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UNIQUENESS FUNCTIONS CONCERNING DIFFERENCE OPERATOR SHARING A POLYNOMIAL WITH FINITE WEIGHT

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Abstract. In this study, we examine the uniqueness results when two difference polynomials of meromorphic (entire) functions share a nonzero polynomial of finite weight, using the notion of weighted sharing of values. Additionally, we examine the various possibilities for $P(z)$. The results of the paper improve and generalize the recent results due to Liu et al. [14] and Liu et al. [15].

1. Introduction and main results

The study of difference equations in the complex domain has experienced great interest in recent decades, not only due to its inherent interest and numerous applications but also motivated by and leaning on different previous theories.

On the one hand, one can point out different works based on Nevanlinna value distribution theory treating meromorphic solutions to difference equations. There has been an increasing interest in studying difference equations and difference product in the complex plane. Halburd and Korhonen [6] established a version of Nevanlinna theory based on difference operators. Bergweiler and Langley [4] considered the value distribution of zeros of difference operators that can be viewed as discrete analogues of zeros of $f'(z)$. In recent years, the difference variant of the Nevanlinna theory has been established in [5, 14, 15]. Using these theories, some mathematicians in the world began to study the uniqueness questions of meromorphic functions sharing values with their shifts and study the value distribution of the nonlinear difference polynomials and produced many fine works; for example, see [2, 3, 8, 10, 11, 16, 17, 18, 19, 20, 21, 23, 24, 25]. To state their results and some related ones in this direction, first of all, we recall the following notations.

It is assumed that the readers are accustomed to Nevanlinna theory and its standard notations for a meromorphic (entire) function f, such as $T(r,f)$ (the Nevanlinna characteristic function), $m(r,f)$ (the proximity function), $N(r,f)$ (the counting function), and $\overline{N}(r,f)$ (the reduced counting function) (for details, we refer the reader to [9, 26]). For such a meromorphic function f, a meromorphic function ω such that $T(r,\omega) = o(T(r,f))$, where $r \to \infty$ outside a possible exceptional set of finite logarithmic measure, is called a small function of f. The family of all small functions of f is denoted by $S(f)$, and $\tilde{S} = S(f) \cup \{\infty\}$.

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Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a;f)$ the set of all a-points of $f(z)$, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If we have for two meromorphic functions $f(z)$ and $g(z)$ that $E_k(a;f) = E_k(a;g)$, then we say that $f(z)$ and $g(z)$ share a with weight k. The IM and CM sharing correspond to the weights 0 and ∞, respectively. If $a(z)$ is a small function, we define that $f(z)$ and $g(z)$ share $a(z)$ IM or $a(z)$ CM or with weight κ depending on whether $f(z)-a(z)$ and $g(z)-a(z)$ share $(0,0)$ or $(0,\infty)$ or $(0,\kappa)$, respectively.

As usual, the order $\rho(f)$ and the hyper-order $\rho_2(f)$ of f are defined, respectively, by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

DEFINITION 1. [12] Let $a \in \mathbb{C} \cup \{\infty\}$. For $p \in \mathbb{N}$, we denote by $N(r,a;f| \leqslant p)$, the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,a;f| \leqslant p)$, we denote the corresponding reduced counting function.

In an analogous manner, we can define $N(r,a;f| \geqslant p)$ and $\overline{N}(r,a;f| \geqslant p)$.

To begin the statement of our results, we recall the following theorems, which are proved by K. Liu, X. L. Liu, and T. B. Cao [14] in 2011.

THEOREM 1. *Let f and g be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. If $n \geqslant 14$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 CM, then $f \equiv tg$ or $f.g \equiv t$, where $t^{n+1} = 1$.*

THEOREM 2. *Let f and g be two transcendental meromorphic functions with finite order. Suppose that $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. If $n \geqslant 26$, $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 IM, then $f \equiv tg$ or $f.g \equiv t$, where $t^{n+1} = 1$.*

In 2015, Y. Liu, J. P. Wang, and F. H. Liu [15] improved Theorems 1 and 2 and obtained the following results.

THEOREM 3. *Let $c \in \mathbb{C} \setminus \{0\}$ and let f and g be two transcendental meromorphic functions with finite order, and $n(\geqslant 14)$, $k(\geqslant 3)$ be two positive integers. If $E_k(1, f^n(z)f(z+c)) = E_k(1, g^n(z)g(z+c))$, then $f \equiv t_1 g$ or $fg \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

THEOREM 4. *Let $c \in \mathbb{C} \setminus \{0\}$ and let f and g be two transcendental meromorphic functions with finite order, and $n(\geqslant 16)$ be a positive integer. If $E_2(1, f^n(z)f(z+c)) = E_2(1, g^n(z)g(z+c))$, then $f \equiv t_1 g$ or $fg \equiv t_2$, for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

THEOREM 5. *Let $c \in \mathbb{C} \setminus \{0\}$ and let f and g be two transcendental meromorphic functions with finite order, and $n(\geqslant 26)$ be a positive integer. If $E_1(1, f^n(z)f(z+c)) = E_1(1, g^n(z)g(z+c))$, then $f \equiv t_1 g$ or $fg \equiv t_2$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

We define shift and difference operators of $f(z)$ by $f(z+c)$ and $\Delta_c f(z) = f(z+c) - f(z)$, respectively. Note that $\Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z))$, where c is a nonzero complex number and $n \geqslant 2$ is a positive integer.

