Journal of Classical Analysis
Volume 26, Number 1 (2025), 97–113 doi:10.7153/jca-2025-26-07

ON LACUNARY STATISTICAL φ–CONVERGENCE OF ORDER α IN PARTIAL METRIC SPACES

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Abstract. In the present paper, we introduce the notions of lacunary statistically φ-convergence and lacunary strongly φ-Cesàro summable of order α in a partial metric space (X,φ) and established the relation between them. Beside this, we get a characterization of lacunary statistical φ-convergence sequences of order α in terms of α-lacunary statistical dense subsequence of it.

1. Introduction

In 1951, Fast [11] and Stienhaus [32] introduced the idea of statistical convergence which is, in fact, a generalization of usual notion of convergence. Later on, Buck [7] studied this concept as "convergence in density" in 1953. It is also a part of monograph by Zygmund [35] and referred as almost convergence. Schoenberg [29] introduced and studied this concept independently in connection with summability of sequences in 1959.

The notion of statistical convergence has its main pillar as natural density, which is defined as

DEFINITION 1. [24] For $K \subseteq \mathbb{N}$, the natural density is denoted by $\delta(K)$ and defined as

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}\text{card}(\{m \in K : m \leqslant n\}),$$

provided the limit exists. It is easily verified that $\delta(K) = 0$, for finite subset K of $\mathbb{N}$ and $\delta(K)+\delta(\mathbb{N}-K) = 1$ for every $K \subseteq \mathbb{N}$.

Using the notion of natural density, statistical convergence is defined as

DEFINITION 2. A real valued sequence (z_m) is said to be statistically convergent to $\ell \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$\delta(\{m \in \mathbb{N} : |z_m - \ell| \geqslant \varepsilon\}) = 0,$$

$$\text{i.e.,}\quad \lim_{n\to\infty} \frac{1}{n}\text{card}(\{m \leqslant n : |z_m - \ell| \geqslant \varepsilon\}) = 0$$

Mathematics subject classification (2020): 46A45, 40A05, 40A35.
Keywords and phrases: Lacunary statistical convergence, partial metric space, lacunary statistical boundedness, lacunary density.
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and ℓ is referred as statistical limit of (z_m). We write $z_m \xrightarrow{S} \ell$ and by $S(c)$ we denote the set of all statistically convergent real sequences.

With the passage of time, various generalization of this notion, have been studied by many more mathematicians. One may refer to [8, 9, 13, 15, 16, 19, 21, 22, 26, 27, 30, 31, 33].

Before proceeding for lacunary statistical convergence, we recall lacunary sequence and lacunary density.

Following Freedman et al. [12], a lacunary sequence $\theta = (m_r)_{r=0}^{\infty}$ is an increasing sequence such that $m_r - m_{r-1} \to \infty$, where $m_0 = 0$, $m_r \geqslant 0$. Here we notate $J_r = (m_{r-1}, m_r]$, $l_r = m_r - m_{r-1}$ and $t_r = \frac{m_r}{m_{r-1}}$.

There is a strong relation between the space $|\sigma_1|$ of strongly Cesàro summable sequences where

$$|\sigma_1| = \left\{ (z_m) : \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} |z_m - \ell| = 0 \text{ for some } \ell \right\}$$

and the space N_θ, where

$$N_\theta = \left\{ (z_m) : \lim_{r\to\infty} \frac{1}{l_r} \sum_{m\in J_r} |z_m - \ell| = 0 \text{ for some } \ell \right\}.$$

Fridy and Orhan [14] in 1993 studied a new variant of statistical convergence, called lacunary statistical convergence which is defined as

DEFINITION 3. A real valued sequence (z_m) is said to be lacunary statistical convergent to $\ell \in \mathbb{R}$ or we can say $z_m \to \ell(S_\theta)$ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{l_r} \text{card}(\{m \in J_r : |z_m - \ell| \geqslant \varepsilon\}) = 0.$$

We notate the class of all lacunary statistical convergent sequences of reals by $S_\theta(c)$.

Aral and Şengül [1], Aral et al. [2, 3], Bhardwaj et al. [6], Fridy and Orhan [15], Li [19], Mohiuddine and Aiyub [22], Şengül and Et [30], Tripathy et al. [34] and some others have structured some significant sequence spaces with the use of lacunary sequences and hence enriched the theory of lacunary statistical convergence.

In 1994 Matthews [20], introducing the idea of partial metric space which is defined as

DEFINITION 4. Let $X \neq \emptyset$. A function $\varphi : X \times X \to \mathbb{R}$ satisfying the following

(φ_1) $0 \leqslant \varphi(u,u) \leqslant \varphi(u,v)$

(φ_2) $\varphi(u,u) = \varphi(u,v) = \varphi(v,v) \iff u = v$

(φ_3) $\varphi(u,v) = \varphi(v,u)$

(φ_4) $\varphi(u,v) \leqslant \varphi(u,w) + \varphi(w,v) - \varphi(w,w)$ for all $u,v,w \in X$, is said to be a partial metric on X and (X,φ) is called a partial metric space.

It can be observed in view of axiom (φ_1) of partial metric space, $|\varphi(u_m,u) - \varphi(u,u)|$ and $\varphi(u_m,u) - \varphi(u,u)$ are the same thing, for any sequence (u_m) in X and $u \in X$.

In comparison to a metric on X, we can say a partial metric φ is precisely a metric $\varphi : X \times X \to \mathbb{R}$ such that for all $u \in X$, $\varphi(u,u) = 0$. That is, in the definition of partial metric space, only one side axiom of metric is preserved, i.e., $\forall\, u,v \in X, \varphi(u,v) = 0 \Rightarrow u = v$ and other half that is, $u = v \Rightarrow \varphi(u,v) = 0$ need not hold good. For a detailed description of partial metric space, one may refer [5, 23, 25, 28].

Nuray [25], Bayram et al. [5] and Kumar et al. [18] stepped into partial metric space via statistical convergence and introduced notion of statistical convergence in partial metric space. We call this notion as statistical φ-convergence.

DEFINITION 5. A sequence (z_m) in a partial metric space (X,φ) is said to be statistically φ-convergent to some $z_0 \in X$ if for given $\varepsilon > 0$,

$$\delta(\{m \in \mathbb{N} : \varphi(z_m,z_0) \geqslant \varphi(z_0,z_0) + \varepsilon\}) = 0$$

and we write it as $z_m \xrightarrow{\varphi} z_0(S)$. By $S(c^{\varphi})$, we notate the class of all statistically φ-convergent sequence from (X,φ).

