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NONNEGATIVE SOLUTIONS OF DISCRETE SECOND–ORDER UNDAMPED STURM—LIOUVILLE BOUNDARY VALUE PROBLEMS

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Abstract. This article considers a second-order undamped difference equation satisfying the Sturm–Liouville boundary conditions. We apply suitable fixed point theorems to establish sufficient conditions for the existence of nonnegative solutions to the considered problem. We also provide two examples to illustrate the applicability of established results.

Though there have been many works on the study of nonnegative solutions to the considered problem, most of the works used the application of the Guo–Krasnoselskii fixed point theorem. This work's novelty lies in applying new fixed point theorems that yield multiple fixed points in conical shells to the considered problem. This work improves some previous results and reports some new results.

1. Introduction

The mathematical modeling of many problems from economics, computer science, control systems, mechanical engineering, biological neural networks, and others leads to considering initial and boundary value problems for nonlinear difference equations [2, 4, 9, 10, 11]. In the last decades, many authors have studied such problems using various methods, such as fixed point theory, fixed point index theory, variational methods, critical point theory, etc.

In this article, using suitable fixed point theorems, we establish sufficient conditions for the existence of nonnegative solutions to the following discrete second-order undamped Sturm–Liouville boundary value problem:

$$\left(\Delta^2 y\right)(\zeta-1)-\lambda y(\zeta)=\mathfrak{f}(\zeta,y(\zeta)),\quad \zeta\in\mathbb{N}_1^T, \tag{1}$$

$$\alpha y(0)-\beta\left(\Delta y\right)(0)=0,\quad \gamma y(T)+\delta\left(\Delta y\right)(T)=0, \tag{2}$$

where $T\in\mathbb{N}_3$; α, β, $\gamma\geqslant 0$, λ, $\delta>0$ with $\alpha^2+\beta^2>0$, $\delta\geqslant\gamma$; $y:\mathbb{N}_0^{T+1}\to\mathbb{R}$ and $\mathfrak{f}:\mathbb{N}_1^T\times[0,\infty)\to[0,\infty)$ is a continuous function.

Second-order difference equations of the form (1) arise in many real-world problems such as undamped vibrating string, the motion of a particle under a central force, diffusion of a gas into a liquid, etc. Researchers have been studying the existence of nonnegative solutions for discrete boundary value problems of the form (1)–(2) during the past few decades. However, most authors focused on applying the Guo–Krasnoselskii fixed point theorem only [7]. Ferhan [13, 14] and Aykut at al. [3] proved

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the existence of positive solutions for (1)–(2) using the properties of differentiable operators along a specific cone in a Banach space and the Guo–Krasnoselskii fixed point theorem when $\lambda = 0$ and $\alpha\gamma T + \alpha\delta + \beta\gamma > 0$. In their monograph, Agarwal et al. [1] presented the existence theory of the positive solutions for various classes of boundary value problems involving second and higher-order difference equations using the Guo–Krasnoselskii fixed point theorem. Recently, in their brief monograph, Henderson et al. [8] presented the existence of positive solutions for some classes of second-order nonlinear finite difference equations and systems of second-order nonlinear finite difference equations, subject to various multi point boundary conditions using the Guo–Krasnoselskii fixed point theorem.

There has been an increasing interest in multiple fixed point theorems [12] and their applications to boundary value problems for differential and finite difference equations. The applications of these fixed point theorems for (1)–(2) are not yet reported. Motivated by these facts and [6], in this article, we establish the existence of nonnegative solutions for the boundary value problem (1)–(2) using a few prominent conical shell fixed point theorems. This work improves some previous results and reports some new results.

This article is organized as follows. Section 2 constructs the Green's function associated with (1)–(2) and estimates its bounds. In Section 3, we establish sufficient conditions for the existence of nonnegative solutions for (1)–(2). Section 4 presents two examples to manifest the validity of the results obtained in Section 3.

2. Preliminaries

We use the following preliminaries throughout the article. Denote by $\mathbb{N}_i = \{i, i+1, i+2, \dots\}$ and $\mathbb{N}_i^j = \{i, i+1, i+2, \dots, j\}$ for any real numbers i and j such that $j - i \in \mathbb{N}_1$.

DEFINITION 1. [5] For $y : \mathbb{N}_0^{T+1} \to \mathbb{R}$, the first-order forward difference of y is defined by

$$(\Delta y)(\zeta) = y(\zeta+1) - y(\zeta), \quad \zeta \in \mathbb{N}_0^T,$$

and the second-order forward difference of y is defined by

$$\left(\Delta^2 y\right)(\zeta) = y(\zeta+2) - 2y(\zeta+1) + y(\zeta), \quad \zeta \in \mathbb{N}_0^{T-1}.$$

DEFINITION 2. By a solution to (1)–(2) we mean a function $y : \mathbb{N}_0^{T+1} \to \mathbb{R}$ such that y satisfies both (1) and (2).

3. Green's function and its properties

In this section, we construct the Green's function for the linear boundary value problem

$$\left(\Delta^2 y\right)(\zeta - 1) - \lambda y(\zeta) = h(\zeta), \quad \zeta \in \mathbb{N}_1^T, \tag{3}$$

$$\alpha y(0) - \beta (\Delta y)(0) = 0, \quad \gamma y(T) + \delta (\Delta y)(T) = 0, \tag{4}$$

associated with (1)–(2). Here $h : \mathbb{N}_1^T \to \mathbb{R}$. Denote by

$$A = \frac{1}{2}\left[\lambda + 2 + \sqrt{\lambda^2 + 4\lambda}\right].$$

Then,

$$\frac{1}{A} = \frac{1}{2}\left[\lambda + 2 - \sqrt{\lambda^2 + 4\lambda}\right].$$

Clearly, $A > 1$. For convenience, we use the following notations:

$$\begin{aligned}
L_1 &= \alpha - \beta(A-1), \\
L_2 &= \alpha + \beta\left(1 - \frac{1}{A}\right), \\
L_3 &= A^T\left[\gamma + \delta(A-1)\right], \\
L_4 &= \frac{1}{A^T}\left[\gamma - \delta\left(1 - \frac{1}{A}\right)\right], \\
\xi &= L_2 L_3 - L_1 L_4, \\
\phi(\zeta) &= L_2 A^{\zeta} - L_1 A^{-\zeta}, \quad \zeta \in \mathbb{N}_0^{T+1}, \\
\psi(\zeta) &= L_3 A^{-\zeta} - L_4 A^{\zeta}, \quad \zeta \in \mathbb{N}_0^{T+1}.
\end{aligned}$$

Using Lemmas 1 and 2, we obtain the expression for a general solution of the nonhomogeneous second-order difference equation (3).

LEMMA 1. *A general solution of the homogeneous second-order difference equation*

$$\left(\Delta^2 y\right)(\zeta - 1) - \lambda y(\zeta) = 0, \quad \zeta \in \mathbb{N}_1, \tag{5}$$

is given by

$$y(\zeta) = C_1 A^{\zeta} + C_2 A^{-\zeta}, \quad \zeta \in \mathbb{N}_0, \tag{6}$$

where C_1 and C_2 are arbitrary constants.

