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ON CERTAIN NEW MODULAR RELATIONS FOR RAMANUJAN'S CONTINUED FRACTIONS OF SIXTEEN ORDER

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Dedicated to Prof. Chandrashekar Adiga, whose pioneering research in number theory has always been a constant source of inspiration.

Abstract. In this paper, we derive several beautiful modular relations for the Ramanujan's continued fractions of sixteen order by using Ramanujan's theta functions.

1. Introduction

Throughout the paper, we use the customary notation

$$(a;q)_0 := 1,$$
$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad n \geqslant 1$$

and

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \qquad |q|<1.$$

Ramanujan's general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1. \tag{1}$$

The well-known Jacobi triple product identity [1, Entry 19] in Ramanujan's notation is

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2}$$

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The three most interesting special cases of (1) are [1, Entry 22]:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty}, \tag{3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{4}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{5}$$

For convenience, we define

$$f_n := f(-q^n) = (q^n;q^n)_\infty,$$

for a positive integer n.

LEMMA 1. *We have*

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \psi(q) = \frac{f_2^2}{f_1}, \qquad f(q) = \frac{f_2^3}{f_1 f_4}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4},$$

$$\varphi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(-q) = \frac{f_1 f_4}{f_2} \qquad \textit{and} \qquad \chi(-q) = \frac{f_1}{f_2}.$$

This lemma is a consequence of (2) and Entry 24 of [1, p. 34].
The following lemma can be found in [1, Entry 30(ii) and (iii)]:

LEMMA 2. *We have*

$$f(a,b) + f(-a,-b) = 2f(a^3b, ab^3), \tag{6}$$

$$f(a,b) - f(-a,-b) = 2af(b/a, a^5b^3). \tag{7}$$

The following identity is an easy consequence of Entry 31 [1] when $n = 2$:

$$f(a,b) = f(a^3b, ab^3) + a\, f(b/a, a^5b^3). \tag{8}$$

The Rogers–Ramanujan continued fraction $R(q)$ is defined by

$$R(q) := \frac{q^{1/5}}{1} \underset{+}{} \frac{q}{1} \underset{+}{} \frac{q^2}{1} \underset{+}{} \frac{q^3}{1} \underset{+}{} \cdots, \qquad |q| < 1, \tag{9}$$

which first appeared in a paper by Rogers [11] in 1894. This continued fraction has many representations, for example it can be expressed in terms of infinite products as follows:

$$R(q) = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{10}$$

Identity (10) has been proved by both Rogers [11] and discovered by Ramanujan [9, Vol. II, Chapter 16, Section 15], [1]. Ramanujan [10, p. 50] also gave 2- and 5-dissections of this continued fraction and its reciprocal which were first proved by Andrews [2] and Hirschhorn [7], respectively. Furthermore, in his 'lost' notebook [10, p. 36], Ramanujan stated four q-series representations for $R(q)$ (see [3, Entry 4.5.1, p. 121]). Chaudhary [4, 5] has studied the modular relations for the Rogers–Ramanujan type identities. Recently, Chaudhary and Choi [6] have derived several modular relations for Rogers–Ramanujan type identities.

The celebrated Ramanujan–G$\ddot{o}$llnitz–Gordon continued fraction is defined by

$$H(q) := \frac{q^{1/2}}{1+q} \, \frac{q^2}{+ \, 1+q^3} \, \frac{q^4}{+ \, 1+q^5} \, \frac{q^6}{+ \, 1+q^7} + \cdots, \qquad |\, q \,| < 1. \tag{11}$$

On page 229 of his second notebook [9], Ramanujan recorded a product representation of $H(q)$, namely

$$H(q) = q^{1/2} \frac{(q;q^8)_\infty (q^7;q^8)_\infty}{(q^3;q^8)_\infty (q^5;q^8)_\infty}. \tag{12}$$

Surekha [13], Vanitha [14] and Park [8] studied the following two continued fractions of order sixteen:

$$I_1(q) := \frac{q^{1/2}\,(1-q^3)}{1-q^4} \, \frac{q^4(1-q)\,(1-q^7)}{+ \, (1-q^4)\,(1+q^8)} \, \frac{q^4\,(1-q^9)\,(1-q^{15})}{+ \, (1-q^4)\,(1+q^{16})} + \cdots, \tag{13}$$

and

$$I_2(q) := \frac{q^{3/2}\,(1-q)}{1-q^4} \, \frac{q^4(1-q^3)\,(1-q^5)}{+ \, (1-q^4)\,(1+q^8)} \, \frac{q^4\,(1-q^{11})\,(1-q^{13})}{+ \, (1-q^4)\,(1+q^{16})} + \cdots, \tag{14}$$

which can be expressed in terms of Ramanujan's theta function as

$$I_1(q) = \frac{q^{1/2} f\left(-q^3,-q^{13}\right)}{f\left(-q^5,-q^{11}\right)} \quad \text{and} \quad I_2(q) = \frac{q^{3/2} f\left(-q,-q^{15}\right)}{f\left(-q^7,-q^9\right)}. \tag{15}$$

Surekha [13] proved modular relations for $I_1(q)$ and $I_2(q)$, and found $2-,4-,8-$, and 16-dissections for $1/I_1(q)$. She also showed that the sign of the coefficients in the power series expansion of $q^{-1/2}I_1(q)$ and its reciprocal are periodic with period 16. Mostly, a similar work have been done by Vanitha [14] for the continued fraction $q^{-3/2}I_2(q)$ and its reciprocal. Further, she gave combinatorial interpretations for the coefficients in the power series expansion of this continued fraction. Park [8] studied the continued fractions $I_1(q)$ and $I_2(q)$ by adopting the theory of modular functions, and he proved the modularities of $I_1(q)$ and $I_2(q)$ to find the relation with the generator of the field of modular functions on $\Gamma_0(16)$. Moreover he proved that the values $2(I_1^2(q)+1/I_1^2(q))$ and $2(I_2^2(q)+1/I_2^2(q))$ are algebraic integers for certain imaginary quadratic quantity of q. Recently, Srivastava and Chaudhary [12] have derived some identities, they are relationships between q-product identities, combinatorial partition identities and continued fractions identities.

