_\alpha a) \leqslant C_0$ for all (p,∞,L)-atom $a(\cdot)$.*

Proof. Given an (p,∞,L)-atom $a(\cdot)$ centered at m_0, we shall prove that $I_\alpha a(\cdot)$ is a $(q,q_0,r,0)$-molecule centered at m_0. For them, let $Q = \{j \in \mathbb{Z}^n : |j - m_0|_\infty \leqslant N\}$ be the discrete cube on which $a(\cdot)$ is supported. Now, we put

$$Q^* = \left\{ j \in \mathbb{Z}^n : |j - m_0|_\infty \leqslant 4\lfloor \sqrt{n} \rfloor N \right\},$$

and write

$$U_1 = \left(\sum_{j \in Q^*} |(I_\alpha a)(j)\, |j - m_0|^{nr}|^{q_0} \right)^{1/q_0} \text{ and } U_2 = \left(\sum_{j \notin Q^*} |(I_\alpha a)(j)\, |j - m_0|^{nr}|^{q_0} \right)^{1/q_0},$$

where $\frac{n}{n-\alpha} < q_0 < \infty$ is fixed. Then, we define $\frac{1}{p_0} := \frac{1}{q_0} + \frac{\alpha}{n}$. By Remark 1, we have that $a(\cdot)$ is an (p, p_0, L)-atom. Now, [13, Proposition (a)] gives

$$U_1 \leqslant C \|I_\alpha a\|_{\ell^{q_0}} N^{nr} \leqslant C \|a\|_{\ell^{p_0}} N^{nr}.$$

Following the ideas to obtain (14) in [12, Proof of Theorem 3.3], we obtain

$$U_2 \leqslant C N^{L+1} \sum_{k \in Q} |a(k)| \left(\sum_{j \notin Q^*} |j - m_0|^{(nr+\alpha-n-L-1)q_0} \right)^{1/q_0}.$$

From Hölder's inequality and Lemma 1 with $\varepsilon = -n + (n + L + 1 - \alpha - nr)q_0 > 0$, we get

$$U_2 \leqslant C \|a\|_{\ell^{p_0}} N^{L+1+n-n/p_0} N^{nr+\alpha-n-L-1+n/q_0} = C \|a\|_{\ell^{p_0}} N^{nr}.$$

Thus, we have

$$\| I_\alpha a(\cdot) | \cdot - m_0|^{nr} \|_{\ell^{q_0}} \leqslant C(U_1 + U_2) \leqslant C \|a\|_{\ell^{p_0}} N^{nr}.$$

Then, for $\theta = (1/q - 1/q_0)/r = (1/p - 1/p_0)/r$, it follows that

$$\mathcal{N}(I_\alpha a) \leqslant C \|a\|_{\ell^{p_0}}^{1-\theta} \|a\|_{\ell^{p_0}}^{\theta} N^{nr\theta} = C \|a\|_{\ell^{p_0}} N^{n(1/p-1/p_0)} \leqslant C_0,$$

where C_0 does not depend on the atom $a(\cdot)$. This gives the size condition (m1).

For $R > 0$ and Φ as in Lemma 2, we put

$$(I_{\alpha,R}\, a)(j) = \sum_{k \neq \mathbf{0}} |k|^{\alpha-n}\, \widehat{\Phi}(k/R)\, a(j-k),$$

and

$$\widehat{(I_{\alpha,R}\, a)}(x) = \sum_{j \in \mathbb{Z}^n} (I_{\alpha,R}\, a)(j)\, e^{2\pi i (j \cdot x)}.$$