Proof. In order to prove that (6) is a general solution of (5), it is enough to show that $y_1(\zeta) = A^{\zeta}$ and $y_2(\zeta) = A^{-\zeta}$ are two linearly independent solutions of (5) on $\mathbb{N}_0$. We have

$$\begin{aligned}
\left(\Delta^2 y_1\right)(\zeta - 1) &= y_1(\zeta+1) - 2y_1(\zeta) + y_1(\zeta-1) \\
&= A^{\zeta+1} - 2A^{\zeta} + A^{\zeta-1} = A^{\zeta}\left[A - 2 + \frac{1}{A}\right] = \lambda y_1(\zeta), \quad \zeta \in \mathbb{N}_1,
\end{aligned}$$

and

$$\begin{aligned}
\left(\Delta^2 y_2\right)(\zeta - 1) &= y_2(\zeta+1) - 2y_2(\zeta) + y_2(\zeta-1) \\
&= A^{-\zeta-1} - 2A^{-\zeta} + A^{-\zeta+1} = A^{-\zeta}\left[\frac{1}{A} - 2 + A\right] = \lambda y_2(\zeta), \quad \zeta \in \mathbb{N}_1,
\end{aligned}$$

implying that $y_1(\zeta)=A^{\zeta}$ and $y_2(\zeta)=A^{-\zeta}$ are solutions of (5) on $\mathbb{N}_0$. Further, the Wronskian of y_1 and y_2 is given by

$$\begin{aligned}
W(y_1,y_2)(\zeta) &= \begin{vmatrix} y_1(\zeta) & y_2(\zeta) \\ (\Delta y_1)(\zeta) & (\Delta y_2)(\zeta) \end{vmatrix} \\
&= \begin{vmatrix} y_1(\zeta) & y_2(\zeta) \\ y_1(\zeta+1)-y_1(\zeta) & y_2(\zeta+1)-y_2(\zeta) \end{vmatrix} \\
&= \begin{vmatrix} A^{\zeta} & A^{-\zeta} \\ A^{\zeta+1}-A^{\zeta} & A^{-\zeta-1}-A^{-\zeta} \end{vmatrix} \\
&= A^{\zeta}\left[A^{-\zeta-1}-A^{-\zeta}\right]-A^{-\zeta}\left[A^{\zeta+1}-A^{\zeta}\right] \\
&= \frac{1}{A}-A\neq 0, \quad \zeta\in\mathbb{N}_1,
\end{aligned}$$

implying that $y_1(\zeta)=A^{\zeta}$ and $y_2(\zeta)=A^{-\zeta}$ are linearly independent on $\mathbb{N}_0$. The proof is complete. □

LEMMA 2. *Let $h:\mathbb{N}_1\to\mathbb{R}$. A general solution of the nonhomogeneous second-order difference equation*

$$\left(\Delta^2 y\right)(\zeta-1)-\lambda y(\zeta)=h(\zeta), \quad \zeta\in\mathbb{N}_1, \tag{7}$$

is given by

$$y(\zeta)=C_1A^{\zeta}+C_2A^{-\zeta}+\frac{A}{(A^2-1)}\sum_{s=1}^{\zeta-1}\left(A^{\zeta-s}-A^{s-\zeta}\right)h(s), \quad \zeta\in\mathbb{N}_0, \tag{8}$$

where C_1 and C_2 are arbitrary constants.

Proof. In view of Lemma 1, it is enough to show that

$$v(\zeta)=\frac{A}{(A^2-1)}\sum_{s=1}^{\zeta-1}\left(A^{\zeta-s}-A^{s-\zeta}\right)h(s), \quad \zeta\in\mathbb{N}_0,$$

is a particular solution of (3). For this purpose, we claim that

$$\left(\Delta^2 v\right)(\zeta-1)-\lambda v(\zeta)=h(\zeta), \quad \zeta\in\mathbb{N}_1. \tag{9}$$

To see this, for $\zeta\in\mathbb{N}_1$, consider

$$\begin{aligned}
\left(\Delta^2 v\right)(\zeta-1) &= v(\zeta+1)-2v(\zeta)+v(\zeta-1) \\
&= \frac{A}{(A^2-1)}\left[\sum_{s=1}^{\zeta}\left(A^{\zeta+1-s}-A^{s-\zeta-1}\right)h(s)-2\sum_{s=1}^{\zeta-1}\left(A^{\zeta-s}-A^{s-\zeta}\right)h(s)\right. \\
&\quad \left.+\sum_{s=1}^{\zeta-2}\left(A^{\zeta-1-s}-A^{s-\zeta+1}\right)h(s)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A}{(A^2-1)}\left[\left(A-\frac{1}{A}\right)h(\zeta)+\left(A^2-\frac{1}{A^2}\right)h(\zeta-1)\right.\\
&\quad -2\left(A-\frac{1}{A}\right)h(\zeta-1)+\sum_{s=1}^{\zeta-2}\left(\left(A^{\zeta+1-s}-A^{s-\zeta-1}\right)\right.\\
&\quad \left.\left.-2\left(A^{\zeta-s}-A^{s-\zeta}\right)+\left(A^{\zeta-1-s}-A^{s-\zeta+1}\right)\right)h(s)\right]\\
&= \frac{A}{(A^2-1)}\left[\left(A-\frac{1}{A}\right)h(\zeta)+\left(A-\frac{1}{A}\right)\left(A-2+\frac{1}{A}\right)h(\zeta-1)\right.\\
&\quad \left.+\sum_{s=1}^{\zeta-2}\left(A^{\zeta-s}-A^{s-\zeta}\right)\left(A-2+\frac{1}{A}\right)h(s)\right]\\
&= \frac{A}{(A^2-1)}\left(A-\frac{1}{A}\right)h(\zeta)\\
&\quad +\left(A-2+\frac{1}{A}\right)\frac{A}{(A^2-1)}\sum_{s=1}^{\zeta-1}\left(A^{\zeta-s}-A^{s-\zeta}\right)h(s)\\
&= h(\zeta)+\lambda v(\zeta),
\end{aligned}
$$

implying that (9) holds. The proof is complete. □

In order to find the unique solution to the boundary value problem (3)–(4), we have to apply the boundary conditions (4) which involve the first-order forward difference of y. For this purpose, we compute Δy in the following remark.

REMARK 1. Consider (8). Then, for $\zeta \in \mathbb{N}_0$,

$$
\begin{aligned}
(\Delta y)(\zeta) &= y(\zeta+1)-y(\zeta)\\
&= C_1\left(A^{\zeta+1}-A^{\zeta}\right)+C_2\left(A^{-\zeta-1}-A^{-\zeta}\right)\\
&\quad +\frac{A}{(A^2-1)}\left[\sum_{s=1}^{\zeta}\left(A^{\zeta+1-s}-A^{s-\zeta-1}\right)h(s)-\sum_{s=1}^{\zeta-1}\left(A^{\zeta-s}-A^{s-\zeta}\right)h(s)\right]\\
&= C_1A^{\zeta}(A-1)+C_2A^{-\zeta}\left(\frac{1}{A}-1\right)+\frac{A}{(A^2-1)}\left(A-\frac{1}{A}\right)h(\zeta)\\
&\quad +\frac{A}{(A^2-1)}\left[\sum_{s=1}^{\zeta-1}\left(\left(A^{\zeta+1-s}-A^{s-\zeta-1}\right)-\left(A^{\zeta-s}-A^{s-\zeta}\right)\right)h(s)\right]\\
&= C_1A^{\zeta}(A-1)+C_2A^{-\zeta}\left(\frac{1}{A}-1\right)+\frac{A}{(A^2-1)}\left(A-\frac{1}{A}\right)h(\zeta)\\
&\quad +\frac{A}{(A^2-1)}\left[\sum_{s=1}^{\zeta-1}\left(A^{\zeta-s}(A-1)+A^{s-\zeta}\left(1-\frac{1}{A}\right)\right)h(s)\right]
\end{aligned}
$$

$$= C_1 A^{\zeta}(A-1) + C_2 A^{-\zeta}\left(\frac{1}{A}-1\right)$$
$$+ \frac{A}{(A^2-1)}\left[\sum_{s=1}^{\zeta}\left(A^{\zeta-s}(A-1) + A^{s-\zeta}\left(1-\frac{1}{A}\right)\right)h(s)\right]. \tag{10}$$

In Lemmas 3, 4 and 5, we construct the Green's function associated with (3)–(4) and state its properties.