In his second notebook [9] and lost notebook [10], Ramanujan stated many interesting q-continued fractions and some of their explicit values. For example, the extremely beautiful entry found in Chapter 16 of the second notebook [9], is Entry 11, which we state as:

Suppose that either q, a and b are complex numbers with $|q|<1$ or q, a and b are complex numbers with $a=bq^m$ for some integer m. Then

$$\begin{aligned}&\frac{(-a;q)_\infty(b;q)_\infty-(a;q)_\infty(-b;q)_\infty}{(-a;q)_\infty(b;q)_\infty+(a;q)_\infty(-b;q)_\infty}\\&\quad=\frac{a-b}{1-q\,+}\;\frac{(a-bq)(aq-b)}{1-q^3\quad+}\;\frac{q(a-bq^2)(aq^2-b)}{1-q^5\quad+\cdots}.\end{aligned}\tag{16}$$

Setting $q\to q^8$, $(a,b)=(q^7,-q)$ in the right-hand side of (16), we obtain

$$A(q^2):=\frac{q(1+q^6)}{1-q^8\;+}\;\frac{q^8(1+q^2)(1+q^{14})}{1-q^{24}\quad+}\;\frac{q^{16}(1+q^{10})(1+q^{22})}{1-q^{40}\quad+\cdots}.\tag{17}$$

Setting $q\to q^8$, $(a,b)=(q^5,-q^3)$ in the right-hand side of (16), we obtain

$$B(q^2):=\frac{q^3(1+q^2)}{1-q^8\;+}\;\frac{q^8(1+q^6)(1+q^{10})}{1-q^{24}\quad+}\;\frac{q^{16}(1+q^{14})(1+q^{18})}{1-q^{40}\quad+\cdots}.\tag{18}$$

Using the left-hand side of (16) with the help of (6) and (7) we find that, respectively

$$A(q^2)=q\frac{f(q^6,q^{26})}{f(q^{10},q^{22})}\qquad\text{and}\qquad B(q^2)=q^3\frac{f(q^2,q^{30})}{f(q^{14},q^{18})}.$$

Thus, we have

$$A(q)=q^{1/2}\frac{f(q^3,q^{13})}{f(q^5,q^{11})}\qquad\text{and}\qquad B(q)=q^{3/2}\frac{f(q,q^{15})}{f(q^7,q^9)}.\tag{19}$$

Using (15) and (19), we find that

$$A(-q)=iI_1(q),\qquad B(-q)=-iI_2(q),\tag{20}$$

$$I_1(-q)=iA(q)\qquad\text{and}\qquad I_2(-q)=-iB(q),\tag{21}$$

where $i=\sqrt{-1}$.

We need the following lemma to prove some of our main results:

LEMMA 3. *We have*

$$\frac{1}{f(q^{14},q^{18})\,f(q^{10},q^{22})}=\frac{f_1f_4}{f_2^2f_8^2}\left\{1+A\left(q^2\right)\right\}\left\{1+B\left(q^2\right)\right\}.$$

Proof. Using (8) with $a=q$ and $b=q^7$, we find that

$$1+A(q^2)=\frac{f\left(q,q^7\right)}{f\left(q^{10},q^{22}\right)}. \tag{22}$$

Again, using (8) with $a=q^3$ and $b=q^5$, we obtain

$$1+B(q^2)=\frac{f\left(q^3,q^5\right)}{f\left(q^{14},q^{18}\right)}. \tag{23}$$

Multiplying (22) and (23) together, we deduce the desired result. □

The main purpose of this paper is to derive several beautiful modular relations for the Ramanujan's continued fractions of sixteen order $A(q)$ and $B(q)$.

2. Main results

In this section, we derive several beautiful modular relations involving $A(q)$ and $B(q)$.

THEOREM 1. *We have*

$$q\frac{f_1 f\left(q^2,q^{14}\right)}{{f_2}^2}=\frac{A\left(q^2\right)+B\left(q^2\right)}{1+A\left(q^2\right)+B\left(q^2\right)+A\left(q^2\right)B\left(q^2\right)}, \tag{24}$$

$$\frac{f_1 f\left(q^6,q^{10}\right)}{{f_2}^2}=\frac{1+A\left(q^2\right)B\left(q^2\right)}{1+A\left(q^2\right)+B\left(q^2\right)+A\left(q^2\right)B\left(q^2\right)}, \tag{25}$$

and

$$\frac{\sqrt{q}f\left(q,q^7\right)}{f\left(q^3,q^5\right)}=\frac{A\left(q\right)+B\left(q\right)}{1+A\left(q\right)B\left(q\right)}. \tag{26}$$

Proof. Using Lemma 1 and Lemma 3, we may rewrite (24) and (25) as

$$A\left(q^2\right)+B\left(q^2\right)=\frac{q\psi\left(q^4\right)f\left(q^2,q^{14}\right)}{f\left(q^{14},q^{18}\right)f\left(q^{22},q^{10}\right)},$$

and

$$1+A\left(q^2\right)B\left(q^2\right)=\frac{\psi\left(q^4\right)f\left(q^6,q^{10}\right)}{f\left(q^{14},q^{18}\right)f\left(q^{22},q^{10}\right)}.$$

Employing the definitions of $A(q)$ and $B(q)$ in the above equalities, we obtain

$$\psi\left(q^4\right)f\left(q^2,q^{14}\right)=f\left(q^6,q^{26}\right)f\left(q^{14},q^{18}\right)+q^2 f\left(q^{22},q^{10}\right)f\left(q^{30},q^2\right), \tag{27}$$

and

$$\psi\left(q^{4}\right) f\left(q^{6}, q^{10}\right)=f\left(q^{14}, q^{18}\right) f\left(q^{22}, q^{10}\right)+q^{4} f\left(q^{30}, q^{2}\right) f\left(q^{26}, q^{6}\right). \tag{28}$$