DEFINITION 6. Let (z_m) be sequence in (X,φ) and $z_0 \in X$. If for given $\varepsilon > 0$, there exists a positive integer m_0 such that following holds

$$\varphi(z_m,z_0) \leqslant \varphi(z_0,z_0) + \varepsilon \text{ for all } m \geqslant m_0$$

then we say (z_m) is φ-convergent to z_0. We write c^{φ} for the class of all φ-convergent sequences.

DEFINITION 7. A sequence (z_m) in a partial metric space (X,φ) is said to be φ-bounded if there exists some $z_0 \in X$ and $M > 0$ such that $\varphi(z_m,z_0) < \varphi(z_0,z_0) + M$ for all $m \geqslant 1$. We write b^{φ} as the class of all φ-bounded sequences.

DEFINITION 8. Let (X,φ) be a *p.m.s.* and (z_m) be a sequence in X. We say (z_m) is statistically φ-bounded if there exist some $z_0 \in X$ and $M > 0$ such that

$$\delta(\{m \in \mathbb{N} : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant M\}) = 0.$$

We write $S(b^{\varphi})$ as the class of all statistically φ-bounded sequences.

We recall [4, 10, 17], a scalar sequence space E is

(i) Solid (normal) if $(\eta_m) \in E$ whenever $|\eta_m| \leqslant |\xi_m|$, $m \geqslant 1$, for $(\xi_m) \in E$.

(ii) Symmetric if $(\eta_m) \in E$ implies $(\eta_{\sigma_m}) \in E$, where (σ_m) is permutation on m.

Motivating from above definition, for an arbitrary sequence space E in *p.m.s.* (X,φ) we say E is solid (normal) if $(\eta_m) \in E$ whenever $\varphi(\eta_m,a) \leqslant \varphi(\xi_m,a)$, $m \geqslant 1$ for $(\xi_m) \in E$ for all $a \in X$.

In this paper, we study the sequence space by using sequences from an arbitrary non-empty set X, equipped with a partial metric. Throughout the paper, (X,φ) will denote the partial metric space abbreviated as *p.m.s.*

2. Lacunary statistical φ-convergence and lacunary statistical φ-boundedness of order α in *p.m.s.*

In this section we study the lacunary statistical φ-convergence of order α for sequences from an arbitrary partial metric space (X,φ) and its relation with lacunary strongly φ-Cesàro summability of order α.

DEFINITION 9. A sequence (z_m) in *p.m.s.* (X,φ) is said to be statistically φ-convergent of order α $(0<\alpha\leqslant 1)$ to $z_0 \in X$ if for $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{n^\alpha} \text{card}(\{m \leqslant n : \varphi(z_m,z_0) \geqslant \varphi(z_0,z_0)+\varepsilon\}) = 0$$

and we write $z_m \xrightarrow{\varphi} z_0(S^\alpha)$. We shall denote the set of all statistically φ-convergent sequences of order α by $S^\alpha(c^\varphi)$.

For $\alpha = 1$, we call a statistically φ-convergent sequence of order α simply as a statistically φ-convergent sequence and corresponding space is denoted by $S(c^\varphi)$.

DEFINITION 10. Let $\theta = (m_r)$ be a lacunary sequence and $0<\alpha\leqslant 1$. A sequence (z_m) in *p.m.s.* (X,φ) is said to be lacunary statistically φ-convergent of order α to $z_0 \in X$ if for $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{l_r^\alpha} \text{card}(\{m \in J_r : \varphi(z_m,z_0) \geqslant \varphi(z_0,z_0)+\varepsilon\}) = 0$$

and we write $z_m \xrightarrow{\varphi} z_0(S_\theta^\alpha)$. By $S_\theta^\alpha(c^\varphi)$ we shall denote the class of all lacunary statistically φ-convergent sequences of order α.

DEFINITION 11. For a given lacunary sequence $\theta = (m_r)$, the α-lacunary density (or θ_α-density) of $K \subseteq \mathbb{N}$ is defined as $\delta_\theta^\alpha(K) = \lim_{r\to\infty} \frac{1}{h_r^\alpha} \text{card}(\{m \in J_r : m \in K\})$.

DEFINITION 12. For a given lacunary sequence $\theta = (m_r)$, if the set of indicies $m's$ for which (z_m) does not satisfy property P has zero α-lacunary density, then we say (z_m) satisfies P for "almost all m with respect to θ_α" abbreviated as "*a.a. m w.r.t.* θ_α."

Lacunary statistical φ-convergence of order α in *p.m.s.* (X,φ), now may be redefined as

DEFINITION 13. Let $\theta = (m_r)$ be a lacunary sequence and $0 < \alpha \leqslant 1$. A sequence (z_m) in *p.m.s.* (X,φ) is said to be lacunary statistically φ-convergent of order α to $z_0 \in X$ if for $\varepsilon > 0$,

$$\delta_\theta^\alpha(\{m \in J_r : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant \varepsilon\}) = 0,$$

$$\text{i.e.,} \quad |\varphi(z_m,z_0) - \varphi(z_0,z_0)| < \varepsilon \quad a.a.\ m\ w.r.t. \quad \theta_\alpha.$$

THEOREM 1. *Let $\alpha,\beta \in (0,1]$ such that $\alpha \leqslant \beta$. Then $S_\theta^\alpha(c^\varphi) \subset S_\theta^\beta(c^\varphi)$, converse may not be true in general.*

Proof. For given $\varepsilon > 0$, we have

$$\begin{aligned} 0 &\leqslant \frac{1}{l_r^\beta}\operatorname{card}(\{m \in J_r : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant \varepsilon\}) \\ &\leqslant \frac{1}{l_r^\alpha}\operatorname{card}(\{m \in J_r : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant \varepsilon\}). \end{aligned}$$

Taking limit $r \to \infty$, we get required result.