LEMMA 3. *Assume* $\xi \neq 0$. *The linear boundary value problem* (3)–(4) *has a unique solution given in the form*

$$y(\zeta) = \sum_{s=1}^{T} \mathscr{G}(\zeta,s)h(s), \quad \zeta \in \mathbb{N}_0^{T+1}, \tag{11}$$

where

$$\mathscr{G}(\zeta,s) = \frac{A}{(A^2-1)}\begin{cases} \frac{\phi(s)\psi(\zeta)}{\xi}, & s \in \mathbb{N}_1^{\zeta}, \\ \frac{\phi(\zeta)\psi(s)}{\xi}, & s \in \mathbb{N}_{\zeta}^{T}. \end{cases} \tag{12}$$

Proof. From Lemma 2, a general solution of the second-order difference equation (3) is given by

$$y(\zeta) = C_1 A^{\zeta} + C_2 A^{-\zeta} + \frac{A}{(A^2-1)}\sum_{s=1}^{\zeta-1}\left(A^{\zeta-s} - A^{s-\zeta}\right)h(s), \quad \zeta \in \mathbb{N}_0^{T+1}, \tag{13}$$

where C_1 and C_2 are arbitrary constants. It follows from Remark 1 that

$$(\Delta y)(\zeta) = C_1 A^{\zeta}(A-1) + C_2 A^{-\zeta}\left(\frac{1}{A}-1\right)$$
$$+ \frac{A}{(A^2-1)}\left[\sum_{s=1}^{\zeta}\left(A^{\zeta-s}(A-1) + A^{s-\zeta}\left(1-\frac{1}{A}\right)\right)h(s)\right], \quad \zeta \in \mathbb{N}_0^{T} \tag{14}$$

Using $\alpha y(0) - \beta(\Delta y)(0) = 0$ in (13)–(14), we obtain

$$L_1 C_1 + L_2 C_2 = 0. \tag{15}$$

Using $\gamma y(T) + \delta(\Delta y)(T) = 0$ in (13)–(14), we obtain

$$L_3 C_1 + L_4 C_2 = \frac{A}{(A^2-1)}\sum_{s=1}^{T}\left[L_3 A^{-s} - L_4 A^{s}\right]. \tag{16}$$

From (15) and (16), we have

$$C_1 = \frac{1}{\xi}\frac{L_2 A}{(A^2-1)}\sum_{s=1}^{T}\left[L_3 A^{-s} - L_4 A^{s}\right], \tag{17}$$

and

$$C_2 = -\frac{1}{\xi}\frac{L_1 A}{(A^2-1)}\sum_{s=1}^{T}\left[L_3 A^{-s} - L_4 A^s\right]. \tag{18}$$

Substituting the equalities (17) and (18) in (13), we obtain (12). □

LEMMA 4. *Assume* $\alpha \geqslant \beta(A-1)$ *and* $\gamma \geqslant \delta\left(1-\frac{1}{A}\right)$. *Then, the following properties hold:*

1. $0 \leqslant L_1 \leqslant L_2$ *and* $0 \leqslant L_4 \leqslant L_3$;
2. $\xi \geqslant 0$;
3. ϕ *is a nondecreasing function on* $\mathbb{N}_0^{T+1}$ *with* $\phi(0) \geqslant 0$;
4. ψ *is a nonincreasing function on* $\mathbb{N}_0^{T+1}$ *with* $\psi(T+1) \geqslant 0$,

where

$$\begin{aligned}
L_1 &= \alpha - \beta(A-1),\\
L_2 &= \alpha + \beta\left(1-\frac{1}{A}\right),\\
L_3 &= A^T\left[\gamma + \delta(A-1)\right],\\
L_4 &= \frac{1}{A^T}\left[\gamma - \delta\left(1-\frac{1}{A}\right)\right],\\
\xi &= L_2L_3 - L_1L_4,\\
\phi(\zeta) &= L_2A^{\zeta} - L_1A^{-\zeta}, \quad \zeta \in \mathbb{N}_0^{T+1},\\
\psi(\zeta) &= L_3A^{-\zeta} - L_4A^{\zeta}, \quad \zeta \in \mathbb{N}_0^{T+1}.
\end{aligned}$$

Proof. It follows from the hypothesis that $0 \leqslant L_1$ and $0 \leqslant L_4$. Consider

$$\begin{aligned}
L_2 - L_1 &= (\alpha - \beta(A-1)) - \left(\alpha + \beta\left(1-\frac{1}{A}\right)\right)\\
&= \beta\left(1-\frac{1}{A}\right) \geqslant 0,
\end{aligned}$$

implying that $L_1 \leqslant L_2$. Now, consider

$$\begin{aligned}
L_3 - L_4 &= A^T\left[\gamma + \delta(A-1)\right] - \frac{1}{A^T}\left[\gamma - \delta\left(1-\frac{1}{A}\right)\right]\\
&= \gamma\left(A^T - \frac{1}{A^T}\right) + \delta\left[A^T(A-1) + \frac{1}{A^T}\left(1-\frac{1}{A}\right)\right]\\
&\geqslant \delta\left(1-\frac{1}{A}\right)\left(A^T - \frac{1}{A^T}\right) + \delta\left[A^T(A-1) + \frac{1}{A^T}\left(1-\frac{1}{A}\right)\right]\\
&= \gamma A^T\left(1-\frac{1}{A}\right) \geqslant 0,
\end{aligned}$$

implying that $L_4 \leqslant L_3$. The proof of (1) is complete. To prove (2), consider

$$\xi = L_2L_3 - L_1L_4 \geqslant L_1L_3 - L_1L_3 = 0.$$

The proof of (2) is complete. Clearly, $\phi(0) = L_2 - L_1 \geqslant 0$. In order to show that ϕ is a nondecreasing function on $\mathbb{N}_0^{T+1}$, it is enough to show that $(\Delta\phi)(\zeta)$ is nonnegative on $\mathbb{N}_0^T$. For $\zeta \in \mathbb{N}_0^T$, we have

$$\begin{aligned}
(\Delta\phi)(\zeta) &= \phi(\zeta+1) - \phi(\zeta) \\
&= \left(L_2A^{\zeta+1} - L_1A^{-\zeta-1}\right) - \left(L_2A^{\zeta} - L_1A^{-\zeta}\right) \\
&= L_2A^{\zeta}(A-1) + L_1A^{-\zeta}\left(1 - \frac{1}{A}\right) \geqslant 0,
\end{aligned}$$

implying that ϕ is a nondecreasing function on $\mathbb{N}_0^{T+1}$. The proof of (3) is complete. Consider

$$\begin{aligned}
\psi(T+1) &= L_3A^{-T-1} - L_4A^{T+1} \\
&= \frac{1}{A}\left[\gamma + \delta(A-1)\right] - A\left[\gamma - \delta\left(1 - \frac{1}{A}\right)\right] \\
&= (\delta - \gamma)\left(A - \frac{1}{A}\right) \geqslant 0.
\end{aligned}$$