Thus to prove (24) and (25), it is suffices to establish (27) and (28), respectively. Now, we start with the following product:

$$\begin{aligned}\psi(q^{4})\psi(q) = f(q^{4},q^{12})f(q,q^{3}) &= \left(\sum_{h=-\infty}^{\infty} q^{8h^{2}-4h}\right)\left(\sum_{l=-\infty}^{\infty} q^{2l^{2}+l}\right)\\ &= \sum_{h,l=-\infty}^{\infty} q^{8h^{2}-4h+2l^{2}+l}.\end{aligned} \tag{29}$$

In the equality (29), we make the change of indices by setting

$$2h+l=4H+\alpha \quad \text{and} \quad -2h+l=4L+\beta,$$

the values of α and β are selected from the set $\{0,\pm 1,2\}$. Then

$$h=H-L+(\alpha-\beta)/4 \quad \text{and} \quad l=2H+L+(\alpha+\beta)/2.$$

It follows easily that $\alpha=\beta$, and so $h=H-L$ and $l=2H+L+\alpha$, where $-1\leqslant\alpha\leqslant 2$. Thus, there is one-to-one correspondence between the set

$$\{(h,l): -\infty<h,l<\infty, h,l\in\mathbb{Z}\}$$

and the set

$$\{(H,L,\alpha): -\infty<H,L<\infty, -1\leqslant\alpha\leqslant 2, H,L,\alpha\in\mathbb{Z}\}.$$

The identity (29) can be written as

$$\begin{aligned}\psi(q^{4})\psi(q) &= \sum_{\alpha=-1}^{2} q^{2\alpha^{2}+\alpha} \sum_{H=-\infty}^{\infty} q^{16H^{2}+(-2+8\alpha)H} \sum_{L=-\infty}^{\infty} q^{16L^{2}+(6+8\alpha)L}\\ &= \sum_{\alpha=-1}^{2} q^{2\alpha^{2}+\alpha} f\left(q^{14+8\alpha}, q^{18-8\alpha}\right) f\left(q^{22+8\alpha}, q^{10-8\alpha}\right)\\ &= qf\left(q^{6},q^{26}\right) f\left(q^{14},q^{18}\right)+f\left(q^{14},q^{18}\right) f\left(q^{22},q^{10}\right)\\ &\quad +q^{3} f\left(q^{22},q^{10}\right) f\left(q^{30},q^{2}\right)+q^{10} f\left(q^{30},q^{2}\right) f\left(q^{38},q^{-6}\right).\end{aligned} \tag{30}$$

We make the same argument for the product $\psi(q^{4})\psi(-q)$, to find that

$$\begin{aligned}\psi(q^{4})\psi(-q) &= f\left(q^{14},q^{18}\right) f\left(q^{22},q^{10}\right)-qf\left(q^{6},q^{26}\right) f\left(q^{14},q^{18}\right)\\ &\quad -q^{3} f\left(q^{22},q^{10}\right) f\left(q^{30},q^{2}\right)+q^{4} f\left(q^{30},q^{2}\right) f\left(q^{26},q^{6}\right).\end{aligned} \tag{31}$$

Adding (30) and (31) and use the identity $\psi(q)+\psi(-q)=2f\left(q^{6},q^{10}\right)$, which follows easily from (6), we obtain (27). Subtracting (31) from (30) and use the identity $\psi(q)-\psi(-q)=2qf\left(q^{2},q^{14}\right)$, which follows easily from (7), we obtain (28). This completes the proof of (24) and (25). The identity (26) can be obtained by dividing (24) by (25) and changing q^2 to q. $\square$

As consequence of Theorem 1, it is easy to obtain the following identities:

THEOREM 2. *We have*

$$\frac{\sqrt{q}f\left(q^{10},q^{22}\right)}{f\left(q^{14},q^{18}\right)}=\frac{\left(A(q)+B(q)\right)\left(1+B\left(q^{2}\right)\right)}{\left(1+A(q)B(q)\right)\left(1+A\left(q^{2}\right)\right)}, \tag{32}$$

$$\frac{q^{-3/2}f\left(q^{6},q^{26}\right)}{f\left(q^{2},q^{30}\right)}=\frac{A\left(q^{2}\right)\left(A(q)+B(q)\right)\left(1+B\left(q^{2}\right)\right)}{B\left(q^{2}\right)\left(1+A(q)B(q)\right)\left(1+A\left(q^{2}\right)\right)}, \tag{33}$$

$$\frac{q^{-5/2}f\left(q^{10},q^{22}\right)}{f\left(q^{2},q^{30}\right)}=\frac{\left(A(q)+B(q)\right)\left(1+B\left(q^{2}\right)\right)}{B\left(q^{2}\right)\left(1+A(q)B(q)\right)\left(1+A\left(q^{2}\right)\right)}, \tag{34}$$

and

$$\frac{q^{3/2}f\left(q^{6},q^{26}\right)}{f\left(q^{14},q^{18}\right)}=\frac{A\left(q^{2}\right)\left(A(q)+B(q)\right)\left(1+B\left(q^{2}\right)\right)}{\left(1+A(q)B(q)\right)\left(1+A\left(q^{2}\right)\right)}. \tag{35}$$

THEOREM 3. *For any $n\geqslant 1$, define*

$$S_n:=S(q^n):=\frac{\left(A(q^n)+B(q^n)\right)\left(1+B\left(q^{2n}\right)\right)}{\left(1+A(q^n)B(q^n)\right)\left(1+A\left(q^{2n}\right)\right)},$$

$$T_n:=T(q^n):=\frac{A\left(q^{2n}\right)\left(A(q^n)+B(q^n)\right)\left(1+B\left(q^{2n}\right)\right)}{B\left(q^{2n}\right)\left(1+A(q^n)B(q^n)\right)\left(1+A\left(q^{2n}\right)\right)},$$

$$U_n:=U(q^n):=\frac{\left(A(q^n)+B(q^n)\right)\left(1+B\left(q^{2n}\right)\right)}{B\left(q^{2n}\right)\left(1+A(q^n)B(q^n)\right)\left(1+A\left(q^{2n}\right)\right)},$$

and

$$V_n:=V(q^n):=\frac{A\left(q^{2n}\right)\left(A(q^n)+B(q^n)\right)\left(1+B\left(q^{2n}\right)\right)}{\left(1+A(q^n)B(q^n)\right)\left(1+A\left(q^{2n}\right)\right)}.$$