For reverse inclusion, consider the following example:

Let $X = \mathbb{R}$ with partial metric φ defined as $\varphi(\xi,\eta) = |\xi - \eta|$; $\xi,\eta \in \mathbb{R}$. Construct a sequence (z_m) such that

$$z_m = \begin{cases} \left[\sqrt{l_r}\right] & \text{at the first } \left[\sqrt{l_r}\right] \text{ integers on } J_r \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } r = 1,2,3,\dots .$$

This implies, $\operatorname{card}(\{m \in J_r : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant \varepsilon\}) \leqslant \left[\sqrt{l_r}\right]$. Thus if we consider $\frac{1}{2} < \beta \leqslant 1$, then we have

$$\lim_{r\to\infty} \frac{1}{l_r^\beta}\operatorname{card}(\{m \in J_r : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant \varepsilon\}) \leqslant \lim_{r\to\infty} \frac{\left[\sqrt{l_r}\right]}{l_r^\beta} \longrightarrow 0.$$

On the other hand for $0 < \alpha < \frac{1}{2}$, we have

$$\lim_{r\to\infty} \frac{1}{l_r^\alpha}\operatorname{card}(\{m \in J_r : |\varphi(z_m,0) - \varphi(0,0)| \geqslant \varepsilon\}) \leqslant \lim_{r\to\infty} \frac{\left[\sqrt{l_r}\right]}{l_r^\alpha} \nrightarrow 0.$$

Hence inclusion is strict for $\alpha < \beta$ with $0 < \alpha < \frac{1}{2}$ and $\frac{1}{2} < \beta \leqslant 1$. $\square$

COROLLARY 1. *Let $\theta = (m_r)$ be lacunary sequence and $0 < \alpha \leqslant \beta \leqslant 1$. Then we have following*

(i) $S_\theta^\alpha(c^\varphi) = S_\theta^\beta(c^\varphi)$ *iff* $\alpha = \beta$.

(ii) $S_\theta^\alpha(c^\varphi) = S_\theta(c^\varphi)$ *iff* $\alpha = 1$.

DEFINITION 14. Let $\theta = (m_r)$ be a lacunary sequence and $0 < \alpha \leqslant 1$. A sequence (z_m) in *p.m.s.* (X,φ) is said to be lacunary statistically φ-bounded of order α if there exist some $z_0 \in X$ and $M > 0$ such that

$$\lim_{r\to\infty} \frac{1}{l_r^\alpha} \mathrm{card}(\{m \in J_r : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant M\}) = 0,$$

$$\text{i.e.,} \quad |\varphi(z_m,z_0) - \varphi(z_0,z_0)| < M \quad \textit{a.a. } m \textit{ w.r.t. } \theta_\alpha$$

and we write $S_\theta^\alpha(b^\varphi)$ for the class of all lacunary statistically bounded sequences of order α.

THEOREM 2. *(i)* $S_\theta^\alpha(b^\varphi)$ *is not symmetric.*

(ii) $S_\theta^\alpha(b^\varphi)$ *is normal.*

Proof.

(i) By taking $\theta = (2^r)$, $X = \mathbb{R}$, $\varphi(\xi,\eta) = |\xi - \eta|$ and for $\alpha = 1$.

Let $(z_m) = (1,0,0,2,0,0,0,0,3,0,0,0,0,0,0,4,\ldots)$. Then $(z_m) \in S_\theta^\alpha(b^\varphi)$. Consider (z'_m) be a sequence which is obtained by rearrangement of (z_m) as follows

$$\begin{aligned}(z'_m) &= (z_1,z_2,z_4,z_3,z_9,z_5,z_{16},z_6,z_{25},z_7,z_{36},z_8,z_{49},z_{10},\ldots) \\ &= (1,0,2,0,3,0,4,0,4,0,5,0,6,0,7,0,\ldots).\end{aligned}$$

Then for any $M > 0$, we have

$$\lim_{n\to\infty} \frac{1}{l_r^\alpha} \mathrm{card}(\{m \in J_r : |\varphi(z'_m,z_0) - \varphi(z_0,z_0)| \geqslant M\}) \neq 0$$

so $(z'_m) \notin S_\theta^\alpha(b^\varphi)$ and hence $S_\theta^\alpha(b^\varphi)$ is not symmetric.

(ii) Let $(z_m) \in S_\theta^\alpha(b^\varphi)$ and (z'_m) be a sequence such that $\varphi(z'_m,a) \leqslant \varphi(z_m,a)$ for all $m \in \mathbb{N}$ and for all $a \in X$. As $(z_m) \in S_\theta^\alpha(b^\varphi)$ so there exists some $z_0 \in X$ such that

$$\lim_{n\to\infty} \frac{1}{l_r^\alpha} \mathrm{card}(\{m \in J_r : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant M\}) = 0.$$

Clearly

$$\mathrm{card}(\{m : |\varphi(z'_m,z_0) - \varphi(z_0,z_0)| \geqslant M\}) \leqslant \mathrm{card}(\{m : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant M\})$$

so $(z'_m) \in S_\theta^\alpha(b^\varphi)$. Hence $S_\theta^\alpha(b^\varphi)$ is normal. □

THEOREM 3. $c^\varphi \subset S_\theta^\alpha(c^\varphi)$, *inclusion is proper.*

Proof. Let (z_m) be a sequence in (X,φ) which is φ-convergent to $z_0 \in X$. Then for given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ we have $\varphi(z_m,z_0) < \varphi(z_0,z_0)+\varepsilon$ for all $m \geqslant m_0$. As α-lacunary density of a finite set is zero, so the result holds.

For proper inclusion, consider the following example:

Let $X = \mathbb{R}$ and φ be the partial metric defined by $\varphi(\xi,\eta) = |\xi-\eta|$; $\xi,\eta \in \mathbb{R}$ and $\alpha = 1$. Take sequence

$$z_i = \begin{cases} m_{r-1}+1 & \text{for } i = m_{r-1}+1, \\ 0 & \text{otherwise.} \end{cases} \qquad i \in J_r,\ r = 1,2,3\ldots.$$

Let, if possible, (z_i) is φ-convergent to some $z_0 \in X$. Then for given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ we have $\varphi(z_i,z_0) < \varphi(z_0,z_0)+\varepsilon$ for all $i \geqslant m_0$, i.e., $|m_{r-1}+1-z_0| < \varepsilon$ for all $m_{r-1}+1 \geqslant m_0$ for all $r = 1,2,3,\ldots$ a contradiction as $m_r \to \infty$. However, (z_i) is lacunary statistically φ-convergent of order α to $0 \in X$, because for $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{l_r^\alpha} \text{card}(\{i \in J_r : \varphi(z_i,0) \geqslant \varphi(0,0)+\varepsilon\}) = \lim_{r\to\infty} \frac{1}{l_r^\alpha} = 0. \quad \square$$

THEOREM 4. *In p.m.s. (X,φ), lacunary statistically φ-convergence of order α implies lacunary statistically φ-boundedness of order α. Converse may not be true in general.*

Proof. Let (z_m) be a lacunary statistically φ-convergent of order α to some $z_0 \in X$. Then for given $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{l_r^\alpha} \text{card}(\{m \in J_r : \varphi(z_m,z_0) > \varphi(z_0,z_0)+\varepsilon\}) = 0.$$

Now for sufficiently large M, we may assert that,

$$\text{card}(\{m \in J_r : \varphi(z_m,z_0) > \varphi(z_0,z_0)+M\}) \leqslant \text{card}(\{m \in J_r : \varphi(z_m,z_0) > \varphi(z_0,z_0)+\varepsilon\})$$

and hence the result follows.