In order to show that ψ is a nonincreasing function on $\mathbb{N}_0^{T+1}$, it is enough to show that $(\Delta\psi)(\zeta)$ is nonpositive on $\mathbb{N}_0^T$. For $\zeta \in \mathbb{N}_0^T$, we have

$$\begin{aligned}
(\Delta\psi)(\zeta) &= \psi(\zeta+1) - \psi(\zeta) \\
&= \left(L_3A^{-\zeta-1} - L_4A^{\zeta+1}\right) - \left(L_3A^{-\zeta} - L_4A^{\zeta}\right) \\
&= -L_4A^{\zeta}(A-1) - L_3A^{-\zeta}\left(1 - \frac{1}{A}\right) \leqslant 0,
\end{aligned}$$

implying that ψ is a nonincreasing function on $\mathbb{N}_0^{T+1}$. The proof of (4) is complete. $\square$

LEMMA 5. *Assume* $\alpha \geqslant \beta(A-1)$ *and* $\gamma \geqslant \delta\left(1 - \frac{1}{A}\right)$ *such that* $\xi > 0$. *Then, the following properties hold:*

1. $\mathscr{G}(\zeta,s) \geqslant 0$ *for all* $(\zeta,s) \in \mathbb{N}_0^{T+1} \times \mathbb{N}_1^T$;
2. $\mathscr{G}(\zeta,s) \leqslant \mathscr{G}(s,s)$ *for all* $(\zeta,s) \in \mathbb{N}_0^{T+1} \times \mathbb{N}_1^T$;
3. $\mathscr{G}(\zeta,s) \geqslant \sigma\mathscr{G}(s,s)$ *for all* $(\zeta,s) \in \mathbb{N}_1^T \times \mathbb{N}_1^T$, *where*

$$\sigma = \min\left\{\frac{\phi(1)}{\phi(T)}, \frac{\psi(T)}{\psi(1)}\right\}.$$

Clearly, $0 < \sigma < 1$.

Proof. The proofs of (1) follows from Lemma 4. For $s \in \mathbb{N}_1^{\zeta}$, we have

$$\frac{\mathscr{G}(\zeta,s)}{\mathscr{G}(s,s)} = \frac{\psi(\zeta)}{\psi(s)} \leqslant 1.$$

Similarly, for $s \in \mathbb{N}_{\zeta}^{T}$, we have

$$\frac{\mathscr{G}(\zeta,s)}{\mathscr{G}(s,s)} = \frac{\phi(\zeta)}{\phi(s)} \leqslant 1.$$

The proof of (2) is complete. For $s \in \mathbb{N}_{\zeta}^{T}$, we have

$$\frac{\mathscr{G}(\zeta,s)}{\mathscr{G}(s,s)} = \frac{\phi(\zeta)}{\phi(s)} \geqslant \frac{\phi(1)}{\phi(T)}.$$

Similarly, for $s \in \mathbb{N}_1^{\zeta}$, we have

$$\frac{\mathscr{G}(\zeta,s)}{\mathscr{G}(s,s)} = \frac{\psi(\zeta)}{\psi(s)} \geqslant \frac{\psi(T)}{\psi(1)}.$$

The proof of (3) is complete. □

4. Existence of nonnegative solutions

The existence results for nonnegative solutions to the boundary value problem (1)–(2) are stated and proven in this section. By Lemma 3, we note that y is a solution of the summation equation

$$y(\zeta) = \sum_{s=1}^{T} \mathscr{G}(\zeta,s)\mathfrak{f}(s,y(s)), \quad \zeta \in \mathbb{N}_0^{T+1}, \tag{19}$$

if and only if y is a solution of (1)–(2). Any solution $y : \mathbb{N}_0^{T+1} \to \mathbb{R}$ of (1)–(2) can be viewed as a real $(T+2)$-tuple vector. Consider a T-dimensional Banach space

$$Y = \left\{ y : \mathbb{N}_0^{T+1} \to \mathbb{R} \;\middle|\; \alpha y(0) - \beta (\Delta y)(0) = 0,\ \gamma y(T) + \delta (\Delta y)(T) = 0 \right\} \subseteq \mathbb{R}^T,$$

with the norm

$$\|y\| = \max_{\zeta \in \mathbb{N}_1^T} |y(\zeta)|,$$

and cone $\mathfrak{K}$ in Y given by

$$\mathfrak{K} = \left\{ y \in Y \;\middle|\; y(\zeta) \geqslant 0 \text{ for } \zeta \in \mathbb{N}_1^T \text{ and } \min_{\zeta \in \mathbb{N}_1^T} y(\zeta) \geqslant \sigma \|y\| \right\}.$$

Define the operator $\mathfrak{S} : Y \to Y$ by

$$(\mathfrak{S}y)(\zeta) = \sum_{s=1}^{T} \mathscr{G}(\zeta,s)\mathfrak{f}(s,y(s)), \quad \zeta \in \mathbb{N}_0^{T+1}. \tag{20}$$

It's obvious that y is a fixed point of $\mathfrak{S}$ if and only if it solves (1)–(2). $\mathfrak{S}$ is trivially completely continuous because it is a summation operator on a discrete finite set. In order to find nonnegative solutions of (1)–(2), we now look for nonnegative fixed points of $\mathfrak{S}$. In this direction, Definition 3 and Theorems 1-3 will be crucial.

DEFINITION 3. [12] Let $Y = (Y, \|\cdot\|)$ be a real Banach space and $\mathfrak{K} \subset Y$ be a cone. Let a, b be two numbers with $0 < a < b < \infty$ and g be a concave nonnegative functional on $\mathfrak{K}$. We define the following sets

$$\mathfrak{K}_a = \{y \in \mathfrak{K} : \|y\| \leqslant a\},$$
$$\mathfrak{K}(g,a,b) = \{y \in \mathfrak{K} : a \leqslant g(y),\ \|y\| \leqslant b\}.$$

Clearly, $\mathfrak{K}_a$ and $\mathfrak{K}(g,a,b)$ are bounded, closed, and convex in $\mathfrak{K}$.

THEOREM 1. [12] *Let $c > 0$ be a number. Suppose $\mathfrak{S} : \mathfrak{K}_c \to \mathfrak{K}$ is completely continuous, and suppose we have a concave nonnegative functional g with $g(y) \leqslant \|y\|$ for all $y \in \mathfrak{K}$, and numbers $0 < a < b \leqslant c < \infty$ satisfying the following conditions:*

(i) $\{y \in \mathfrak{K}(g,a,b) : g(y) > a\} \neq \emptyset$ *and* $g(\mathfrak{S}y) > a$ *if* $y \in \mathfrak{K}(g,a,b)$*;*

(ii) $\mathfrak{S}y \in \mathfrak{K}_c$ *if* $y \in \mathfrak{K}(g,a,c)$*;*

(iii) $g(\mathfrak{S}y) > a$ *for all* $y \in \mathfrak{K}(g,a,c)$ *with* $\|\mathfrak{S}y\| > b$.

Then, $\mathfrak{S}$ has a fixed point in $\mathfrak{K}(g,a,c)$.