Then

$$\begin{aligned}
&\frac{\varphi\left(q^{30}\right)\psi\left(q^{4}\right)}{f\left(q^{70},q^{90}\right)f\left(q^{42},q^{54}\right)}\\
&=\left\{1+B\left(q^{6}\right)B\left(q^{10}\right)\right\}+S_3S_5\left\{1+A\left(q^{6}\right)A\left(q^{10}\right)\right\}, \qquad (36)\\
&\frac{\varphi\left(q^{12}\right)\psi\left(q^{2}\right)}{f\left(q^{42},q^{54}\right)f\left(q^{14},q^{18}\right)}\\
&=\left\{1+B\left(q^{2}\right)B\left(q^{6}\right)\right\}+S_1S_3\left\{1+A\left(q^{2}\right)A\left(q^{6}\right)\right\}, \qquad (37)\\
&\frac{\varphi\left(q^{14}\right)\psi\left(q^{4}\right)}{f\left(q^{98},q^{126}\right)f\left(q^{14},q^{18}\right)}\\
&=\left\{1+B\left(q^{2}\right)B\left(q^{14}\right)\right\}+S_1S_7\left\{1+A\left(q^{2}\right)A\left(q^{14}\right)\right\}, \qquad (38)
\end{aligned}$$

$$\frac{\psi\left(q^{6}\right)\varphi\left(q^{4}\right)}{f\left(q^{54},q^{42}\right)f\left(q^{10},q^{22}\right)}=\left\{1+A\left(q^{2}\right)B\left(q^{6}\right)\right\}+\frac{S_3}{U_1}\left\{1+\frac{A\left(q^{6}\right)}{B\left(q^{2}\right)}\right\}, \qquad (39)$$

$$\frac{2\psi\left(q^{6}\right)\psi\left(q^{8}\right)}{f\left(q^{66},q^{30}\right)f\left(q^{14},q^{18}\right)}=\left\{1+A\left(q^{6}\right)B\left(q^{2}\right)\right\}+\frac{V_1}{S_3}\left\{1+\frac{B\left(q^{6}\right)}{A\left(q^{2}\right)}\right\}, \qquad (40)$$

$$\frac{\psi\left(q^{4}\right)\psi\left(q\right)}{f\left(q^{14},q^{18}\right)f\left(q^{22},q^{10}\right)}=\left\{1+A\left(q^{2}\right)B\left(q^{2}\right)\right\}+\frac{V_1}{S_1}\left\{1+\frac{B\left(q^{2}\right)}{A\left(q^{2}\right)}\right\}, \qquad (41)$$

$$\frac{2\psi\left(q^{24}\right)\psi\left(q^{2}\right)}{f\left(q^{54},q^{42}\right)f\left(q^{2},q^{30}\right)}=\left\{1+\frac{B\left(q^{6}\right)}{B\left(q^{2}\right)}\right\}+S_3T_1\left\{1+\frac{A\left(q^{6}\right)}{A\left(q^{2}\right)}\right\}, \qquad (42)$$

$$\frac{\psi\left(q^{28}\right)\varphi\left(q^{2}\right)}{f\left(q^{126},q^{98}\right)f\left(q^{2},q^{30}\right)}=\left\{1+\frac{B\left(q^{14}\right)}{B\left(q^{2}\right)}\right\}+S_7T_1\left\{1+\frac{A\left(q^{14}\right)}{A\left(q^{2}\right)}\right\}, \qquad (43)$$

$$\frac{\psi\left(q^{60}\right)\varphi\left(q^{2}\right)}{f\left(q^{110},q^{50}\right)f\left(q^{18},q^{78}\right)}=\left\{1+\frac{A\left(q^{10}\right)}{A\left(q^{6}\right)}\right\}+\frac{1}{S_5T_3}\left\{1+\frac{B\left(q^{10}\right)}{B\left(q^{6}\right)}\right\}, \qquad (44)$$

$$\frac{\varphi\left(q^{8}\right)\psi\left(q\right)}{f^2\left(q^{18},q^{14}\right)}=\left\{1+B^2\left(q^{2}\right)\right\}+S_1^2\left\{1+A^2\left(q^{2}\right)\right\}, \qquad (45)$$

and

$$\frac{\psi\left(q^{16}\right)\psi\left(q\right)}{f\left(q^{26},q^{6}\right)f\left(q^{10},q^{22}\right)}=1+\frac{1}{S_1T_1}. \qquad (46)$$

Proof. Identity (36) can be written in the form

$$\begin{aligned}
\frac{\varphi\left(q^{30}\right)\psi\left(q^{4}\right)}{f\left(q^{70},q^{90}\right)f\left(q^{42},q^{54}\right)}=&\left\{1+B\left(q^{6}\right)B\left(q^{10}\right)\right\}+\frac{\left(A\left(q^{3}\right)+B\left(q^{3}\right)\right)\left(1+B\left(q^{6}\right)\right)}{\left(1+A\left(q^{3}\right)B\left(q^{3}\right)\right)\left(1+A\left(q^{6}\right)\right)}\\
&\times\frac{\left(A\left(q^{5}\right)+B\left(q^{5}\right)\right)\left(1+B\left(q^{10}\right)\right)}{\left(1+A\left(q^{5}\right)B\left(q^{5}\right)\right)\left(1+A\left(q^{10}\right)\right)}\left\{1+A\left(q^{6}\right)A\left(q^{10}\right)\right\}.
\end{aligned}$$