For further generalization of $\Delta_c(f)$, we now define the linear difference operator of an entire (meromorphic) function f as $L_c(f) = f(z+c) + c_0 f(z)$, where c_0 is a finite complex constant. Clearly, for the particular choice of the constant $c_0 = -1$, we get $L_c(f) = \Delta_c(f)$.

As Theorems 1, 2, 3, 4, and 5 deal with only the shared value of 1 CM for the shift operator, it would be desirable to explore the problem of the shared polynomial with finite weight for the difference operator as well as the linear difference operator of functions. Moreover, it becomes interesting to investigate how far the conclusions of Theorems 1, 2, 3, 4, and 5 hold for these functions and the various possibilities for $P(z)$ (where $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ is a non-zero polynomial). We prove the following theorems and corollaries in this regard:

THEOREM 6. (Main) *Let f and g be two transcendental meromorphic functions of hyper order $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let $c \in \mathbb{C} \setminus \{0\}$ be a complex constant such that $L_c(f) \not\equiv 0$ and $L_c(g) \not\equiv 0$. Let $p(z)$ be a non-zero polynomial with finite degree, n, m be two positive integers, and $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ be a non-zero polynomial. Suppose $f^n P(f) L_c(f) - p(z)$ and $g^n P(g) L_c(g) - p(z)$ share $(1, \kappa)$, and if one of the following conditions holds;*

(I) $\kappa \geqslant 2$ *and* $n > m + 22$.

(II) $\kappa = 1$ *and* $n > \frac{3}{2} m + 25$.

(III) $\kappa = 0$ *and* $n > 4m + 40$,

then one of the following conclusions holds;

1. *If $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ is a non-zero polynomial, then one of the following holds;*

   (a) $f = tg$, for a constant t such that $t^d = 1$, where $d = gcd\{n+1, n+2, \ldots, n+m+1-i, \ldots, n+m+1\}$.

   (b) f and g satisfying the algebraic equation $R(f, g) = 0$, where

$$R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) L_c(\omega_1) - \omega_2^n P(\omega_2) L_c(\omega_2).$$

2. *If $P(z)$ reduces to a non-zero monomial, namely $P(z) = a_i z^i \not\equiv 0$, for $i \in \{0, 1, 2, \ldots, m\}$, then one of the following holds;*

   (a) If $p(z)$ is a non-zero polynomial, then

$$a_i^2 f^{n+i} L_c(f) g^{n+i} L_c(g) \equiv p^2.$$

(b) If $p(z)$ is a non-zero constant d, then

(i) For $c_0 = 0$, $f = e^{\mathscr{U}}$ and $g = te^{-\mathscr{U}}$, where d and t are constants such that $t^{n+i+1} = d^2$ and $\mathscr{U}$ is a non-constant polynomial.

(ii) For $c_0 \neq 0$, $f = c_1 e^{\alpha(z)}$, $g(z) = c_2 e^{-\alpha(z)}$, where α, c_1, c_2 and d are non-zero constants satisfying

$$a_i^2 (c_1 c_2)^{n+i+1} (e^{\alpha c} + c_0)(e^{-\alpha c} + c_0) = d^2.$$

3. *If $P(z) = (z^m - 1)$, then one of the conclusion (1) holds, and if $c_0 = 0$, then $f \equiv tg$ for a constant t such that $t^m = 1$.*

4. *If $P(z) = (z-1)^m$ $(m \geqslant 2)$, then one of the conclusion (1) holds.*

5. *If $P(z) = x_0$, some constant, then $f^n L_c(f) \equiv g^n L_c(g)$ and if $c_0 = 0$, then $f = tg$, for a constant t such that $t^{n+1} = 1$.*

If $L_c(f) = \Delta_c(f)$, then one can easily get the following theorem.

THEOREM 7. *Under the assumptions of Theorem 6. Suppose $f^n P(f)\Delta_c(f) - p(z)$ and $g^n P(g)\Delta_c(g) - p(z)$ share $(1, \kappa)$, and if one of the following conditions holds;*

(I) $\kappa \geqslant 2$ and $n > m + 22$.

(II) $\kappa = 1$ and $n > \frac{3}{2}m + 25$.

(III) $\kappa = 0$ and $n > 4m + 40$,

then one of the following conclusions holds;

1. *If $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ is a non-zero polynomial, then one of the following holds;*

   (a) $f = tg$, for a constant t such that $t^d = 1$, where $d = gcd\{n+1, n+2, \ldots, n+m+1-i, \ldots, n+m+1\}$.

   (b) f and g satisfying the algebraic equation $R(f,g) = 0$, where

$$R(\omega_1, \omega_2) = \omega_1^n P(\omega_1)\Delta_c(\omega_1) - \omega_2^n P(\omega_2)\Delta_c(\omega_2).$$

2. *If $P(z)$ reduces to a non-zero monomial, namely $P(z) = a_i z^i \not\equiv 0$, for $i \in \{0, 1, 2, \ldots, m\}$, then one of the following holds;*

   (a) If $p(z)$ is a non-zero polynomial, then

$$a_i^2 f^{n+i} \Delta_c(f) g^{n+i} \Delta_c(g) \equiv p^2.$$

   (b) If $p(z)$ is a non-zero constant d, then $f = c_1 e^{\alpha(z)}$, $g(z) = c_2 e^{-\alpha(z)}$, where α, c_1, c_2 and d are non-zero constants satisfying

$$a_i^2 (c_1 c_2)^{n+i+1} (e^{\alpha c} + e^{-\alpha c} - 2) = -d^2.$$

3. *If* $P(z) = (z^m - 1)$, *then one of the conclusion* (1) *holds, and if* $\Delta_c = f(z+c)$, *then* $f \equiv tg$ *for a constant* t *such that* $t^m = 1$.