THEOREM 2. [12] *Let $c > 0$ be a number. Suppose $\mathfrak{S} : \mathfrak{K}_c \to \mathfrak{K}$ is completely continuous, and suppose we have a concave nonnegative functional g with $g(y) \leqslant \|y\|$ for all $y \in \mathfrak{K}$, and numbers a and d, with $0 < d < a < c < \infty$ satisfying the following conditions:*

(i) $\{y \in \mathfrak{K}(g,a,c) : g(y) > a\} \neq \emptyset$ *and* $g(\mathfrak{S}y) > a$ *if* $y \in \mathfrak{K}(g,a,c)$*;*

(ii) $\|\mathfrak{S}y\| < d$ *if* $y \in \mathfrak{K}_d$*; and either*

(iii) $\|\mathfrak{S}y\| - g(\mathfrak{S}y) \leqslant c - a$ *for each* $y \in \mathfrak{K}_c$ *such that* $\|\mathfrak{S}y\| > c$*; or*

(iv) $g(\mathfrak{S}y) > \frac{a}{c}\|\mathfrak{S}y\|$ *for each* $y \in \mathfrak{K}_c$ *such that* $\|\mathfrak{S}y\| > c$.

Then, $\mathfrak{S}$ has minimum two fixed points in $\mathfrak{K}_c$.

THEOREM 3. [12] *Let $c > 0$ be a number. Suppose $\mathfrak{S} : \mathfrak{K}_c \to \mathfrak{K}_c$ is completely continuous, and suppose we have a concave nonnegative functional g with $g(y) \leqslant \|y\|$ for all $y \in \mathfrak{K}$, and numbers $0 < d < a < b \leqslant c < \infty$, satisfying the following conditions:*

(i) $\{y \in \mathfrak{K}(g,a,b) : g(y) > a\} \neq \emptyset$ *and* $g(\mathfrak{S}y) > a$ *if* $y \in \mathfrak{K}(g,a,b)$*;*

(ii) $\|\mathfrak{S}y\| < d$ *if* $y \in \mathfrak{K}_d$*;*

(iii) $g(\mathfrak{S}y) > a$ *for all* $y \in \mathfrak{K}(g,a,c)$ *with* $\|\mathfrak{S}y\| > b$.

Then, $\mathfrak{S}$ *has minimum three fixed points in* $\mathfrak{K}_c$.

LEMMA 6. $\mathfrak{S}(\mathfrak{K}) \subset \mathfrak{K}$.

Proof. Let $y \in \mathfrak{K}$. Then, by Lemma 5, we have $\big(\mathfrak{S}y\big)(\zeta) \geqslant 0$ for all $\zeta \in \mathbb{N}_1^T$. Now, consider

$$\begin{aligned}
\min_{\zeta\in\mathbb{N}_1^T} \big(\mathfrak{S}y\big)(\zeta) &= \min_{\zeta\in\mathbb{N}_1^T}\left[\sum_{s=1}^{T}\mathscr{G}(\zeta,s)\mathfrak{f}(s,y(s))\right]\\
&\geqslant \sum_{s=1}^{T}\left[\min_{\zeta\in\mathbb{N}_1^T}\mathscr{G}(\zeta,s)\right]\mathfrak{f}(s,y(s))\\
&\geqslant \sigma\sum_{s=1}^{T}\mathscr{G}(s,s)\mathfrak{f}(s,y(s))\\
&\geqslant \sigma\sum_{s=1}^{T}\left[\max_{\zeta\in\mathbb{N}_1^T}\mathscr{G}(\zeta,s)\right]\mathfrak{f}(s,y(s))\\
&\geqslant \sigma\max_{\zeta\in\mathbb{N}_1^T}\left[\sum_{s=1}^{T}\mathscr{G}(\zeta,s)\mathfrak{f}(s,y(s))\right]\\
&= \sigma\max_{\zeta\in\mathbb{N}_1^T}\big(\mathfrak{S}y\big)(\zeta) = \sigma\,\|\mathfrak{S}y\|.
\end{aligned}$$

Therefore, $\mathfrak{S}y \in \mathfrak{K}$. The proof is complete. □

Denote by

$$M = \sum_{s=1}^{T}\mathscr{G}(s,s).$$

Let us define a concave nonnegative functional $g : \mathfrak{K} \to [0,\infty)$ by

$$g(y) = \min_{\zeta\in\mathbb{N}_1^T} y(\zeta), \quad \zeta \in \mathfrak{K}.$$

Clearly, $g(y) \leqslant \|y\|$ for all $y \in \mathfrak{K}$.

THEOREM 4. *Suppose there exist numbers* $0 < b < \frac{b}{\sigma} \leqslant c < \infty$ *satisfying the following conditions:*

(C 1) $\mathfrak{f}(\zeta,y) \leqslant \dfrac{c}{M}, \quad (\zeta,y) \in \mathbb{N}_1^T \times [b,c]$;

(C 2) $\mathfrak{f}(\zeta,y) > \dfrac{b}{M\sigma}, \quad (\zeta,y) \in \mathbb{N}_1^T \times \left[b,\dfrac{b}{\sigma}\right]$.

Then, (1)–(2) *has a nonnegative solution* y *in* $\mathfrak{K}(g,b,c)$.

Proof. We demonstrate that every requirement stated in Theorem 1 is met. It is evident that $\mathfrak{S} : \mathfrak{K}_c \to \mathfrak{K}$ is completely continuous. For all $y \in \mathfrak{K}(g,b,c)$, we have $b \leqslant g(y)$ and $\|y\| \leqslant c$. That is,

$$b \leqslant y(\zeta) \leqslant c, \quad \zeta \in \mathbb{N}_1^T.$$

From assumption (C 1), we get

$$\begin{aligned}
\|\mathfrak{S}y\| &= \max_{\zeta \in \mathbb{N}_1^T} \big(\mathfrak{S}y\big)(\zeta) \\
&= \max_{\zeta \in \mathbb{N}_1^T} \left[\sum_{s=1}^{T} \mathscr{G}(\zeta,s)\mathfrak{f}(s,y(s))\right] \\
&\leqslant \frac{c}{M} \max_{\zeta \in \mathbb{N}_1^T} \left[\sum_{s=1}^{T} \mathscr{G}(\zeta,s)\right] \\
&\leqslant \frac{c}{M} \sum_{s=1}^{T} \left[\max_{\zeta \in \mathbb{N}_1^T} \mathscr{G}(\zeta,s)\right] \\
&\leqslant \frac{c}{M} \left[\sum_{s=1}^{T} \mathscr{G}(s,s)\right] = c,
\end{aligned}$$

implying that $\mathfrak{S}y \in \mathfrak{K}_c$. Consequently, Theorem 1's condition (ii) is met. We now demonstrate that Theorem 1's condition (i) is true. Let $y^* \equiv \frac{b}{\sigma}$. Since

$$g\left(y^*\right) = g\left(\frac{b}{\sigma}\right) = \frac{b}{\sigma} > b \quad \text{and} \quad \|y^*\| = \frac{b}{\sigma},$$

we obtain $y^* \in \mathfrak{K}\left(g,b,\frac{b}{\sigma}\right)$, and

$$\left\{y \in \mathfrak{K}\left(g,b,\frac{b}{\sigma}\right) : g(y) > b\right\} \neq \emptyset.$$

Also, if $y \in \mathfrak{K}\left(g,b,\frac{b}{\sigma}\right)$, then

$$b \leqslant y(\zeta) \leqslant \frac{b}{\sigma}, \quad \zeta \in \mathbb{N}_1^T.$$

From assumption (C 2), we obtain

$$\begin{aligned}
g\left(\mathfrak{S}y\right) &= \min_{\zeta \in \mathbb{N}_1^T} \big(\mathfrak{S}y\big)(\zeta) \\
&= \min_{\zeta \in \mathbb{N}_1^T} \left[\sum_{s=1}^{T} \mathscr{G}(\zeta,s)\mathfrak{f}(s,y(s))\right] \\
&> \frac{b}{M\sigma} \min_{\zeta \in \mathbb{N}_1^T} \left[\sum_{s=1}^{T} \mathscr{G}(\zeta,s)\right]
\end{aligned}$$

$$\geqslant \frac{b}{M\sigma}\sum_{s=1}^{T}\left[\min_{\zeta\in\mathbb{N}_1^T}\mathscr{G}(\zeta,s)\right]$$
$$\geqslant \frac{b}{M}\left[\sum_{s=1}^{T}\mathscr{G}(s,s)\right]=b.$$