Employing (32) in the above identity, with $q \to q^3$ and $q \to q^5$, and then multiply both sides into $f\left(q^{70},q^{90}\right) f\left(q^{42},q^{54}\right)$, after some simplification, we obtain

$$\varphi\left(q^{15}\right)\psi\left(q^2\right) = f\left(q^{35},q^{45}\right) f\left(q^{21},q^{27}\right) + q^{12} f\left(q^{75},q^{5}\right) f\left(q^{45},q^{3}\right) + q^2 f\left(q^{25},q^{55}\right) f\left(q^{15},q^{33}\right) + q^6 f\left(q^{15},q^{65}\right) f\left(q^{9},q^{39}\right). \quad (47)$$

Thus to prove (36), it is suffices to establish (47). Now, we start with the following product:

$$2\varphi\left(q^{15}\right)\psi\left(q^2\right) = f(q^{15},q^{15}) f(1,q^2) = \left(\sum_{h=-\infty}^{\infty} q^{15h^2}\right)\left(\sum_{l=-\infty}^{\infty} q^{l^2+l}\right) = \sum_{h,l=-\infty}^{\infty} q^{15h^2+l^2+l}. \quad (48)$$

In the equality (48), we set

$$3h+l = 8H+\alpha \quad \text{and} \quad -5h+l = 8L+\beta,$$

where α and β have values selected from the complete set of residues modulo 8 given by $\{0,\pm1,\pm2,\pm3,4\}$. Then

$$h = H-L+(\alpha-\beta)/8 \quad \text{and} \quad l = 5H+3L+(5\alpha+3\beta)/8.$$

It follows easily that $\alpha=\beta$, and so $h=H-L$ and $l=5H+3L+\alpha$, where $-3 \leqslant \alpha \leqslant 4$. Thus, there is one-to-one correspondence between the set

$$\{(h,l) : -\infty < h,l < \infty, h,l \in \mathbb{Z}\},$$

and the set

$$\{(H,L,\alpha) : -\infty < H,L < \infty, -3 \leqslant \alpha \leqslant 4, H,L,\alpha \in \mathbb{Z}\}.$$

The identity (48) can be written as

$$\begin{aligned} 2\varphi\left(q^{15}\right)\psi\left(q^2\right) &= \sum_{\alpha=-3}^{4} q^{\alpha^2+\alpha} \sum_{H=-\infty}^{\infty} q^{40H^2+5(1+2\alpha)H} \sum_{L=-\infty}^{\infty} q^{24L^2+3(1+2\alpha)L} \\ &= \sum_{\alpha=-3}^{4} q^{\alpha^2+\alpha} f\left(q^{5(9+2\alpha)},q^{5(7-2\alpha)}\right) f\left(q^{3(9+2\alpha)},q^{3(7-2\alpha)}\right) \\ &= q^6 f\left(q^{15},q^{65}\right) f\left(q^{9},q^{39}\right) + q^2 f\left(q^{25},q^{55}\right) f\left(q^{15},q^{33}\right) \\ &\quad + f\left(q^{35},q^{45}\right) f\left(q^{21},q^{27}\right) + f\left(q^{45},q^{35}\right) f\left(q^{27},q^{21}\right) \\ &\quad + q^2 f\left(q^{55},q^{25}\right) f\left(q^{33},q^{15}\right) + q^6 f\left(q^{65},q^{15}\right) f\left(q^{39},q^{9}\right) \\ &\quad + q^{12} f\left(q^{75},q^{5}\right) f\left(q^{45},q^{3}\right) + q^{20} f\left(q^{85},q^{-5}\right) f\left(q^{51},q^{-3}\right), \end{aligned}$$

which is same as (47), this complete the proof of (36). The proofs of (37)–(46) can be derived by a similar way. So, we omit the details. □

THEOREM 4. *For S_n, U_n and V_n as defined in Theorem 3, we have*

$$\frac{\varphi\left(q^{39}\right)\varphi(q)-\varphi\left(-q^{39}\right)\varphi(-q)}{4q\,f\left(q^{234},q^{182}\right)f\left(q^{54},q^{42}\right)}$$
$$=\left\{1+B\left(q^{6}\right)B\left(q^{26}\right)\right\}+S_3S_{13}\left\{1+A\left(q^{6}\right)A\left(q^{26}\right)\right\}, \tag{49}$$

$$\frac{\varphi\left(q^{15}\right)\varphi(q)-\varphi\left(-q^{15}\right)\varphi(-q)}{4q\,f\left(q^{210},q^{270}\right)f\left(q^{14},q^{18}\right)}$$
$$=\left\{1+B\left(q^{2}\right)B\left(q^{30}\right)\right\}+S_1S_{15}\left\{1+A\left(q^{2}\right)A\left(q^{30}\right)\right\}, \tag{50}$$

$$\frac{\varphi\left(q^{55}\right)\varphi(q)-\varphi\left(-q^{55}\right)\varphi(-q)}{4q\,f\left(q^{198},q^{154}\right)f\left(q^{90},q^{70}\right)}$$
$$=\left\{1+B\left(q^{10}\right)B\left(q^{22}\right)\right\}+S_5S_{11}\left\{1+A\left(q^{10}\right)A\left(q^{22}\right)\right\}, \tag{51}$$

$$\frac{\varphi\left(q^{63}\right)\varphi(q)-\varphi\left(-q^{63}\right)\varphi(-q)}{4q\,f\left(q^{162},q^{126}\right)f\left(q^{126},q^{98}\right)}$$
$$=\left\{1+B\left(q^{14}\right)B\left(q^{18}\right)\right\}+S_7S_9\left\{1+A\left(q^{14}\right)A\left(q^{18}\right)\right\}, \tag{52}$$

$$\frac{\psi\left(q^{7}\right)\psi(q)+\psi\left(-q^{7}\right)\psi(-q)}{2f\left(q^{126},q^{98}\right)f\left(q^{10},q^{22}\right)}$$
$$=\left\{1+A\left(q^{2}\right)B\left(q^{14}\right)\right\}+\frac{S_7}{U_1}\left\{1+\frac{A\left(q^{14}\right)}{B\left(q^{2}\right)}\right\}, \tag{53}$$

and

$$\frac{\left\{\psi\left(q^{15}\right)\psi(q)+\psi\left(-q^{15}\right)\psi(-q)\right\}}{2f\left(q^{66},q^{30}\right)f\left(q^{70},q^{90}\right)}$$
$$=\left\{1+A\left(q^{6}\right)B\left(q^{10}\right)\right\}+\frac{V_5}{S_3}\left\{1+\frac{B\left(q^{6}\right)}{A\left(q^{10}\right)}\right\}. \tag{54}$$