4. *If* $P(z) = (z-1)^m$ $(m \geqslant 2)$, *then one of the conclusion* (1) *holds.*

5. *If* $P(z) = x_0$, *some constant, then* $f^n \Delta_c(f) \equiv g^n \Delta_c(g)$ *and if* $\Delta_c = f(z+c)$, *then* $f = tg$, *for a constant* t *such that* $t^{n+1} = 1$.

It is well-established that for meromorphic functions f and g, they reduce to entire functions in the absence of poles. As a consequence, the following corollaries are apparent:

COROLLARY 1. *Let* f *and* g *be two transcendental entire functions of hyper order* $\rho_2(f) < 1$ *and* $\rho_2(g) < 1$. *Let* $c \in \mathbb{C} \setminus \{0\}$ *be a complex constant such that* $L_c(f) \not\equiv 0$ *and* $L_c(g) \not\equiv 0$. *Let* $p(z)$ *be a non-zero polynomial with finite degree,* n, m *be two positive integers, and* $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ *be a non-zero polynomial. Suppose* $f^n P(f) L_c(f) - p(z)$ *and* $g^n P(g) L_c(g) - p(z)$ *share* $(1, \kappa)$, *and if one of the following conditions holds;*

(I) $\kappa \geqslant 2$ *and* $n > m + 10$.

(II) $\kappa = 1$ *and* $n > \frac{3}{2}m + 25$.

(III) $\kappa = 0$ *and* $n > 4m + 40$,

then one of the conclusions of Theorem 6 holds.

COROLLARY 2. *Under the assumptions of Corollary 1. Suppose* $f^n P(f) \Delta_c(f) - p(z)$ *and* $g^n P(g) \Delta_c(g) - p(z)$ *share* $(1, \kappa)$, *and if one of the following conditions holds;*

(I) $\kappa \geqslant 2$ *and* $n > m + 10$.

(II) $\kappa = 1$ *and* $n > \frac{3}{2}m + 25$.

(III) $\kappa = 0$ *and* $n > 4m + 40$,

e conclusions of Theorem 7 holds.

EXAMPLE 1. Let $P(z) = a_m z^m$, $f(z) = e^z$, $g(z) = tf(z)$, where $t^{n+m+1} = 1$, $n, m \in \mathbb{N}$. Let c and $p(z)$ are non-zero complex constants, $c_0 = 0$, now it is easy to see that $f^n P(f) L_c(f)$ and $g^n P(g) L_c(g)$ share 1 CM. Clearly, f and g satisfy the conclusion (2) (b) (i) of Theorem 6. A simple computation shows that f and g satisfy the conclusion (2) (b) (i) of Theorem 7.

EXAMPLE 2. Let $P(z) = a_m z^m$, $f(z) = e^{2\pi z}$, $g(z) = t\frac{1}{f(z)}$, where $t^{n+m+1} = 1$, $n, m \in \mathbb{N}$ and c is a non-zero complex constant. Then $f^n P(f) f(z+c)$ and $g^n P(g) g(z+c)$ share 1 CM. Clearly, f and g satisfy the conclusion (2) (b) (i) of Theorem 6. A simple computation shows that f and g satisfy the conclusion (2) (b) (i) of Theorem 7.

EXAMPLE 3. Let $P(z)=a_mz^m$, $f(z)=e^z$, $g(z)=\frac{1}{f(z)}$, $p(z)=2$, $c=-\log(-1)$. Then $f^nP(f)(f(z+c)-f(z))$ and $g^nP(g)(g(z+c)-g(z))$ share 2 CM. Clearly, f and g satisfy the conclusion (2) (b) (i) of Theorem 6.

EXAMPLE 4. Let $P(z)=(z^m-1)$, c is a non-zero constant such that $e^c\neq 1$, $p(z)=1$. Suppose t is a non-zero constant such that $t^{n+1}=t^m=1$. Let $f(z)=e^z$, $g(z)=te^z$. Then $f^nP(f)(f(z+c))$ and $g^nP(g)(g(z+c))$ share 1 CM. Clearly, f and g satisfy the conclusion (3) of Theorem 6.

2. Auxiliary lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We define the function H as

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{1}$$

LEMMA 1. [27] *Let f be a non-constant meromorphic function and $P(f)=a_0+a_1f+a_2f^2+\cdots+a_nf^n$, where $a_0,a_1,a_2,\ldots,a_n$ are constants and $a_n\neq 0$. Then*

$$T(r,P(f))=nT(r,f)+O(1).$$

LEMMA 2. [5, 22] *Let f be a transcendental meromorphic function of hyper order $\rho_2(f)<1$, and let c be a non-zero complex constant. Then we have*

(i) $T(r,f(z+c))\leqslant T(r,f(z))+S(r,f)$,

(ii) $N(r,f(z+c))\leqslant N(r,f(z))+S(r,f)$,

(iii) $N\left(r,\frac{1}{f(z+c)}\right)\leqslant N(r,\frac{1}{f(z)})+S(r,f)$.