This demonstrates that Theorem 1's condition (i) is met. Lastly, we verify that Theorem 1's condition (iii) is likewise true. Assume that $\|\mathfrak{S}y\| > \frac{b}{\sigma}$ for every $y\in\mathfrak{K}(g,b,c)$. Then,

$$g(\mathfrak{S}y)=\min_{\zeta\in\mathbb{N}_1^T}\big(\mathfrak{S}y\big)(\zeta)\geqslant\sigma\|\mathfrak{S}y\|>b.$$

Therefore, Theorem 1's condition (iii) is met. Thus, (1)–(2) has a nonnegative solution y in $\mathfrak{K}(g,b,c)$ according to Theorem 1. □

THEOREM 5. *Suppose there exist numbers $0<a<b<\frac{2b}{\sigma}<\infty$ satisfying the following conditions:*

(B 1) $\mathfrak{f}(\zeta,y)<\dfrac{a}{M},\quad (\zeta,y)\in\mathbb{N}_1^T\times[0,a];$

(B 2) $\mathfrak{f}(\zeta,y)>\dfrac{b}{M\sigma},\quad (\zeta,y)\in\mathbb{N}_1^T\times\left[b,\dfrac{2b}{\sigma}\right].$

Then, (1)–(2) *has minimum two nonnegative solutions y_1 and y_2 in $\mathfrak{K}_{\frac{2b}{\sigma}}$.*

Proof. We demonstrate that every condition stated in Theorem 2 is met. It is evident that $\mathfrak{S}:\mathfrak{K}_c\to\mathfrak{K}$ is completely continuous. For all $y\in\mathfrak{K}_a$, we have $\|y\|\leqslant a$. From assumption (B 1), we get

$$\|\mathfrak{S}y\|=\max_{\zeta\in\mathbb{N}_1^T}\big(\mathfrak{S}y\big)(\zeta)$$
$$=\max_{\zeta\in\mathbb{N}_1^T}\left[\sum_{s=1}^{T}\mathscr{G}(\zeta,s)\mathfrak{f}(s,y(s))\right]$$
$$<\frac{a}{M}\max_{\zeta\in\mathbb{N}_1^T}\left[\sum_{s=1}^{T}\mathscr{G}(\zeta,s)\right]$$
$$\leqslant\frac{a}{M}\sum_{s=1}^{T}\left[\max_{\zeta\in\mathbb{N}_1^T}\mathscr{G}(\zeta,s)\right]$$
$$\leqslant\frac{a}{M}\left[\sum_{s=1}^{T}\mathscr{G}(s,s)\right]=a.$$

Consequently, Theorem 2's condition (ii) is met. We now demonstrate that Theorem 2 holds under condition (i). Let $y^*\equiv\frac{2b}{\sigma}$. Since

$$g(y^*)=g\left(\frac{2b}{\sigma}\right)=\frac{2b}{\sigma}>b\quad\text{and}\quad\|y^*\|=\frac{2b}{\sigma},$$

we obtain $y^* \in \mathfrak{K}\left(g,b,\frac{2b}{\sigma}\right)$, and

$$\left\{y \in \mathfrak{K}\left(g,b,\frac{2b}{\sigma}\right) : g(y) > b\right\} \neq \emptyset.$$

Also, if $y \in \mathfrak{K}\left(g,b,\frac{2b}{\sigma}\right)$, then

$$b \leqslant y(\zeta) \leqslant \frac{2b}{\sigma}, \quad \zeta \in \mathbb{N}_1^T.$$

From assumption (B 2), we obtain

$$\begin{aligned}
g(\mathfrak{S}y) &= \min_{\zeta \in \mathbb{N}_1^T} (\mathfrak{S}y)(\zeta) \\
&= \min_{\zeta \in \mathbb{N}_1^T} \left[\sum_{s=1}^{T} \mathscr{G}(\zeta,s)\mathfrak{f}(s,y(s))\right] \\
&> \frac{b}{M\sigma} \min_{\zeta \in \mathbb{N}_1^T} \left[\sum_{s=1}^{T} \mathscr{G}(\zeta,s)\right] \\
&\geqslant \frac{b}{M\sigma} \sum_{s=1}^{T} \left[\min_{\zeta \in \mathbb{N}_1^T} \mathscr{G}(\zeta,s)\right] \\
&\geqslant \frac{b}{M} \left[\sum_{s=1}^{T} \mathscr{G}(s,s)\right] = b.
\end{aligned}$$

This demonstrates that Theorem 2's condition (i) is met. Lastly, we verify that Theorem 2's condition (iv) is likewise true. Assume that $\|\mathfrak{S}y\| > \frac{2b}{\sigma}$ and that $y \in \mathfrak{K}_{\frac{2b}{\sigma}}$. Then,

$$g(\mathfrak{S}y) = \min_{\zeta \in \mathbb{N}_1^T} (\mathfrak{S}y)(\zeta) \geqslant \sigma\|\mathfrak{S}y\| > \frac{\sigma}{2}\|\mathfrak{S}y\| = \frac{b}{\left(\frac{2b}{\sigma}\right)}\|\mathfrak{S}y\|.$$

Therefore, Theorem 2's condition (iv) is met. Thus, in $\mathfrak{K}_{\frac{2b}{\sigma}}$, (1)–(2) has at least two nonnegative solutions, denoted as y_1 and y_2, according to Theorem 2. □

THEOREM 6. *Suppose there exist numbers* $0 < a < \frac{b}{\sigma} \leqslant c < \infty$ *satisfying the following conditions:*

(A 1) $\mathfrak{f}(\zeta,y) \leqslant \frac{c}{M}, \quad (\zeta,y) \in \mathbb{N}_1^T \times [0,c];$

(A 2) $\mathfrak{f}(\zeta,y) < \frac{a}{M}, \quad (\zeta,y) \in \mathbb{N}_1^T \times [0,a];$

(A 3) $\mathfrak{f}(\zeta,y) > \frac{b}{M\sigma}, \quad (\zeta,y) \in \mathbb{N}_1^T \times \left[b,\frac{b}{\sigma}\right].$

Then, (1)–(2) *has minimum three nonnegative solutions* y_1, y_2 *and* y_3 *in* $\mathfrak{K}_c$ *such that*

$$\|y_1\| < a, \quad g(y_2) > b, \tag{21}$$

$$\|y_3\| > a \quad \text{with} \quad g(y_3) < b. \tag{22}$$

Proof. We demonstrate that every condition stated in Theorem 3 is met. We have $\|y\| \leqslant c$ for every $y \in \mathfrak{K}_c$. From assumption (A 1), we get

$$\begin{aligned}\|\mathfrak{S}y\| &= \max_{\zeta\in\mathbb{N}_1^T} \big(\mathfrak{S}y\big)(\zeta) \\ &= \max_{\zeta\in\mathbb{N}_1^T} \left[\sum_{s=1}^{T} \mathscr{G}(\zeta,s)\mathfrak{f}(s,y(s))\right] \\ &\leqslant \frac{c}{M} \max_{\zeta\in\mathbb{N}_1^T} \left[\sum_{s=1}^{T} \mathscr{G}(\zeta,s)\right] \\ &\leqslant \frac{c}{M} \sum_{s=1}^{T} \left[\max_{\zeta\in\mathbb{N}_1^T} \mathscr{G}(\zeta,s)\right] \\ &\leqslant \frac{c}{M} \left[\sum_{s=1}^{T} \mathscr{G}(s,s)\right] = c.\end{aligned}$$