For converse part, let $X = \mathbb{R}$ and φ be the partial metric defined by $\varphi(\xi,\eta) = |\xi-\eta|$; $\xi,\eta \in \mathbb{R}$ and $\alpha = 1$. Consider a sequence (z_m) in X as

$$z_m = \begin{cases} -1 & \text{at first } \left[\frac{l_r}{2}\right] \text{ integers on } J_r \\ 1 & \text{otherwise.} \end{cases}$$

Now $\frac{1}{l_r}\text{card}(\{m \in J_r : \varphi(z_m,1) > \varphi(1,1)+\varepsilon\}) = \left[\frac{l_r}{2l_r}\right]$ or $\left[\frac{l_r+1}{2l_r}\right]$ and hence (z_m) is not lacunary statistically φ-convergent to 1. Similar can be proved for -1. As every bounded sequence is lacunary statistically φ-bounded of order α, hence the result follows. $\square$

REMARK 1. Subsequence of a lacunary statistically φ-convergent sequence of order α need not be lacunary statistically φ-convergent of order α. For this consider the following example:

Let $X = \mathbb{R}$ be the partial metric space equipped with partial metric φ defined as $\varphi(\xi,\eta) = |\xi - \eta|$; $\xi,\eta \in \mathbb{R}$.

$$z_i = \begin{cases} m_{r-1}+1 & \text{for } i = m_{r-1}+1, \\ 0 & \text{otherwise} \end{cases} \quad \text{on } J_r,\ r = 1,2,3\ldots.$$

Then (z_i) is lacunary statistically φ-convergent of order α to 0.

Now $(m_0+1, m_1+1, m_2+1,\ldots)$ is a subsequence of (z_i) which is not lacunary statistically φ-convergent of order α.

Before characterizing the lacunary statistically φ-convergent of order α we have the following definition.

DEFINITION 15. A sequence (z_{m_n}), $n \in \mathbb{N}$ is said to be α-lacunary statistical dense if $\delta_\theta^\alpha(B) = 1$, where $B = \{m_1 < m_2 < m_3 < \ldots\}$,

$$\text{i.e., } \lim_{r\to\infty} \frac{1}{l_r^\alpha} \text{card}(\{m \in J_r : m \in B\} = 1.$$

The following theorem characterizes the lacunary statistically φ-convergent sequence of order α in term of α-lacunary statistically dense φ-convergent subsequences.

THEOREM 5. *A sequence (z_m) is lacunary statistically φ-convergent of order α iff every α-lacunary statistically dense subsequence of (z_m) is lacunary statistically φ-convergent of order α.*

Proof. Let (z_m) is lacunary statistically φ-convergent to z_0 of order α. So for given $\varepsilon > 0$, we have

$$\lim_{r\to\infty} \frac{1}{l_r^\alpha} \text{card}(\{m \in J_r : \varphi(z_m, z_0) > \varphi(z_0, z_0) + \varepsilon\}) = 0, \text{ i.e., } \delta_\theta^\alpha(A) = 0$$

where $A = \{m \in \mathbb{N} : \varphi(z_m, z_0) > \varphi(z_0, z_0) + \varepsilon\}$ and $(z_{m_n})_{n\in\mathbb{N}}$ is lacunary statistically dense subsequence of (z_m) which is not lacunary φ-statistically convergent of order α, i.e.,

$$\liminf_{r\to\infty} \frac{1}{l_r^\alpha} \text{card}(\{m_n \in J_r : \varphi(z_{m_n}, z_0) > \varphi(z_0, z_0) + \varepsilon\}) = d, \text{ where } d \in (0,1).$$

Now

$$\text{card}(\{m \in J_r : \varphi(z_m, z_0) > \varphi(z_0, z_0) + \varepsilon\}) \geqslant \text{card}(\{m_n \in J_r : \varphi(z_{m_n}, z_0) > \varphi(z_0, z_0) + \varepsilon\}).$$

So $\liminf_{r\to\infty} \frac{1}{l_r^\alpha} \text{card}(\{m \in J_r : \varphi(z_m, z_0) > \varphi(z_0, z_0) + \varepsilon\}) \geqslant d \neq 0$, a contradiction to given.

Converse part follows from the fact that every sequence is a α-lacunary statistical dense subsequence of itself. □

THEOREM 6. *A sequence (z_m) in (X,φ) is lacunary statistically φ-convergent of order α to some $z_0 \in X$ iff there exists a sequence (y_m), φ-convergent to z_0 such that $y_m = z_m$ a.a. m w.r.t. θ_α.*

Proof. Let (z_m) be a lacunary statistically φ-convergent to some $z_0 \in X$. So for each $\varepsilon > 0$, we get $\delta_\theta^\alpha(K) = 0$ where $K = \{m \in \mathbb{N} : \varphi(z_m, z_0) > \varphi(z_0, z_0) + \varepsilon\}$. Let

$$y_m = \begin{cases} z_m & \text{for } m \in \mathbb{N} - K \\ z_0 & \text{for } m \in K. \end{cases}$$

Now $\{m \in \mathbb{N} : y_m \neq z_m\} \subseteq K$ and so $y_m = z_m$ *a.a. m w.r.t.* θ_α. Also

$$\varphi(y_m, z_0) = \begin{cases} \varphi(z_m, z_0) & \text{for } m \in \mathbb{N} - K \\ \varphi(z_0, z_0) & \text{for } m \in K \end{cases}$$

implies $\varphi(y_m, z_0) < \varphi(z_0, z_0) + \varepsilon$ for all $m \geqslant 1$. Thus (y_m) is φ-convergent sequence and $y_m = z_m$ *a.a. m w.r.t.* θ_α.