LEMMA 3. [13] *Let F and G be two non-constant meromorphic functions sharing $(1,2)$. Then one of the following cases holds:*

(i) $T(r)\leqslant N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+S(r)$,

(ii) $F=G$,

(iii) $F.G=1$, *where $T(r)$ denotes the maximum of $T(r,F)$ and $T(r,G)$ and $S(r)=o\{T(r)\}$ as $r\to\infty$, possibly outside a set of finite linear measure.*

LEMMA 4. [1] *Let F and G be two non-constant meromorphic functions sharing $(1,1)$ and $H\not\equiv 0$. Then*

$$T(r,F)\leqslant N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+\frac{1}{2}\overline{N}(r,F)$$
$$+S(r,F)+S(r,G).$$

LEMMA 5. [1] *Let F and G be two non-constant meromorphic functions sharing $(1,0)$ and $H \not\equiv 0$. Then*

$$T(r,F) \leqslant N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+2\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)$$
$$+2\overline{N}(r,F)+\overline{N}(r,G)+S(r,F)+S(r,G).$$

LEMMA 6. *Let f be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$, and let $c \in \mathbb{C}\setminus\{0\}$ be a complex constant, and n,m be two positive integers such that $L_c(f) \not\equiv 0$, where $L_c(f) = f(z+c)+c_0 f(z)$ is a linear difference operator, $P(z) = a_m z^m + a_{m-1}z^{m-1}+\cdots+a_1 z+a_0$ be a non-zero polynomial and $F(z) = f^n P(f)L_c(f)$, then*

$$(n+m-2)T(r,f)+S(r,f) \leqslant T(r,F) \leqslant (n+m+2)T(r,f)+S(r,f).$$

Proof. Since f is a meromorphic function with $\rho_2(f) < 1$. From Lemmas 1 and 2, we have

$$\begin{aligned} T(r,F) &= T\left(r, f^n P(f)L_c(f)\right)+S(r,f) \\ &\leqslant (n+m+2)T(r,f)+S(r,f). \end{aligned} \tag{2}$$

On the other hand, from Lemmas 1 and 2, we have

$$\begin{aligned} (n+m+1)T(r,f) &= T(r,f^{n+1}P(f))+S(r,f) \\ &\leqslant T(r,f^n P(f)L_c(f))+T\left(r,\frac{f}{L_c(f)}\right)+S(r,f) \\ &\leqslant T(r,F)+m\left(r,\frac{L_c(f)}{f}\right)+N\left(r,\frac{L_c(f)}{f}\right) \\ &\leqslant T(r,F)+3T(r,f)+S(r,f) \\ (n+m-2)T(r,f) &\leqslant T(r,F)+S(r,f). \end{aligned} \tag{3}$$

From (2) and (3), we can see that

$$(n+m-2)T(r,f)+S(r,f) \leqslant T(r,F) \leqslant (n+m+2)T(r,f)+S(r,f). \quad \square$$

LEMMA 7. [4] *Let f be a meromorphic function of finite order ρ and let c be a fixed nonzero complex constant. Then, for each $\varepsilon > 0$,*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\varepsilon}).$$

LEMMA 8. *Let f and g be two transcendental meromorphic functions of hyper order $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let $c \in \mathbb{C}\setminus\{0\}$ be a complex constant, and let n and m be two positive integers such that $n > m+11$. Suppose $L_c(f) \not\equiv 0$ and $L_c(g) \not\equiv 0$, let $F = f^n P(f)L_c(f) - p(z)$ and $G = g^n P(g)L_c(g) - p(z)$, where $p(z)$ is a non-zero polynomial with finite degree and $P(z)$ is defined as in Theorem 6. If $H \equiv 0$, then one of the following conclusions holds;*

1. *If* $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ *is a non-zero polynomial, then one of the following holds;*

   (a) $f = tg$, *for a constant* t *such that* $t^d = 1$, *where* $d = gcd\{n+1, n+2, \ldots, n+m+1-i, \ldots, n+m+1\}$.

   (b) f *and* g *satisfying the algebraic equation* $R(f,g) = 0$, *where*

$$R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) L_c(\omega_1) - \omega_2^n P(\omega_2) L_c(\omega_2).$$

2. *If* $P(z)$ *reduces to a non-zero monomial, namely* $P(z) = a_i z^i \not\equiv 0$, *for* $i \in \{0, 1, 2, \ldots, m\}$, *then one of the following holds;*

   (a) *If* $p(z)$ *is a non-zero polynomial, then*

$$a_i^2 f^{n+i} L_c(f) g^{n+i} L_c(g) \equiv p^2.$$

   (b) *If* $p(z)$ *is a non-zero constant* d, *then*

   (i) *For* $c_0 = 0$, $f = e^{\mathscr{U}}$ *and* $g = te^{-\mathscr{U}}$, *where* d *and* t *are constants such that* $t^{n+i+1} = d^2$ *and* $\mathscr{U}$ *is a non-constant polynomial.*

   (ii) *For* $c_0 \neq 0$, $f = c_1 e^{\alpha(z)}$, $g(z) = c_2 e^{-\alpha(z)}$, *where* α, c_1, c_2 *and* d *are non-zero constants satisfying*

$$a_i^2 (c_1 c_2)^{n+i+1} (e^{\alpha c} + c_0)(e^{-\alpha c} + c_0) = d^2.$$

3. *If* $P(z) = (z^m - 1)$, *then one of the conclusion* (1) *holds and if* $c_0 = 0$, *then* $f \equiv tg$ *for a constant* t *such that* $t^m = 1$.

4. *If* $P(z) = (z-1)^m$ $(m \geqslant 2)$, *then one of the conclusion* (1) *holds.*

5. *If* $P(z) = x_0$, *some constant, then* $f^n L_c(f) \equiv g^n L_c(g)$ *and if* $c_0 = 0$, *then* $f = tg$, *for a constant* t *such that* $t^{n+1} = 1$.