Hence, $\mathfrak{S} : \mathfrak{K}_c \to \mathfrak{K}_c$ is completely continuous. For all $y \in \mathfrak{K}_a$, we have $\|y\| \leqslant a$. From assumption (A 2), we get

$$\begin{aligned}\|\mathfrak{S}y\| &= \max_{\zeta\in\mathbb{N}_1^T} \big(\mathfrak{S}y\big)(\zeta) \\ &= \max_{\zeta\in\mathbb{N}_1^T} \left[\sum_{s=1}^{T} \mathscr{G}(\zeta,s)\mathfrak{f}(s,y(s))\right] \\ &< \frac{a}{M} \max_{\zeta\in\mathbb{N}_1^T} \left[\sum_{s=1}^{T} \mathscr{G}(\zeta,s)\right] \\ &\leqslant \frac{a}{M} \sum_{s=1}^{T} \left[\max_{\zeta\in\mathbb{N}_1^T} \mathscr{G}(\zeta,s)\right] \\ &\leqslant \frac{a}{M} \left[\sum_{s=1}^{T} \mathscr{G}(s,s)\right] = a.\end{aligned}$$

Consequently, Theorem 3's condition (ii) is met. We now demonstrate that Theorem 3 holds under condition (i). Let $y^* \equiv \frac{b}{\sigma}$. Since

$$g\left(y^*\right) = g\left(\frac{b}{\sigma}\right) = \frac{b}{\sigma} > b \quad \text{and} \quad \|y^*\| = \frac{b}{\sigma},$$

we obtain $y^* \in \mathfrak{K}\left(g, b, \frac{b}{\sigma}\right)$, and

$$\left\{y \in \mathfrak{K}\left(g, b, \frac{b}{\sigma}\right) : g(y) > b\right\} \neq \emptyset.$$

Also, if $y \in \mathfrak{K}\left(g, b, \frac{b}{\sigma}\right)$, then

$$b \leqslant y(\zeta) \leqslant \frac{b}{\sigma}, \quad \zeta \in \mathbb{N}_1^T.$$

From assumption (A 3), we obtain

$$
\begin{aligned}
g(\mathfrak{S}y) &= \min_{\zeta\in\mathbb{N}_1^T} \big(\mathfrak{S}y\big)(\zeta) \\
&= \min_{\zeta\in\mathbb{N}_1^T} \left[\sum_{s=1}^{T} \mathscr{G}(\zeta,s)\mathfrak{f}(s,y(s))\right] \\
&> \frac{b}{M\sigma} \min_{\zeta\in\mathbb{N}_1^T} \left[\sum_{s=1}^{T} \mathscr{G}(\zeta,s)\right] \\
&\geqslant \frac{b}{M\sigma} \sum_{s=1}^{T} \left[\min_{\zeta\in\mathbb{N}_1^T} \mathscr{G}(\zeta,s)\right] \\
&\geqslant \frac{b}{M} \left[\sum_{s=1}^{T} \mathscr{G}(s,s)\right] = b.
\end{aligned}
$$

This demonstrates that Theorem 3's condition (i) is met. Lastly, we verify that Theorem 3's condition (iii) is likewise true. Assume that $\|\mathfrak{S}y\| > \frac{b}{\sigma}$ for every $y \in \mathfrak{K}(g,b,c)$. Then,

$$
g(\mathfrak{S}y) = \min_{\zeta\in\mathbb{N}_1^T} \big(\mathfrak{S}y\big)(\zeta) \geqslant \sigma\|\mathfrak{S}y\| > b.
$$

Therefore, Theorem 3's condition (iii) is met. Thus, according to Theorem 3, there are at least three nonnegative solutions to (1)–(2) in $\mathfrak{K}_c$, such that (21) and (22) hold. These solutions are y_1, y_2, and y_3. □

REMARK 2. Since α, β, $\gamma \geqslant 0$, $\delta > 0$ with $\alpha^2+\beta^2 > 0$ and $\delta \geqslant \gamma$, it follows from the boundary conditions (2) that we have $y(0) \geqslant 0$ and $y(T+1) \geqslant 0$ for each $y \in \mathfrak{K}$. If we remove the restriction $\delta \geqslant \gamma$, then $y(T+1)$ need not be nonnegative.

5. Applications

APPLICATION 1. (Dirichlet conditions) Consider (1)–(2) with $\lambda = \frac{1}{2}$, $T = 4$, $\alpha = \gamma = \delta = 1$, $\beta = 0$ and

$$
\mathfrak{f}(\zeta,\xi) = \frac{37}{25(\zeta+1)} \begin{cases} \dfrac{1}{120} + \sin\xi, & \zeta \in \mathbb{N}_1^4, \quad 0 \leqslant \xi \leqslant \dfrac{1}{35}, \\ \dfrac{1}{120} + 150\left(\xi - \dfrac{1}{35}\right) + \sin\xi, & \zeta \in \mathbb{N}_1^4, \quad \dfrac{1}{35} \leqslant \xi \leqslant \dfrac{1}{27}, \\ \dfrac{1}{120} + \dfrac{80}{63} + \sin\xi, & \zeta \in \mathbb{N}_1^4, \quad \dfrac{1}{27} \leqslant \xi \leqslant 360. \end{cases}
$$

Clearly, $T \in \mathbb{N}_3$; α, β, $\gamma \geqslant 0$, λ, $\delta > 0$ with $\alpha^2+\beta^2 > 0$, $\delta \geqslant \gamma$; $u : \mathbb{N}_0^5 \to \mathbb{R}$ and $f : \mathbb{N}_1^4 \times [0,\infty) \to [0,\infty)$ is a continuous function. We obtain $A = 2$, $L_1 = L_2 = 1$, $L_3 = 32$, $L_4 = \frac{1}{32}$, $\xi = \frac{1023}{32}$,

$$
\phi(\zeta) = 2^{\zeta} - 2^{-\zeta}, \quad \zeta \in \mathbb{N}_0^5,
$$

and

$$\psi(\zeta)=2^{5-\zeta}-2^{\zeta-5},\quad \zeta\in\mathbb{N}_0^5.$$

The Green's function of (3)–(4) is

$$\mathscr{G}(\zeta,s)=\frac{64}{3069}\begin{cases}(2^s-2^{-s})\left(2^{5-\zeta}-2^{\zeta-5}\right), & s\in\mathbb{N}_1^{\zeta},\\ \left(2^{\zeta}-2^{-\zeta}\right)(2^{5-s}-2^{s-5}), & s\in\mathbb{N}_{\zeta}^4.\end{cases}$$

Then,

$$\sigma=\min\left\{\frac{\phi(1)}{\phi(4)},\frac{\psi(4)}{\psi(1)}\right\}=\frac{8}{85},$$

and

$$M=\sum_{s=1}^{T}\mathscr{G}(s,s)=\frac{64}{3069}\sum_{s=1}^{4}\left[32+\frac{1}{32}-2^{2s-5}-2^{5-2s}\right]=\frac{760}{341}.$$

If we choose $a=\frac{1}{35}$, $b=\frac{1}{27}$ and $c=360$, then $0<a<\frac{b}{\sigma}\leqslant c<\infty$ and

(A 1) $\mathfrak{f}(\zeta,y)\leqslant\dfrac{3069}{19},\quad(\zeta,y)\in\mathbb{N}_1^4\times[0,360]$;

(A 2) $\mathfrak{f}(\zeta,y)<\dfrac{341}{26600},\quad(\zeta,y)\in\mathbb{N}_1^4\times\left[0,\dfrac{1}{35}\right]$;

(A 3) $\mathfrak{f}(\zeta,y)>\dfrac{5797}{32832},\quad(\zeta,y)\in\mathbb{N}_1^4\times\left[\dfrac{1}{27},\dfrac{85}{216}\right]$.