Proof. Identity (49) can be written in the form

$$\frac{\varphi\left(q^{39}\right)\varphi(q)-\varphi\left(-q^{39}\right)\varphi(-q)}{4q\,f\left(q^{234},q^{182}\right)f\left(q^{54},q^{42}\right)}$$
$$=\left\{1+B\left(q^{6}\right)B\left(q^{26}\right)\right\}+\frac{\left(A\left(q^{3}\right)+B\left(q^{3}\right)\right)\left(1+B\left(q^{6}\right)\right)}{\left(1+A\left(q^{3}\right)B\left(q^{3}\right)\right)\left(1+A\left(q^{6}\right)\right)}$$
$$\times\frac{\left(A\left(q^{13}\right)+B\left(q^{13}\right)\right)\left(1+B\left(q^{26}\right)\right)}{\left(1+A\left(q^{13}\right)B\left(q^{13}\right)\right)\left(1+A\left(q^{26}\right)\right)}\left\{1+A\left(q^{6}\right)A\left(q^{26}\right)\right\}.$$

Employing (32) in the above identity, with $q \to q^3$ and $q \to q^{13}$, and then multiply both sides into $4qf\left(q^{234},q^{182}\right)f\left(q^{54},q^{42}\right)$, after some simplification, we obtain

$$\begin{aligned}\varphi\left(q^{39}\right)\varphi(q)-\varphi\left(-q^{39}\right)\varphi(-q)&\\ =4qf\left(q^{234},q^{182}\right)f\left(q^{54},q^{42}\right)&+4q^9f\left(q^{286},q^{130}\right)f\left(q^{66},q^{30}\right)\\ +4q^{25}f\left(q^{338},q^{78}\right)f\left(q^{78},q^{18}\right)&+4q^{49}f\left(q^{390},q^{26}\right)f\left(q^{90},q^{6}\right). \quad (55)\end{aligned}$$

Thus, it is suffices to establish (55). Now, we start with the following product:

$$\begin{aligned}\varphi\left(q^{39}\right)\varphi(q)=f(q^{39},q^{39})f(q,q)&=\left(\sum_{h=-\infty}^{\infty}q^{39h^2}\right)\left(\sum_{l=-\infty}^{\infty}q^{l^2}\right)\\ &=\sum_{h,l=-\infty}^{\infty}q^{39h^2+l^2}.\end{aligned}$$

In the above equality, we set

$$3h+l=16H+\alpha \quad \text{and} \quad -13h+l=16L+\beta,$$

where α and β have values selected from the complete set of residues modulo 16 given by $\{0,\pm1,\pm2,\pm3,\pm4,\pm5,\pm6,\pm7,8\}$. Then

$$h=H-L+(\alpha-\beta)/16 \quad \text{and} \quad l=13H+3L+(13\alpha+3\beta)/16.$$

It follows easily that $\alpha=\beta$, and so $h=H-L$ and $l=13H+3L+\alpha$, where $-7\leqslant\alpha\leqslant 8$. Thus, there is one-to-one correspondence between the set

$$\{(h,l):-\infty<h,l<\infty,h,l\in\mathbb{Z}\},$$

and the set

$$\{(H,L,\alpha):-\infty<H,L<\infty,-7\leqslant\alpha\leqslant 8,H,L,\alpha\in\mathbb{Z}\}.$$

Now, we have

$$\begin{aligned}\varphi\left(q^{39}\right)\varphi(q)&=\sum_{\alpha=-7}^{8}q^{\alpha^2}\sum_{H=-\infty}^{\infty}q^{26(8H^2+\alpha H)}\sum_{L=-\infty}^{\infty}q^{6(8L^2+\alpha L)}\\ &=\sum_{\alpha=-7}^{8}q^{\alpha^2}f\left(q^{208+26\alpha},q^{208-26\alpha}\right)f\left(q^{48+6\alpha},q^{48-6\alpha}\right)\end{aligned}$$

$$
\begin{aligned}
&= q^{49} f\left(q^{26}, q^{390}\right) f\left(q^{6}, q^{90}\right) + q^{36} f\left(q^{52}, q^{364}\right) f\left(q^{12}, q^{84}\right) \\
&\quad + q^{25} f\left(q^{78}, q^{338}\right) f\left(q^{18}, q^{78}\right) + q^{16} f\left(q^{104}, q^{312}\right) f\left(q^{24}, q^{72}\right) \\
&\quad + q^{9} f\left(q^{130}, q^{286}\right) f\left(q^{30}, q^{66}\right) + q^{4} f\left(q^{156}, q^{260}\right) f\left(q^{36}, q^{60}\right) \\
&\quad + q f\left(q^{182}, q^{234}\right) f\left(q^{42}, q^{54}\right) + f\left(q^{208}, q^{208}\right) f\left(q^{48}, q^{48}\right) \\
&\quad + q f\left(q^{234}, q^{182}\right) f\left(q^{54}, q^{42}\right) + q^{4} f\left(q^{260}, q^{156}\right) f\left(q^{60}, q^{36}\right) \\
&\quad + q^{9} f\left(q^{286}, q^{130}\right) f\left(q^{66}, q^{30}\right) + q^{16} f\left(q^{312}, q^{104}\right) f\left(q^{72}, q^{24}\right) \\
&\quad + q^{25} f\left(q^{338}, q^{78}\right) f\left(q^{78}, q^{18}\right) + q^{36} f\left(q^{364}, q^{52}\right) f\left(q^{84}, q^{12}\right) \\
&\quad + q^{49} f\left(q^{390}, q^{26}\right) f\left(q^{90}, q^{6}\right) + q^{64} f\left(q^{416}, 1\right) f\left(q^{96}, 1\right).
\end{aligned}
$$