Proof. Since $H \equiv 0$. On Integration (1), we get

$$\frac{1}{F-1} \equiv \frac{bG + a - b}{G - 1}, \tag{4}$$

where a, b are constant and $a \neq 0$.

We now consider the following cases.

Case 1. Let $b \neq 0$ and $a \neq b$. If $b = -1$, then from (4), we have

$$F \equiv \frac{-a}{G - a - 1}.$$

Therefore

$$\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = 3\overline{N}(r, \infty; f).$$

So, from Lemma 6 and the second fundamental theorem, we get

$$T(r,G) \leqslant \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,a+1;G) + S(r,G),$$

which implies that

$$\begin{aligned}(n+m-2)T(r,g) &\leqslant (m+6)T(r,g) + 3T(r,f) + S(r,f) + S(r,g)\\ (n-8)T(r,g) &\leqslant 3T(r,f) + S(r,f) + S(r,g). \end{aligned} \tag{5}$$

Similarly, we can write

$$(n-8)T(r,f) \leqslant 3T(r,g) + S(r,f) + S(r,g). \tag{6}$$

By adding (5) and (6), we get

$$(n-11)\{T(r,f)+T(r,g)\} \leqslant S(r,f) + S(r,g),$$

which is a contradiction since $n > 11$.

If $b \neq -1$, from (4), we obtain that

$$F - \left(1 + \frac{1}{b}\right) \equiv \frac{-a}{b^2\left[G + \frac{a-b}{b}\right]}.$$

So,

$$\overline{N}\left(r, \frac{(b-a)}{b}; G\right) = \overline{N}(r,\infty;F) = 3\overline{N}(r,\infty;f).$$

By using the same argument as in the case when $b = -1$, we can get a contradiction.

Case 2. Let $b \neq 0$ and $a = b$. If $b \neq -1$, from (4), we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G-1}.$$

Therefore,

$$\overline{N}\left(r, \frac{1}{1+b}; G\right) \equiv \overline{N}(r,0;F).$$

So, in view of Lemma 6 and the second fundamental theorem and by using the same argument as in Case 1, we get a contradiction as $n > m+11$.

Therefore, $a = b$ and $b = -1$, then from (4), we have $F.G \equiv 1$, i.e.,

$$(f^n P(f) L_c(f)).(g^n P(g) L_c(g)) \equiv p^2(z). \tag{7}$$

Suppose $P(z)$ is not a non-zero monomial. For the sake of simplicity, let $P(z) = (z-a)$, where $a \in \mathbb{C} \setminus \{0\}$, clearly $\Theta(0,f) + \Theta(a,f) = 2$, which is impossible. Thus

$P(z)$ reduces to a non-zero monomial, namely $P(z)=a_iz^i\neq 0$ for some $i\in\{0,1,2,\ldots,m\}$, and so (7) reduces to

$$a_i^2 f^{n+i}L_c(f)g^{n+i}L_c(g)\equiv p^2. \tag{8}$$

Now, let $p(z)$ be a non-zero constant d. In this case, we see that $f(z)$ and $g(z)$ have no zeros, and so we can take $f(z)$ and $g(z)$ as follows:

$$f(z)=e^{\alpha(z)},\quad g(z)=e^{\beta(z)}, \tag{9}$$

where α and β are non-constant polynomials.

Now, let $p(z)$ be a non-zero constant d, and from (8) and (9), we obtain

$$a_i^2\left[e^{(n+i)\alpha(z)}\left(e^{\alpha(z+c)}+c_0e^{\alpha(z)}\right)\right]\cdot\left[e^{(n+i)\beta(z)}\left(e^{\beta(z+c)}+c_0e^{\beta(z)}\right)\right]\equiv d^2, \tag{10}$$

which implies that

$$a_i^2e^{(n+i)(\alpha+\beta)}\left[e^{\alpha(z+c)+\beta(z+c)}+c_0e^{\alpha(z+c)+\beta(z)}+c_0e^{\alpha(z)+\beta(z+c)}+c_0^2e^{\alpha+\beta}\right]\equiv d^2(z). \tag{11}$$

Keeping in view of (9), we must have

$$(n+i)(\alpha+\beta)+\alpha(z+c)+\beta(z+c)=\mathscr{A}_1, \tag{12}$$
$$(n+i)(\alpha+\beta)+\alpha(z)+\beta(z+c)=\mathscr{A}_2, \tag{13}$$
$$(n+i)(\alpha+\beta)+\alpha(z+c)+\beta(z)=\mathscr{A}_3, \tag{14}$$
$$(n+i+1)(\alpha+\beta)=\mathscr{A}_4, \tag{15}$$

where $\mathscr{A}_1,\mathscr{A}_2,\mathscr{A}_3,\mathscr{A}_4$ are constants. Let $\alpha(z)+\beta(z)=w.$ Then (12) can be written as

$$(n+i)w(z)+w(z+c)=\mathscr{A}_1, \tag{16}$$

for all $z\in\mathbb{C}$. Therefore, from (15), we must have $w=B$, where B is a constant, and therefore, we have

$$\beta=B-\alpha. \tag{17}$$

Keeeping in view of (17), (9) can be written as

$$f(z)=e^{\alpha(z)},\quad g(z)=e^{B}e^{-\alpha(z)}. \tag{18}$$

Now, (11) can be written as

$$a_i^2\left[e^{\mathscr{A}_4}+c_0e^{\mathscr{A}_3}+c_0e^{\mathscr{A}_2}+c_0^2e^{\mathscr{A}_4}\right]\equiv d^2(z). \tag{19}$$

For a non-constant polynomial $\mathscr{U}$, (18) can be written as

$$f(z)=e^{\mathscr{U}},\quad g(z)=e^{B}e^{-\mathscr{U}}. \tag{20}$$

Using the above forms of f and g and keeping in mind that p is a constant, say d, (8) reduces to

$$a_i^2 e^{B(n+1+i)}\left(e^{\mathscr{U}(z+c)-\mathscr{U}(z)}+c_0\right)\left(e^{-(\mathscr{U}(z+c)-\mathscr{U}(z))}+c_0\right)\equiv d^2. \tag{21}$$

If $c_0=0$, (21) reduces to $e^{(n+i+1)B}=d^2$. Set $e^B=t$. Then (20) can be written as $f(z)=e^{\mathscr{U}}$, $g(z)=te^{-\mathscr{U}}$, where t is a constant such that $t^{n+i+1}=d^2$.