Then, by Theorem 6, (1)–(2) has minimum three nonnegative solutions y_1, y_2 and y_3 in $\mathfrak{K}_{360}$ such that

$$\|y_1\|<\frac{1}{35},\quad g(y_2)>\frac{1}{27},\tag{23}$$

$$\|y_3\|>\frac{1}{35}\quad\text{with}\quad g(y_3)<\frac{1}{27}.\tag{24}$$

APPLICATION 2. (Right focal conditions) Consider (1)–(2) with $\lambda=\frac{1}{2}$, $T=4$, $\alpha=\gamma=\delta=1$, $\beta=0$ and

$$\mathfrak{f}(\zeta,\xi)=\frac{19}{10(\zeta+1)}\begin{cases}\dfrac{1}{120}+\sin\xi, & \zeta\in\mathbb{N}_1^4,\quad 0\leqslant\xi\leqslant\dfrac{1}{35},\\ \dfrac{1}{120}+150\left(\xi-\dfrac{1}{35}\right)+\sin\xi, & \zeta\in\mathbb{N}_1^4,\quad \dfrac{1}{35}\leqslant\xi\leqslant\dfrac{1}{27},\\ \dfrac{1}{120}+\dfrac{80}{63}+\sin\xi, & \zeta\in\mathbb{N}_1^4,\quad \dfrac{1}{27}\leqslant\xi\leqslant 360.\end{cases}$$

Clearly, $T\in\mathbb{N}_3$; α, β, $\gamma\geqslant 0$, λ, $\delta>0$ with $\alpha^2+\beta^2>0$, $\delta\geqslant\gamma$; $u:\mathbb{N}_0^5\to\mathbb{R}$ and $f:\mathbb{N}_1^4\times[0,\infty)\to[0,\infty)$ is a continuous function. We obtain $A=2$, $L_1=L_2=1$, $L_3=16$, $L_4=-\frac{1}{32}$, $\xi=\frac{513}{32}$,

$$\phi(\zeta)=2^{\zeta}-2^{-\zeta},\quad \zeta\in\mathbb{N}_0^5,$$

and

$$\psi(\zeta) = 2^{4-\zeta} + 2^{\zeta-5}, \quad \zeta \in \mathbb{N}_0^5.$$

The Green's function of (3)–(4) is

$$\mathscr{G}(\zeta,s) = \frac{64}{1539}\begin{cases} (2^s - 2^{-s})\left(2^{4-\zeta} + 2^{\zeta-5}\right), & s \in \mathbb{N}_1^{\zeta}, \\ \left(2^{\zeta} - 2^{-\zeta}\right)(2^{4-s} + 2^{s-5}), & s \in \mathbb{N}_{\zeta}^4. \end{cases}$$

Then,

$$\sigma = \min\left\{\frac{\phi(1)}{\phi(4)}, \frac{\psi(4)}{\psi(1)}\right\} = \frac{24}{255},$$

and

$$M = \sum_{s=1}^{T} \mathscr{G}(s,s) = \frac{64}{1539}\sum_{s=1}^{4}\left[16 - \frac{1}{32} + 2^{2s-5} - 2^{4-2s}\right] = \frac{164}{57}.$$

If we choose $a = \frac{1}{35}$, $b = \frac{1}{27}$ and $c = 360$, then $0 < a < \frac{b}{\sigma} \leqslant c < \infty$ and

(A 1) $\mathfrak{f}(\zeta,y) \leqslant \frac{5130}{41}, \quad (\zeta,y) \in \mathbb{N}_1^4 \times [0,360];$

(A 2) $\mathfrak{f}(\zeta,y) < \frac{57}{5740}, \quad (\zeta,y) \in \mathbb{N}_1^4 \times \left[0, \frac{1}{35}\right];$

(A 3) $\mathfrak{f}(\zeta,y) > \frac{1615}{11808}, \quad (\zeta,y) \in \mathbb{N}_1^4 \times \left[\frac{1}{27}, \frac{255}{648}\right].$

Then, by Theorem 6, (1)–(2) has minimum three nonnegative solutions y_1, y_2 and y_3 in $\mathfrak{K}_{360}$ such that

$$\|y_1\| < \frac{1}{35}, \quad g(y_2) > \frac{1}{27}, \tag{25}$$

$$\|y_3\| > \frac{1}{35} \text{ with } g(y_3) < \frac{1}{27}. \tag{26}$$

6. Conclusion

In this article, we considered a second-order undamped difference equation satisfying the Sturm–Liouville boundary conditions, and applied suitable fixed point theorems to establish sufficient conditions for the existence of nonnegative solutions. We also provided two examples to illustrate the applicability of established results. This work improves some previous results and reports some new results.

REFERENCES

[1] R. P. AGARWAL, D. O'REGAN AND P. J. Y. WONG, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, 1999.

[2] E. AMOROSO, P. CANDITO AND J. MAWHIN, *Existence of a priori bounded solutions for discrete two-point boundary value problems*, J. Math. Anal. Appl., **519**, 2 (2023), 1–18.

[3] N. AYKUT AND SH. GUSEINOV, *On positive solutions of boundary value problems for nonlinear second order difference equations*, Turk. J. Math., **27**, 4 (2003), 481–507.

[4] M. BARGHOUTHE, M. BERRAJAA AND A. AYOUJIL, *Second order discrete boundary value problem with the* $(p_1(k);p_2(k))$ *-Laplacian*, Bol. Soc. Parana. Mat., **42**, 3 (2024), 1–10.

[5] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales*, Birkhäuser Boston Inc., 2001.

[6] Z. DU, *Positive solutions for a second-order three-point discrete boundary value problem*, J. Appl. Math. Comput., **26**, 1–2 (2008), 219–231.

[7] D. J. GUO AND V. LAKSHMIKANTHAM, *Nonlinear Problems in Abstract Cones*, Academic Press Inc., 1988.

[8] J. HENDERSON AND R. LUCA, *Boundary Value Problems for Second-Order Finite Difference Equations and Systems*, De Gruyter Studies in Mathematics, 2023.

[9] L. KONG AND M. WANG, *On a second order discrete problem*, Matematiche, **77**, 2 (2022), 407–418.

[10] L. KONG AND M. WANG, *Multiple solutions for a nonlinear discrete problem of the second order*, Differ. Equ., **14**, 2 (2022), 189–204.

[11] J. KUANG AND J. LIAO, *Existence of positive solutions for second-order differential equations with two-point boundary value problems involving* p *-Laplacian*, J. Appl. Math. Comput., **70**, 2 (2024), 1523–1542.

[12] R. W. LEGGETT AND L. R. WILLIAMS, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math., **28**, 4 (1979), 673–688.

[13] F. MERDIVENCI, *Two positive solutions of a boundary value problem for difference equations*, J. Differ. Equ., **1**, 3 (1995), 263–270.

[14] F. A. MERDIVENCI, *Existence of positive solutions of nonlinear discrete Sturm–Liouville problems*, Math. Comput. Model., **32**, 5–6 (2000), 599–607.

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