Changing q to $-q$ in the above identity, and then subtracting the resulting identity from the above identity, we arrive at (55), this completes the proof of (49). The proofs of the remaining identities in this theorem, follow in a similar way. □

THEOREM 5. *For S_n, U_n and V_n as defined in Theorem 3, and*

$$
W_n = W(q^n) = \frac{A(q^n) + B(q^n)}{1 + A(q^n) B(q^n)},
$$

we have

$$
\begin{aligned}
&\frac{\varphi(q^6)\,\psi(q^3)\,\left(1 + A(q^2)\right)}{f(q^9, q^{15})\, f(q^{42}, q^{54})} \\
&= \left\{ \left\{ 1 + A\left(q^6\right) B\left(q^2\right) \right\} + \frac{S_3}{S_1} \left\{ 1 + A\left(q^2\right) B\left(q^6\right) \right\} \right\} \\
&\quad + S_3 W_3 \left\{ \left\{ 1 + B\left(q^2\right) B\left(q^6\right) \right\} + \frac{1}{S_3 U_1} \left\{ 1 + \frac{A(q^6)}{A(q^2)} \right\} \right\},
\end{aligned} \tag{56}
$$

and

$$
\begin{aligned}
&\frac{2\,\psi(q^{12})\,\psi(q^3)\,\left(1 + B(q^2)\right)}{f(q^{42}, q^{54})\, f(q^{21}, q^3)} \\
&= \left\{ \left\{ 1 + B\left(q^2\right) B\left(q^6\right) \right\} + S_3 V_1 \left\{ 1 + \frac{A(q^6)}{A(q^2)} \right\} \right\} \\
&\quad + \frac{V_1}{W_3} \left\{ \left\{ 1 + \frac{B(q^6)}{A(q^2)} \right\} + \frac{V_3}{V_1} \left\{ 1 + \frac{B(q^2)}{A(q^6)} \right\} \right\}.
\end{aligned} \tag{57}
$$

Proof. The first identity of the above theorem can be written in the form

$$\begin{aligned}
&\frac{\varphi\left(q^{6}\right)\psi\left(q^{3}\right)\left(1+A\left(q^{2}\right)\right)}{f\left(q^{9},q^{15}\right)f\left(q^{42},q^{54}\right)}\\
&=\left\{\left\{1+A\left(q^{6}\right)B\left(q^{2}\right)\right\}+\frac{\left(A\left(q^{3}\right)+B\left(q^{3}\right)\right)\left(1+B\left(q^{6}\right)\right)}{\left(1+A\left(q^{3}\right)B\left(q^{3}\right)\right)\left(1+A\left(q^{6}\right)\right)}\right.\\
&\quad\left.\times\frac{\left(1+A(q)B(q)\right)\left(1+A\left(q^{2}\right)\right)}{\left(A(q)+B(q)\right)\left(1+B\left(q^{2}\right)\right)}\left\{1+A\left(q^{2}\right)B\left(q^{6}\right)\right\}\right\}\\
&\quad+\frac{\left(A\left(q^{3}\right)+B\left(q^{3}\right)\right)^{2}\left(1+B\left(q^{6}\right)\right)}{\left(1+A\left(q^{3}\right)B\left(q^{3}\right)\right)^{2}\left(1+A\left(q^{6}\right)\right)}\left\{\left\{1+B\left(q^{2}\right)B\left(q^{6}\right)\right\}\right.\\
&\quad+\frac{\left(1+A\left(q^{3}\right)B\left(q^{3}\right)\right)\left(1+A\left(q^{6}\right)\right)}{\left(A\left(q^{3}\right)+B\left(q^{3}\right)\right)\left(1+B\left(q^{6}\right)\right)}\frac{B\left(q^{2}\right)\left(1+A(q)B(q)\right)\left(1+A\left(q^{2}\right)\right)}{\left(A(q)+B(q)\right)\left(1+B\left(q^{2}\right)\right)}\\
&\quad\left.\times\left\{1+\frac{A\left(q^{6}\right)}{A\left(q^{2}\right)}\right\}\right\}.
\end{aligned}$$

Using Theorem 2, (26) and (22) in the above identity, after some simplification, we obtain

$$\begin{aligned}
f\left(q^{6},q^{6}\right)f\left(q^{9},q^{3}\right)f\left(q^{7},q\right)&=f\left(q^{9},q^{15}\right)\left\{qf\left(q^{14},q^{18}\right)f\left(q^{66},q^{30}\right)\right.\\
&\quad+f\left(q^{22},q^{10}\right)f\left(q^{42},q^{54}\right)+q^{7}f\left(q^{30},q^{2}\right)f\left(q^{18},q^{78}\right)\\
&\quad\left.+q^{10}f\left(q^{26},q^{6}\right)f\left(q^{6},q^{90}\right)\right\}+f\left(q^{21},q^{3}\right)\left\{q^{7}f\left(q^{26},q^{6}\right)f\left(q^{78},q^{18}\right)\right.\\
&\quad+q^{4}f\left(q^{30},q^{2}\right)f\left(q^{54},q^{42}\right)+q^{10}f\left(q^{18},q^{14}\right)f\left(q^{6},q^{90}\right)\\
&\quad\left.+q^{3}f\left(q^{10},q^{22}\right)f\left(q^{30},q^{66}\right)\right\}. \qquad (58)
\end{aligned}$$