If $c_0\neq 0$, then from (21), one can say that $e^{\mathscr{U}(z+c)-\mathscr{U}(z)}+c_0$ has no zeros. Then $\Phi(z)=e^{\mathscr{U}(z+c)-\mathscr{U}(z)}\neq 0,\ -c_0,\ \infty$. By Picard's theorem, Φ is a constant, and so $\deg(\mathscr{U}(z))=1$. Therefore, from (20), we may obtain $f(z)=c_1e^{\alpha z}$, $g(z)=c_2e^{-\alpha z}$, where α, c_1 and c_2 are non-zero constants. Using these in (8), we obtain

$$a_i^2(c_1c_2)^{(n+1+i)}\left(e^{\alpha c}+c_0\right)\left(e^{-\alpha c}+c_0\right)\equiv d^2.$$

Case 3. Let $b=0$. From (4), we obtain

$$F\equiv\frac{G+a-1}{a}. \tag{22}$$

If $a\neq 1$, then from (22), we obtain

$$\overline{N}(r,1-a;G)=\overline{N}(r,0;F).$$

We can similarly deduce a contradiction as in Case 2. Therefore $a=1$ and from (22), we obtain $F\equiv G$. i.e.,

$$f^nP(f)L_c(f)\equiv g^nP(g)L_c(g). \tag{23}$$

Subcase 3.1. When $P(z)$ is a non-zero polynomial as in Theorem 6.
Let $h=\frac{f}{g}$. If h is a constant, then substituting $f=gh$ in (23), we get

$$a_mg^m\left(h^{n+m+1}-1\right)+a_{m-1}g^{m-1}\left(h^{n+m}-1\right)+\cdots+a_0\left(h^{n+1}-1\right)\equiv 0,$$

which implies that $h^d=1$, where d is the gcd of the elements of $J=\{p\in I: a_p\neq 0\}$ and $I=\{n+1,n+2,\ldots,n+m+1-i,\ldots,n+m+1\}$.

Thus $f=tg$ for a constant t such that $t^d=1$, where d is the GCD of the elements of $J=\{p\in I: a_p\neq 0\}$ and $I=\{n+1,n+2,\ldots,n+m+1-i,\ldots,n+m+1\}$, $i\in\{0,1,2,\ldots,m\}$.

If h is not a constant, then we know that by (23) that f and g satisfy the algebraic equation $R(f,g)=0$, where $R(\omega_1,\omega_2)=\omega_1^nP(\omega_1)L_c(\omega_1)-\omega_2^nP(\omega_2)L_c(\omega_2)$. We now discuss the following subcases,

Subcase 3.2. When $P(z)=z^m-1$. Then from (23), we have

$$f^n(f^m-1)L_c(f)\equiv g^n(g^m-1)L_c(g). \tag{24}$$

Let $h=\frac{f}{g}$, now the result follows from Subcase 3.1.

Suppose $c_0 = 0$, then (24) becomes

$$f^n(f^m - 1)f(z+c) \equiv g^n(g^m - 1)g(z+c). \tag{25}$$

Let $h = \frac{f}{g}$, then from (25), we have

$$g^n h^n(g^m h^m - 1)g(z+c)h(z+c) \equiv g^n(g^m - 1)g(z+c),$$

which implies that

$$g^m = \frac{h^n h(z+c) - 1}{h^{n+m}h(z+c) - 1}. \tag{26}$$

If 1 is a Picard exceptional value of $h^{n+m}h(z+c)$, by applying Nevanlinna's second fundamental theorem with Lemma 2, we get

$$\begin{aligned} T(r, h^{n+m}h(z+c)) &\leqslant \overline{N}(r,\infty; h^{n+m}h(z+c)) + \overline{N}(r,0; h^{n+m}h(z+c)) \\ &\quad + \overline{N}(r,1; h^{n+m}h(z+c)) \\ &\leqslant 4T(r,h) + S(r,h). \end{aligned} \tag{27}$$

On the other hand, combining the standard Valiron-Mohon'ko theorem with (27) and Lemma 2, we get

$$\begin{aligned} (n+m)T(r,h) &= T(r, h^{n+m}) + S(r,h) \\ &\leqslant T(r, h^{n+m}h(z+c)) + T(r, h(z+c)) + S(r,h) \\ &\leqslant 5T(r,h) + S(r,h), \end{aligned}$$

which contradicts since $n > 5 - m$.