A computation gives

$$\widehat{(I_{\alpha,R}\, a)}(x) = \mu_{\alpha,R}(x)\, \widehat{a}(x).$$

By Lemma 2 and Lemma 4, we have

$$\begin{aligned} |\widehat{(I_\alpha\, a)}(x)| &= \lim_{R \to \infty} |\widehat{(I_{\alpha,R}\, a)}(x)| = \lim_{R \to \infty} |\mu_{\alpha,R}(x)|\, |\widehat{a}(x)| \\ &\leqslant C|x|^{-\alpha} |x|^{L+1} \sum_{j \in \mathbb{Z}^n} |a(j)| |j|^{L+1} e^{2\pi\sqrt{n}|j|}, \end{aligned} \tag{22}$$

for all $x \in [-1/2, 1/2)^n \setminus \{\mathbf{0}\}$. Since $L + 1 - \alpha > 0$, after letting $x \to \mathbf{0}$ in (22), we obtain

$$\left| \sum_{j \in \mathbb{Z}^n} (I_\alpha a)(j) \right| = |\widehat{(I_\alpha\, a)}(\mathbf{0})| = 0,$$

which is the moment condition (m2) for the range $\frac{n-1}{n} < p \leqslant 1$. $\square$

THEOREM 4. *Let* $0 < \alpha < n$ *and let* I_α *be the discrete Riesz potential given by* (21)*. If* $\frac{n-1}{n} < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$*, then there exists a positive constant* C *such that*

$$\|I_\alpha b\|_{H^q(\mathbb{Z}^n)} \leqslant C\|b\|_{H^p(\mathbb{Z}^n)} \tag{23}$$

for all $b \in H^p(\mathbb{Z}^n)$*.*

Proof. The case $n = 1$ was proved in [8, Theorem 4]. From now on, we consider $n \geqslant 2$. For the case $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ (i.e.: $1 < p < q < \infty$), we have that $H^p(\mathbb{Z}^n) = \ell^p(\mathbb{Z}^n)$ and $H^q(\mathbb{Z}^n) = \ell^q(\mathbb{Z}^n)$. So, the $\ell^p(\mathbb{Z}^n) \to \ell^q(\mathbb{Z}^n)$ boundedness of I_α follows from [13, Proposition (a)] or [12, Theorem 3.1].

For $\frac{n}{n+\alpha} < p \leqslant 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ (i.e.: $1 < q < \infty$), we have that $H^p(\mathbb{Z}^n) \subsetneqq \ell^p(\mathbb{Z}^n)$ and $H^q(\mathbb{Z}^n) = \ell^q(\mathbb{Z}^n)$. In this case, [12, Theorem 3.3] gives the $H^p(\mathbb{Z}^n) \to \ell^q(\mathbb{Z}^n)$ boundedness of I_α.

If $\frac{n}{n-1} \leqslant \alpha < n$, we have that $\frac{n}{n+\alpha} \leqslant \frac{n-1}{n}$. This is contemplated in the previous case. Thus, for $0 < \alpha < \frac{n}{n-1}$, $\frac{n-1}{n} < p \leqslant \frac{n}{n+\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ (i.e.: $\frac{n-1}{n} < p < q \leqslant 1$), we have that $H^p(\mathbb{Z}^n) \subsetneqq \ell^p(\mathbb{Z}^n)$ and $H^q(\mathbb{Z}^n) \subsetneqq \ell^q(\mathbb{Z}^n)$. Here is where we apply the molecular reconstruction to establish the $H^p(\mathbb{Z}^n) \to H^q(\mathbb{Z}^n)$ boundedness of I_α. For them, we take $\frac{n}{n-\alpha} < q_0 < \infty$, $r > \frac{1}{q} - \frac{1}{q_0}$ and put $\frac{1}{p_0} := \frac{1}{q_0} + \frac{\alpha}{n}$, since I_α is a bounded linear operator $\ell^{p_0}(\mathbb{Z}^n) \to \ell^{q_0}(\mathbb{Z}^n)$, Proposition 2 and Corollary 1 (with $L = \lfloor n(r + \frac{1}{q_0}) + \alpha \rfloor + 1$) allow us to obtain (23). This finishes the proof. $\square$

REMARK 4. We point out that the lower bound $\frac{n-1}{n}$ in this theorem is a consequence of Theorem 1, which is used in Proposition 1. Using other type of molecules, for instance by adapting to the discrete setting the molecular theory used in [10], one can expect to break this barrier and achieve the full range $0 < p \leqslant 1$. This will be left for future work.

REMARK 5. The method presented in this paper can be used to obtain $H^p(\mathbb{Z}^n) \to H^p(\mathbb{Z}^n)$ estimates for discrete singular operators on $\mathbb{Z}^n$.

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