Conversely, let there exists $m_0 \in \mathbb{N}$ such that $\varphi(y_m, z_0) < \varphi(z_0, z_0) + \varepsilon$ for all $m \geqslant m_0$. The result follows from the inclusion relation

$$\{m : \varphi(z_m, z_0) \geqslant \varphi(z_0, z_0) + \varepsilon\} \subseteq K \cup \{1, 2, 3, \ldots, m_0 - 1\}. \quad \square$$

Following on the similar lines, we have

THEOREM 7. *A sequence (z_m) is lacunary statistically φ-bounded of order α iff there exists a φ-bounded sequence (y_m) such that $y_m = z_m$ a.a. m w.r.t θ_α.*

REMARK 2. Example in Remark 1 asserts that a subsequence of a lacunary statistically φ-bounded sequence of order α need not be lacunary statistically φ-bounded of order α.

Using the same technique as in Theorem 5, we give a characterization of the lacunary statistically φ-boundedness of order α in terms of the α-lacunary statistically dense φ-bounded subsequences of it, in terms of following.

THEOREM 8. *A sequence (z_m) is lacunary statistically φ-bounded of order α iff every α-lacunary statistically dense subsequence of (z_m) is lacunary statistically φ-bounded of order α.*

3. Lacunary strongly φ-Cesàro summable spaces

DEFINITION 16. Let (z_m) be a sequence in *p.m.s.* (X,φ) and $0 < \alpha \leqslant 1$. The sequence (z_m) is strongly φ-Cesàro summable of order α to $z_0 \in X$ if

$$\lim_{n \to \infty} \frac{1}{n^\alpha} \sum_{m=1}^{n} |\varphi(z_m, z_0) - \varphi(z_0, z_0)| = 0.$$

We notate $|\sigma_1|^\alpha(c^\varphi)$ for the set of all strongly φ-Cesàro summable sequences of order α. We write $|\sigma_1|^\alpha(c^\varphi)$ for $\alpha = 1$ as $|\sigma_1|(c^\varphi)$.

DEFINITION 17. Let (z_m) be a sequence in *p.m.s.* (X,φ) and $0<\alpha\leqslant 1$. The sequence (z_m) is lacunary strongly φ-Cesàro summable of order α to $z_0\in X$ if

$$\lim_{r\to\infty}\frac{1}{l_r^{\alpha}}\sum_{m\in J_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|=0.$$

In this case, we write $z_m \xrightarrow{\varphi} z_0(N_\theta^\alpha)$. We denote $N_\theta^\alpha(c^\varphi)$ for the set of all lacunary strongly φ-Cesàro summable sequences of order α. We write $N_\theta^\alpha(c^\varphi)$ for $\alpha=1$ as $N_\theta(c^\varphi)$.

THEOREM 9. *For $\alpha\in(0,1]$, $N_\theta^\alpha(c^\varphi)\subset S_\theta^\alpha(c^\varphi)$, i.e., every lacunary strongly φ-Cesàro summable sequence of order α is lacunary statistically φ-convergent of order α to same limit and inclusion is proper.*

Proof. For $\varepsilon>0$ and $0<\alpha\leqslant 1$, we have

$$\begin{aligned}&\frac{1}{l_r^{\alpha}}\sum_{m\in J_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\\ &\geqslant\frac{1}{l_r^{\alpha}}\sum_{\substack{m\in J_r\\ |\varphi(z_m,z_0)-\varphi(z_0,z_0)|>\varepsilon}}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\\ &\geqslant \varepsilon.\frac{1}{l_r^{\alpha}}\operatorname{card}(\{m\in J_r:|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\geqslant\varepsilon\}\end{aligned}$$

and hence the result follows by above inequality.

For proper inclusion, consider the following example:

Let $X=\mathbb{R}$ and φ be the partial metric defined by $\varphi(\xi,\eta)=|\xi-\eta|$; $\xi,\eta\in\mathbb{R}$. Construct a sequence (z_m) such that

$$z_m=\begin{cases}1,2,\dots,\left[l_r^{\frac{\alpha}{2}}\right] & \text{at the first } \left[l_r^{\frac{\alpha}{2}}\right] \text{ integers on} J_r\\ 0 & \text{otherwise,}\end{cases} \quad \text{for all } r=1,2,3,\dots$$

where $[\cdot]$ denote the greatest integer function. Then for every $\varepsilon>0$,

$$\frac{1}{l_r^{\alpha}}\operatorname{card}(\{m\in J_r:|\varphi(z_m,0)-\varphi(0,0)|\geqslant\varepsilon\})\leqslant\frac{1}{l_r^{\alpha}}\left[l_r^{\frac{\alpha}{2}}\right]\to 0 \text{ as } r\to\infty,$$

and so $(z_m)\in S_\theta^\alpha(c^\varphi)$.

On the other hand,

$$\begin{aligned}\frac{1}{l_r^{\alpha}}\sum_{m\in J_r}|\varphi(z_m,0)-\varphi(0,0)|&=\frac{1}{l_r^{\alpha}}\sum_{m\in J_r}|z_m-0|\\ &=\frac{1}{l_r^{\alpha}}\left[1+2+\dots+\left[l_r^{\frac{\alpha}{2}}\right]\right]\\ &=\frac{1}{l_r^{\alpha}}\left(\frac{\left[l_r^{\frac{\alpha}{2}}\right]\left(\left[l_r^{\frac{\alpha}{2}}\right]+1\right)}{2}\right)\to\frac{1}{2}\neq 0 \text{ as } r\to\infty,\end{aligned}$$

and this implies $(z_m) \notin N_\theta^\alpha(c^\varphi)$. $\square$

THEOREM 10. *For $0 < \alpha \leqslant 1$, the following holds*

(i) If $\liminf t_r > 1$, *then* $S^\alpha(c^\varphi) \subseteq S_\theta^\alpha(c^\varphi)$.

(ii) If $\limsup \dfrac{m_r}{m_{r-1}^\alpha} < \infty$, *then* $S_\theta^\alpha(c^\varphi) \subseteq S^\alpha(c^\varphi)$.

(iii) If $\liminf \dfrac{l_r^\alpha}{m_r} > 0$, *then we have* $S(c^\varphi) \subseteq S_\theta^\alpha(c^\varphi)$.

Proof.

(i) Let $\liminf t_r > 1$. Then for sufficiently large r, $\exists\, \delta > 0$ such that $t_r > 1+\delta$. As $t_r = \dfrac{m_r}{m_{r-1}}$, so $\dfrac{l_r}{m_{r-1}} \geqslant \delta$. This gives $\dfrac{m_{r-1}}{l_r} \leqslant \dfrac{1}{\delta}$. After adding 1 to both sides, we have $\dfrac{m_r}{l_r} \leqslant \dfrac{1+\delta}{\delta}$, i.e., $\dfrac{1}{m_r^\alpha} \geqslant \dfrac{\delta^\alpha}{(1+\delta)^\alpha}\dfrac{1}{l_r^\alpha}$.