Therefore, 1 is not a Picard exceptional value of $h^{n+m}h(z+c)$. Thus there exists z_0 such that $h(z_0)^{n+m}h(z_0+c) = 1$ then by (26), we have $h(z_0)^n h(z_0+c) = 1$. Hence $h(z_0)^m = 1$, and

$$\begin{aligned} \overline{N}(r,0; h^{n+m}h(z+c) - 1) &\leqslant \overline{N}(r,0; h^m - 1) \\ &\leqslant mT(r,h) + O(1). \end{aligned} \tag{28}$$

Denote,

$$M(z) = h(z)^{n+m}h(z+c). \tag{29}$$

We have $T(r,M) \leqslant (n+m+1)T(r,h) + S(r,h)$. Applying Nevanlinna's second fundamental theorem to M and using Lemma 2 and (28), we get

$$\begin{aligned} T(r,M) &\leqslant \overline{N}(r,0;M) + \overline{N}(r,\infty;M) + \overline{N}(r,1;M) + S(r,M) \\ &\leqslant \overline{N}(r,0;M) + \overline{N}(r,\infty;M) + mT(r,h) + S(r,h) \\ &\leqslant (m+4)T(r,h) + S(r,h). \end{aligned} \tag{30}$$

On the other hand, using (29) and (30), we have

$$\begin{aligned}(n+m)T(r,h) &= T(r,h^{n+m})+S(r,h)\\ &\leqslant T(r,M)+T(r,h(z+c))+S(r,h)\\ &\leqslant (m+5)T(r,h)+S(r,h),\end{aligned}$$

which is a contradiction since $n>5$.

Therefore, $h^{n+m}h(z+c)\equiv 1$, and $h^n h(z+c)\equiv 1$. Thus $h^m\equiv 1$. Hence we get $f=tg$, where $t^m=1$.

Subcase 3.3. When $P(z)=(z-1)^m$, then from (23), we have

$$f^n(f-1)^m L_c(f)\equiv g^n(g-1)^m L_c(g). \tag{31}$$

Let $h=\frac{f}{g}$. If $m=1$, then the result follows from Subcase 3.2.

For $m\geqslant 2$, first we suppose that h is not a constant, then from (31), we can say that f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(w_1,w_2)=w_1^n(w_1-1)^m L_c(w_1)-w_2^n(w_2-1)^m L_c(w_2).$$

Next, we suppose that h is a constant, then from (31), we get

$$f^n L_c(f)\sum_{i=0}^{m}(-1)^i\binom{m}{m-i}f^{m-i}=g^n L_c(g)\sum_{i=0}^{m}(-1)^i\binom{m}{m-i}g^{m-i}. \tag{32}$$

Now, substituting $f=gh$ in (32), we get

$$\sum_{i=0}^{m}(-1)^i\binom{m}{m-i}g^{m-i}\left[h^{n+m-i+1}-1\right]\equiv 0,$$

which implies that $h^d=1$, where $d=gcd\{n+1,n+2,\ldots,n+m+1-i,\ldots,n+m+1\}$.

Thus $f=tg$ for a constant t such that $t^d=1$, where $d=gcd\{n+1,n+2,\ldots,n+m+1-i,\ldots,n+m+1\}$.

Subcase 3.4. When $P(z)=x_0$, some constant. Then from (23), we have

$$f^n L_c(f)\equiv g^n L_c(g). \tag{33}$$

Suppose $c_0=0$, then from (33),

$$f^n f(z+c)\equiv g^n g(z+c). \tag{34}$$

Let $h=\frac{f}{g}$. Then by (34), we have

$$h^n(z)=\frac{1}{h(z+c)}. \tag{35}$$

Now by Lemmas 1, 2, and 7, we get

$$
\begin{aligned}
nT(r,h) = T(r,h^n) + S(r,h) = T\left(r, \frac{1}{h(z+c)}\right) + S(r,h) \\
&\leqslant N(r,0;h(z+c)) + m\left(r, \frac{1}{h(z+c)}\right) + S(r,h) \\
&\leqslant N(r,0;h(z)) + m\left(r, \frac{1}{h(z)}\right) + S(r,h) \\
&\leqslant T(r,h(z)) + S(r,h),
\end{aligned}
$$

which is a contradiction since $n > 1$. Therefore, if h is a constant, then substituting $f = gh$ into (34), we get $f = tg$ for a constant t such that $t^{n+1} = 1$. □

3. Proof of the main theorems

Proof of Theorem 6. Let $F = \frac{f^n(z)P(f)L_c(f)}{p(z)}$ and $G = \frac{g^n(z)P(g)L_c(g)}{p(z)}$. Then F and G share $(1,\kappa)$ except the zeros and poles of $p(z)$. Now we consider the following cases.

Case 1. Let H be defined as above. Suppose that $H \not\equiv 0$.

Subcase 1.1. Let $\kappa \geqslant 2$. Let us assume (i) of Lemma 3 holds. i.e.,

$$
T(r,F) \leqslant N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G). \tag{36}
$$

By Lemma 6, we obtain

$$
\begin{aligned}
(n+m-2)T(r,f) &\leqslant N_2\left(r, \frac{1}{f^n(z)P(f)L_c(f)}\right) + N_2\left(r, \frac{1}{g^n(z)P(g)L_c(g)}\right) \\
&\quad + 2\overline{N}(r, f^n(z)P(f)L_c(f)) + 2\overline{N}(r, g^n(z)P(g)L_c(g)) \\
&\quad + S(r,f) + S(r,g) \\
&\leqslant 2\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{P(f)}\right) + N\left(r, \frac{1}{L_c(f)}\right) + 2\overline{N}\left(r, \frac{1}{g}\right) \\
&\quad + N\left(r, \frac{1}{P(g)}\right) + N\left(r, \frac{1}{L_c(g)}\right) + 6\overline{N}(r,f) + 6\overline{N}(r,g) \\
&\quad + S(r,f) + S(r,g) \\
&\leqslant (m+10)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g).
\end{aligned} \tag{37}
$$

Similarly, we can write

$$
(n+m-2)T(r,g) \leqslant (m+10)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \tag{38}
$$

Now adding (37) and (38), we have

$$
(n-m-22)\{T(r,f) + T(r,g)\} \leqslant S(r,f) + S(r,g),
$$

which is a contradiction since $n > m+22$.