Thus to prove the desired result, it is suffices to establish (58). Now, we start with the following product of three theta functions:

$$\begin{aligned}
\varphi\left(q^{6}\right)\psi\left(q^{3}\right)f\left(q^{7},q\right)&=f\left(q^{6},q^{6}\right)f\left(q^{9},q^{3}\right)f\left(q^{7},q\right)\\
&=\left(\sum_{h=-\infty}^{\infty}q^{6h^{2}}\right)\left(\sum_{k=-\infty}^{\infty}q^{6k^{2}+3k}\right)\left(\sum_{l=-\infty}^{\infty}q^{4l^{2}+3l}\right)\\
&=\sum_{h,k,l=-\infty}^{\infty}q^{6h^{2}+6k^{2}+3k+4l^{2}+3l}.
\end{aligned}$$

In the above equality, we use the following change of indices:

$$3h+3k+2l=8H+\alpha,\quad h-k=2K+\beta\quad\text{and}\quad h+k-2l=8L+\gamma,$$

where $\alpha \in \{0, \pm1, \pm2, \pm3, 4\}$, $\beta \in \{0, 1\}$ and $\gamma \in \{0, \pm1, \pm2, \pm3, -4\}$. solving the above system,

$$\begin{aligned} h =& H + K + L + \frac{\alpha + 4\beta + \gamma}{8}, \\ k =& H - K + L + \frac{\alpha - 4\beta + \gamma}{8}, \\ l =& H - 3L + \frac{\alpha - 3\gamma}{8}, \end{aligned}$$

Since h, k, l, H, K and L are all integers, the values of α, β, and δ are associated in the following table. The corresponding values of h, k and l are given in the table which follows form the above equalities:

α	0	±1	±2	±3	4
β	0	1	0	1	0
γ	0	±3	∓2	±1	-4
h	$H+K+L$	$H+K+L^{+1}_{+0}$	$H+K+L$	$H+K+L^{+1}_{+0}$	$H+K+L$
k	$H-K+L$	$H-K+L^{+0}_{-1}$	$H-K+L$	$H-K+L^{+0}_{-1}$	$H-K+L$
l	$H-3L$	$H-3L\mp1$	$H-3L\pm1$	$H-3L$	$H-3L+2$

When α assume the values -3, -2, -1, 0, 1, 2, 3 and 4 in succession, it is easy to see that the corresponding values of $6h^2+6k^2+3k+4l^2+3l$ are respectively associated in the following table:

α	$6h^2+6k^2+3k+4l^2+3l$
-3	$16H^2+12K^2+48L^2-6H+9K-18L+3$
-2	$16H^2+12K^2+48L^2-2H-3K+18L+1$
-1	$16H^2+12K^2+48L^2+2H+9K-42L+10$
0	$16H^2+12K^2+48L^2+6H-3K-6L$
1	$16H^2+12K^2+48L^2+10H+9K+30L+7$
2	$16H^2+12K^2+48L^2+14H-3K-30L+7$
3	$16H^2+12K^2+48L^2+18H+9K+6L+6$
4	$16H^2+12K^2+48L^2+22H-3K-54L+22$

It is obvious, from the equations connecting h, k and l with H, K and L that, there is a one-one correspondence between the set

$$\{(h,k,l) : -\infty < h,k,l < \infty, h,k,l \in \mathbb{Z}\},$$

and the set

$$\{(H,K,L,\alpha) : -\infty < H,K,L < \infty, -3 \leqslant \alpha \leqslant 4, H,K,L,\alpha \in \mathbb{Z}\}.$$

Now, we have

$$
\begin{aligned}
&\varphi\left(q^6\right)\psi\left(q^3\right)f\left(q^7,q\right)\\
=&\sum_{h,k,l=-\infty}^{\infty} q^{6h^2+6k^2+3k+4l^2+3l}\\
=&q^3\sum_{H,K,L=-\infty}^{\infty} q^{16H^2+12K^2+48L^2-6H+9K-18L}\\
&+q\sum_{H,K,L=-\infty}^{\infty} q^{16H^2+12K^2+48L^2-2H-3K+18L}\\
&+q^{10}\sum_{H,K,L=-\infty}^{\infty} q^{16H^2+12K^2+48L^2+2H+9K-42L}\\
&+\sum_{H,K,L=-\infty}^{\infty} q^{16H^2+12K^2+48L^2+6H-3K-6L}\\
&+q^7\sum_{H,K,L=-\infty}^{\infty} q^{16H^2+12K^2+48L^2+10H+9K+30L}\\
&+q^7\sum_{H,K,L=-\infty}^{\infty} q^{16H^2+12K^2+48L^2+14H-3K-30L}\\
&+q^6\sum_{H,K,L=-\infty}^{\infty} q^{16H^2+12K^2+48L^2+18H+9K+6L}\\
&+q^{22}\sum_{H,K,L=-\infty}^{\infty} q^{16H^2+12K^2+48L^2+22H-3K-54L}\\
=&q^3 f\left(q^{10},q^{22}\right)f\left(q^{21},q^3\right)f\left(q^{30},q^{66}\right)+qf\left(q^{14},q^{18}\right)f\left(q^9,q^{15}\right)f\left(q^{66},q^{30}\right)\\
&+q^{10}f\left(q^{26},q^6\right)f\left(q^9,q^{15}\right)f\left(q^6,q^{90}\right)+f\left(q^{22},q^{10}\right)f\left(q^9,q^{15}\right)f\left(q^{42},q^{54}\right)\\
&+q^7f\left(q^{26},q^6\right)f\left(q^{21},q^3\right)f\left(q^{78},q^{18}\right)+q^7f\left(q^{30},q^2\right)f\left(q^9,q^{15}\right)f\left(q^{18},q^{78}\right)\\
&+q^6f\left(q^{34},q^{-2}\right)f\left(q^{21},q^3\right)f\left(q^{54},q^{42}\right)+q^{22}f\left(q^{38},q^{-6}\right)f\left(q^9,q^{15}\right)f\left(q^{-6},q^{102}\right),
\end{aligned}
$$

which is same as (58), this completes the proof of the desired result. A proof of second identity of this theorem follows in a similar way. □

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