For $\varepsilon > 0$ and sufficiently large r, we have

$$
\begin{aligned}
&\frac{1}{m_r^\alpha}\operatorname{card}(\{m \leqslant m_r : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant \varepsilon\}) \\
&\qquad \geqslant \frac{1}{m_r^\alpha}\operatorname{card}(\{m \in J_r : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant \varepsilon\}) \\
&\qquad \geqslant \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{l_r^\alpha}\operatorname{card}(\{m \in J_r : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant \varepsilon\})
\end{aligned}
$$

and hence the result.

(ii) Let $\limsup_r \dfrac{m_r}{m_{r-1}^\alpha} < \infty$. Then there exists $M > 0$ such that $\dfrac{m_r}{m_{r-1}^\alpha} < M$ for all $r \geqslant 1$. Let us suppose, that $(z_m) \in S_\theta^\alpha(c^\varphi)$. Then for $z_0 \in X$ and $\varepsilon > 0$

$$
\lim_{r\to\infty}\frac{1}{l_r^\alpha}\operatorname{card}(\{m \in J_r : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant \varepsilon\}) = 0 \ \text{ i.e., } \lim_{r\to\infty}\frac{M_r}{l_r^\alpha} = 0
$$

where $M_r = \operatorname{card}(\{m \in J_r : |\varphi(z_m,z_0) - \varphi(z_0,z_0)| \geqslant \varepsilon\})$. So for given $\varepsilon > 0$, $\exists\, r_0 \in \mathbb{N}$ such that $\dfrac{M_r}{l_r^\alpha} < \varepsilon\ \forall\, r > r_0$. Let $G = \sup\{M_r : 1 \leqslant r \leqslant r_0\}$ and n be any integer satisfying $m_{r-1} < n \leqslant m_r$. Then we have,

$$\begin{aligned}
&\frac{1}{n^\alpha}\operatorname{card}\left(\{m \leqslant n : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\}\right)\\
&\quad\leqslant \frac{1}{m_{r-1}^\alpha}\operatorname{card}\left(\{m \leqslant m_r : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\}\right)\\
&\quad= \frac{1}{m_{r-1}^\alpha}\left\{M_1+M_2+\ldots+M_{r_0}+M_{r_0+1}+\ldots+M_r\right\}\\
&\quad\leqslant \frac{r_0 G}{m_{r-1}^\alpha}+\frac{1}{m_{r-1}^\alpha}\{M_{r_0+1}+M_{r_0+2}+\ldots+M_r\}\\
&\quad= \frac{r_0 G}{m_{r-1}^\alpha}+\frac{1}{m_{r-1}^\alpha}\left\{l_{r_0+1}\frac{M_{r_0+1}}{l_{r_0+1}}+l_{r_0+2}\frac{M_{r_0+2}}{l_{r_0+2}}+\ldots+l_r\frac{M_r}{l_r}\right\}
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{1}{n^\alpha}\operatorname{card}\left(\{m \leqslant n : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\}\right)\\
&\quad\leqslant \frac{r_0 G}{m_{r-1}^\alpha}+\frac{1}{m_{r-1}^\alpha}\left(\sup_{r>r_0}\frac{M_r}{l_r}\right)\left\{l_{r_0+1}+l_{r_0+2}+\ldots+l_r\right\}\\
&\quad\leqslant \frac{r_0 G}{m_{r-1}^\alpha}+\frac{1}{m_{r-1}^\alpha}\varepsilon\left(m_r-m_{r_0}\right)\\
&\quad= \frac{r_0 G}{m_{r-1}^\alpha}+\varepsilon\left(\frac{m_r}{m_{r-1}^\alpha}-\frac{m_{r_0}}{m_{r-1}^\alpha}\right)\\
&\quad\leqslant \frac{r_0 G}{m_{r-1}^\alpha}+\varepsilon\left(\frac{m_r}{m_{r-1}^\alpha}\right)\\
&\quad\leqslant \frac{r_0 G}{m_{r-1}^\alpha}+\varepsilon.M
\end{aligned}$$

and hence by applying limit in above inequality we get the result.

(iii) Let $\varepsilon > 0$ and $z_0 \in X$. Then we have,

$$\{m \in J_r : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\} \subseteq \{m \leqslant m_r : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\}.$$

This implies

$$\begin{aligned}
&\frac{1}{m_r}\operatorname{card}\left(\{m \in J_r : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\}\right)\\
&\quad\leqslant \frac{1}{m_r}\operatorname{card}\left(\{m \leqslant m_r : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\}\right)
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{l_r^\alpha}{m_r}\frac{1}{l_r^\alpha}\operatorname{card}\left(\{m \in J_r : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\}\right)\\
&\quad\leqslant \frac{1}{m_r}\operatorname{card}\left(\{m \leqslant m_r : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\}\right).
\end{aligned}$$

Taking limit $r \to \infty$ in above inequality, we get result. $\square$

THEOREM 11. *For $0 < \alpha \leqslant 1$, following holds*

(i) If $\liminf t_r > 1$, *then* $|\sigma_1|^\alpha(c^\varphi) \subseteq N_\theta^\alpha(c^\varphi)$.

(ii) If $\limsup \dfrac{m_r}{m_{r-1}^\alpha} < \infty$, *then* $N_\theta^\alpha(c^\varphi) \subseteq |\sigma_1|^\alpha(c^\varphi)$.

(iii) If $\liminf \dfrac{l_r^\alpha}{m_r} > 0$, *then we have* $|\sigma_1|(c^\varphi) \subset N_\theta^\alpha(c^\varphi)$.

Proof.

(i) Let $z \in |\sigma_1|^\alpha(c^\varphi)$. Then for given $\varepsilon > 0$ and sufficiently large r, as in part (i) of Theorem 10, we have,

$$\frac{1}{m_r^\alpha} \geqslant \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{l_r^\alpha}.$$

Now consider

$$\begin{aligned}\frac{1}{m_r^\alpha}\left(\sum_{m\leqslant m_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right) &\geqslant \frac{1}{m_r^\alpha}\left(\sum_{m\in J_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right)\\ &\geqslant \frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{l_r^\alpha}\left(\sum_{m\in J_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right)\end{aligned}$$

and hence result follows by taking limit $r \to \infty$ in above inequality.