Subcase 1.2. Let $\kappa = 1$. Using Lemmas 4 and 6, we obtain

$$T(r,F) \leqslant N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + \frac{1}{2}\overline{N}(r,F) + S(r,F) + S(r,G),$$

which implies that

$$\begin{aligned}(n+m-2)T(r,f) \leqslant\ & N_2\left(r,\frac{1}{f^n(z)P(f)L_c(f)}\right) + N_2\left(r,\frac{1}{g^n(z)P(g)L_c(g)}\right) \\ & + 2\overline{N}(r,f^n(z)P(f)L_c(f)) + 2\overline{N}(r,g^n(z)P(g)L_c(g)) \\ & + \frac{1}{2}\overline{N}\left(r,\frac{1}{f^n(z)P(f)L_c(f)}\right) + \frac{1}{2}\overline{N}(r,f^n(z)P(f)L_c(f)) \\ & + S(r,f) + S(r,g) \\ \leqslant\ & 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{P(f)}\right) + N\left(r,\frac{1}{L_c(f)}\right) + 2\overline{N}\left(r,\frac{1}{g}\right) \\ & + N\left(r,\frac{1}{P(g)}\right) + N\left(r,\frac{1}{L_c(g)}\right) + 6\overline{N}(r,f) + 6\overline{N}(r,g) \\ & + \frac{1}{2}\left[\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{P(f)}\right) + N\left(r,\frac{1}{L_c(f)}\right)\right] + \frac{3}{2}\overline{N}(r,f) \\ & + S(r,f) + S(r,g) \\ \leqslant\ & \left(\frac{3}{2}m+13\right)T(r,f) + (m+10)T(r,g) + S(r,f) + S(r,g). \qquad (39)\end{aligned}$$

Similarly, we can write

$$(n+m-2)T(r,g) \leqslant \left(\frac{3}{2}m+13\right)T(r,g) + (m+10)T(r,f) + S(r,f) + S(r,g). \qquad (40)$$

Now adding (39) and (40), we get

$$\left(n-\frac{3}{2}m-25\right)\{T(r,f)+T(r,g)\} \leqslant S(r,f) + S(r,g),$$

which is a contradiction since $n > \frac{3}{2}m+25$.

Subcase 1.3. Let $\kappa = 0$. Using Lemmas 5 and 6, we obtain

$$T(r,F) \leqslant N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + 2\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + 2\overline{N}(r,F) + \overline{N}(r,G) + S(r,F) + S(r,G),$$

which implies that

$$\begin{aligned}(n+m-2)T(r,f) \leqslant\ & N_2\left(r,\frac{1}{f^n(z)P(f)L_c(f)}\right)+N_2\left(r,\frac{1}{g^n(z)P(g)L_c(g)}\right)\\&+2\overline{N}(r,f^n(z)P(f)L_c(f))+2\overline{N}(r,g^n(z)P(g)L_c(g))\\&+2\overline{N}\left(r,\frac{1}{f^n(z)P(f)L_c(f)}\right)+\overline{N}\left(r,\frac{1}{g^n(z)P(g)L_c(g)}\right)\\&+2\overline{N}(r,f^n(z)P(f)L_c(f))+\overline{N}(r,g^n(z)P(g)L_c(g))+S(r,f)\\&+S(r,g)\\ \leqslant\ & 4\overline{N}\left(r,\frac{1}{f}\right)+3N\left(r,\frac{1}{P(f)}\right)+3N\left(r,\frac{1}{L_c(f)}\right)+3\overline{N}\left(r,\frac{1}{g}\right)\\&+2N\left(r,\frac{1}{P(g)}\right)+2N\left(r,\frac{1}{L_c(g)}\right)+12\overline{N}(r,f)+9\overline{N}(r,g)\\&+S(r,f)+S(r,g)\\ \leqslant\ & (3m+22)T(r,f)+(2m+16)T(r,g)+S(r,f)+S(r,g). \qquad (41)\end{aligned}$$

Similarly, we can write

$$(n+m-2)T(r,g) \leqslant (3m+22)T(r,f)+(2m+16)T(r,g)+S(r,f)+S(r,g). \qquad (42)$$

Now, adding (41) and (42), we get

$$(n-4m-40)\{T(r,f)+T(r,g)\} \leqslant S(r,f)+S(r,g),$$

which is a contradiction since $n > 4m+40$.

Case 2. Suppose $H \equiv 0$. Now the result follows directly from Lemma 8. □

Proof of Theorem 7. The proof of Theorem 7 is straightforward, similar to the proof of Theorem 6. □

Proof of Corollary 1 *and* 2. Since f and g are entire functions, we can take $N(r,f) = S(r,f)$ and $N(r,g) = S(r,g)$. Now the proof is straightforward, similar to the proof of Theorem 6. □

For further study, one could use the following open questions:

OPEN QUESTIONS.

1. Is there another way that reduces the value of n in Theorems 6–7 more significantly?
2. If $f^nP(f)L_c(f)$ and $g^nP(g)L_c(g)$ share any set S instead of a non-zero polynomial $p(z)$, then can the conclusion of Theorem 6 be obtained?

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