(ii) Let $z \in N_\theta^\alpha(c^\varphi)$ and n be any integer such that $m_{r-1} < n \leqslant m_r$. Then we have,

$$\frac{1}{n^\alpha}\left(\sum_{k=1}^{n}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right) < \frac{1}{m_{r-1}^\alpha}\left(\sum_{k=1}^{m_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right).$$

Taking $M_r = \sum_{m\in J_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|$ and proceeding same as that in part (ii) of Theorem 10, we get required inclusion.

(iii) Let $z \in |\sigma_1|(c^\varphi)$ and $z_0 \in X$. Then we have

$$\begin{aligned}\frac{1}{m_r}\left(\sum_{m\in J_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right) &\leqslant \frac{1}{m_r}\left(\sum_{m=1}^{m_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right)\\ \text{i.e., } \frac{l_r^\alpha}{m_r}\frac{1}{l_r^\alpha}\left(\sum_{m\in J_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right) &\leqslant \frac{1}{m_r}\left(\sum_{k=1}^{m_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right).\end{aligned}$$

Taking limit $r \to \infty$ and using $\liminf \dfrac{l_r^\alpha}{m_r} > 0$ in above inequality, we get required inclusion. $\square$

4. Results on lacunary refinement

The present section concludes the paper by showing various inclusion relations, which arises for varying lacunary sequences θ.

DEFINITION 18. By lacunary refinement $\theta^* = (m_r^*)$ of a lacunary sequence $\theta = (m_r)$ we mean $J_r^* \supseteq J_r$ where $J_r^* = (m_{r-1}^*, m_r^*]$ and $J_r = (m_{r-1}, m_r]$.

We use $l_r^* = m_r^* - m_{r-1}^*$ throughout this section.

THEOREM 12. *$(z_m) \notin N_\theta(c^\varphi)$ implies $(z_m) \notin N_{\theta^*}(c^\varphi)$.*

Proof. Let $(z_m) \notin N_\theta(c^\varphi)$. Then for any $z_0 \in X$,

$$\lim_{r\to\infty} \frac{1}{l_r} \sum_{m\in J_r} |\varphi(z_m, z_0) - \varphi(z_0, z_0)| \neq 0.$$

So there exists $\varepsilon > 0$ and a subsequence $\left(m_{r_j}\right)$ of (m_r) such that

$$\frac{1}{l_{r_j}} \sum_{m\in J_{r_j}} |\varphi(z_m, z_0) - \varphi(z_0, z_0)| \geqslant \varepsilon.$$

Writing $J_{r_j} = J_{t+1}^* \cup J_{t+2}^* \cup \ldots \cup J_{t+p}^*$, then we have

$$\frac{\sum_{J_{t+1}^*} |\varphi(z_i, z_0) - \varphi(z_0, z_0)| + \ldots + \sum_{J_{t+p}^*} |\varphi(z_i, z_0) - \varphi(z_0, z_0)|}{l_{t+1}^* + \ldots + l_{t+p}^*} \geqslant \varepsilon$$

implies for some j, we have $\frac{1}{l_{t+j}^*} \sum_{J_{t+j}^*} |\varphi(z_i, z_0) - \varphi(z_0, z_0)| \geqslant \varepsilon$ and hence $(z_m) \notin N_{\theta^*}(c^\varphi)$. $\square$

THEOREM 13. *Let $0 < \alpha \leqslant \beta \leqslant 1$ and $\liminf_{r\to\infty} \frac{l_r}{l_r^*} > 0$. Then*

$$S_{\theta^*}^{\alpha}(c^\varphi) \subseteq S_\theta^{\beta}(c^\varphi).$$

Proof. As $J_r^* \supseteq J_r$ for all $r \in \mathbb{N}$, so for $\varepsilon > 0$, we have

$$\{m \in J_r^* : |\varphi(z_m, z_0) - \varphi(z_0, z_0) \geqslant \varepsilon\} \supseteq \{m \in J_r : |\varphi(z_m, z_0) - \varphi(z_0, z_0)| \geqslant \varepsilon\}.$$

This implies

$$\begin{aligned}
&\frac{1}{l_r^{*\alpha}}\operatorname{card}\left(\{m \in J_r^* : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\}\right)\\
&\qquad \geqslant \frac{1}{l_r^{*\alpha}}\operatorname{card}\left(\{m \in J_r : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\}\right)\\
&\qquad \geqslant \frac{1}{l_r^{*\beta}}\operatorname{card}\left(\{m \in J_r : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\}\right)\\
&\qquad = \left(\frac{l_r}{l_r^*}\right)^{\beta}\frac{1}{l_r^{\beta}}\operatorname{card}\left(\{m \in J_r : |\varphi(z_m,z_0)-\varphi(z_0,z_0)| \geqslant \varepsilon\}\right).
\end{aligned}$$

Hence the proof. □

COROLLARY 2. *If* $0<\alpha\leqslant 1$ *and* $\liminf \frac{l_r}{l_r^*}>0$*, then*

(i) $S_{\theta^*}^{\alpha}(c^{\varphi}) \subseteq S_{\theta}(c^{\varphi})$.

(ii) $S_{\theta^*}(c^{\varphi}) \subseteq S_{\theta}(c^{\varphi})$.

THEOREM 14. *Let* $0<\alpha\leqslant\beta\leqslant 1$. *If* $\liminf\limits_{r\to\infty}\frac{l_r}{l_r^*}>0$*, then we have*

$$N_{\theta^*}^{\alpha}(c^{\varphi}) \subseteq N_{\theta}^{\beta}(c^{\varphi}).$$

Proof. Let $(z_m)\in N_{\theta^*}^{\alpha}(c^{\varphi})$. Then for given $\varepsilon>0$ and $z_0\in X$, we have

$$\begin{aligned}
&\frac{1}{l_r^{*\alpha}}\left(\sum_{m\in J_r^*}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right)\\
&\qquad = \frac{1}{l_r^{*\alpha}}\left(\sum_{k\in J_r^*-J_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)| + \sum_{m\in J_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right)\\
&\qquad \geqslant \frac{l_r^{\beta}}{l_r^{*\alpha}}\frac{1}{l_r^{\beta}}\left(\sum_{m\in J_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right)\\
&\qquad \geqslant \left(\frac{l_r}{l_r^*}\right)^{\beta}\frac{1}{l_r^{\beta}}\left(\sum_{m\in J_r}|\varphi(z_m,z_0)-\varphi(z_0,z_0)|\right)
\end{aligned}$$

and hence the result follows by taking limit $r\to\infty$ and using $\liminf\frac{l_r}{l_r^*}>0$ in above inequality. □

COROLLARY 3. *If* $\liminf\limits_{r\to\infty}\frac{l_r}{l_r^*}>0$ *and* $0<\alpha\leqslant 1$*, then following holds* *(i)* $N_{\theta^*}^{\alpha}(c^{\varphi}) \subseteq N_{\theta}(c^{\varphi})$.

(ii) $N_{\theta^*}(c^{\varphi}) \subseteq N_{\theta}(c^{\varphi})$.

Acknowledgement. The authors would like to thank the referees for their critical analysis of the manuscript, helpful comments and suggestions, which improved the presentation of this paper.

REFERENCES

[1] N. D. ARAL, H. Ş. KANDEMIR, *On f-lacunary statistical convergence of order β of double sequences for difference sequences of fractional order*, Facta Universitatis Ser. Math. Inform., **38**, 2 (2023), 329–343.

[2] N. D. ARAL, H. Ş. KANDEMIR AND M. ET, *Strongly lacunary convergence of order β of difference sequences of fractional order in neutrosophic normed spaces*, Filomat, **37**, 19 (2023), 6443–6451.

[3] N. D. ARAL, H. Ş. KANDEMIR AND M. ET, *Lacunary harmonic summability in neutrosophic normed spaces*, Filomat, **38**, 19 (2024), 6763–6771.

[4] F. BAŞAR, *Summability Theory and its Applications*, 2^{nd} ed., CRC Press/Taylor & Francis Group, Boca Raton London New York, (2022).

[5] E. BAYRAM, Ç. BEKTAŞ AND Y. ALTIN, *On statistical convergence of order α in partial metric spaces*, Georgian Math. J., **31**, 4 (2024), 557–565.

[6] V. K. BHARDWAJ, S. GUPTA, S. A. MOHIUDDINE AND A. KILIÇMAN, *On lacunary statistical boundedness*, J. Inequal. Appl., **1**, (2014), 1–11.

[7] R. C. BUCK, *Generalized asymptotic density*, Amer. J. Math., **75**, 2 (1953), 335–346.

[8] J. CONNOR, *The statistical and strong p-Cesàro convergence of sequences*, Analysis, **8**, (1–2) (1988), 47–64.

[9] J. CONNOR, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull., **32** (2) (1989), 194–198.

[10] R. G. COOKE, *Infinite matrices and sequence spaces*, Macmillan, London, (1950).

[11] H. FAST, *Sur la convergence statistique*, Colloq. Math., **2**, (3–4) (1951), 241–244.

[12] A. R. FREEDMAN, J. J. SEMBER AND M. RAPHAEL, *Some Cesàro-type summability spaces*, Proc. Londan math. Soc., **37**, 3 (1978), 508–520.

[13] J. A. FRIDY, *On statistical convergence*, Analysis, **5**, 4 (1985), 301–314.

[14] J. A. FRIDY AND C. ORHAN, *Lacunary statistical summability*, J. Math. anal. Appl., **173**, 2 (1993), 497–504.

[15] J. A. FRIDY AND C. ORHAN, *Lacunary statistical convergence*, Pacific J. Math., **160**, 1 (1993), 43–51.

[16] S. GUPTA AND V. K. BHARDWAJ, *On deferred f-statistical convergence*, Kyungpook Math. J., **58**, (2018), 91–103.

[17] P. K. KAMTHAN AND M. GUPTA, *Sequence Spaces and Series*, Marcel Deker. Inc., New York and Basel, (1981).

[18] M. KUMAR, RITU AND S. GUPTA, *On statistical boundedness in partial metric spaces*, South East Asian J. Math. Math. Sci., **20**, 2 (2024), 235–246.

[19] J. LI, *Lacunary statistical convergence and inclusion properties between lacunary methods*, Int. J. Math. Math. Sci., **23**, 3 (2000), 175–180.

[20] S. G. MATTHEWS, *Partial metric topology*, Annals of the New York Academy of Sciences, **728**, 1 (1994), 183–197.

[21] S. A. MOHIUDDINE AND M. A. ALGHAMDI, *Statistical summability through a lacunary sequence in locally solid Riesz spaces*, J. Inequal Appl., **1**, (2012), 1–9.

[22] S. A. MOHIUDDINE AND M. AIYUB, *Lacunary statistical convergence in random 2-normed spaces*, Appl. Math. Inf. Sci, **6**, 3 (2012), 581–585.

[23] S. J. O. NEIL, *Two topologies are better than one*, University of Warwick. Department of Computer Science. (Department of Computer Science Research Report), CS-RR-283 (1995).

[24] I. NIVEN AND H. S. ZUCKERMAN, *An Introduction to Theory of Numbers*, Fourth. Ed., New York John Willey and Sons (1980).
[25] F. NURAY, *Statistical convergence in partial metric spaces*, Korean J. Math., **30**, 1 (2022), 155–160.
[26] D. RATH AND B. C. TRIPATHY, *On statistically convergent and statistically Cauchy sequences*, Indian J. Pure Appl. Math., **25**, (1994), 381–381.
[27] T. ŠALÁT, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30**, 2 (1980), 139–150.
[28] B. SAMET, C. VETRO AND F. VETRO, *From metric to partial metric spaces*, Fixed Point Theory Appl., **1**, (2013), 118–134.
[29] I. J. SCHOENBERG, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66**, 5 (1959), 361–375.
[30] H. ŞENGÜL AND M. ET, *On lacunary statistical convergence of order* α, Acta Math. Sci., **34**, 2 (2014), 473–482.
[31] N. SHARMA AND S. KUMAR, *Statistical convergence and Cesàro summability of difference sequences relative to modulus function*, J. Class. Anal., **23**, 1 (2024), 51–62.
[32] H. STEINHAUS, *Sur la convergence ordinaireet la convergence asymptotique*, Colloq. Math, **2**, 1 (1951), 73–74.
[33] B. C. TRIPATHY, *On statistically convergent and statistically bounded sequences*, Bull. Malays. Math. Soc., **20**, 1 (1997), 31–33.
[34] B. C. TRIPATHY, B. HAZARIKA AND B. CHOUDHARY, *Lacunary I-convergent sequences*, Kyungpook Math. Journal, **52**, 4 (2012), 473–482.
[35] A. ZYGMUND, *Trigonometric Series*, Cambridge Univ. Press, UK 1979.

(Received October 27, 2024)

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