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PROPERTIES FOR THE CLASS OF UNIVALENT FUNCTIONS DEFINED BY DOUBLE SUBORDINATION ASSOCIATED WITH PETAL SHAPED DOMAIN

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Abstract. In this paper, with the help of double subordination the authors introduce a new subclass of analytic univalent functions $\mathscr{C}_{\mathfrak{q}}^{*}$ defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and examine the initial bounds for the first four coefficients of functions that belong to this newly defined class. Furthermore, we investigate the upper bounds of Fekete-Szegö inequality, Hankel determinant of different orders, Zalcman conjecture, logarithmic coefficients and Inverse coefficients for such family. Our results are new as per the existing literature.

1. Introduction and motivation

Let $\mathfrak{A}$ represents the family of all holomorphic functions of the form:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. By $\mathscr{S}$, we mean the subclass of $\mathfrak{A}$ which are univalent in $\mathbb{D}$ and normalized by $h(0) = h'(0) - 1 = 0$. For $f_1,\ f_2 \in \mathfrak{A}$, we say f_1 is said to be subordinate to f_2, denoted by $f_1 \prec f_2$ $\big($or, more precisely $f_1(z) \prec f_2(z)\big)$ if there exists a Schwarz function β analytic in $\mathbb{D}$ with $\beta(0) = 0$ and $|\beta(z)| < 1$ for all $z \in \mathbb{D}$, such that $f_1(z) = f_2(\beta(z))$ $(z \in \mathbb{D})$. If f_2 is univalent in $\mathbb{D}$, then

$$f_1(z) \prec f_2(z) \Leftrightarrow f_1(0) = f_2(0) \text{ and } f_1(\mathbb{D}) \subset f_2(\mathbb{D}). \tag{2}$$

The well-investigated subfamilies of univalent functions i.e class of starlike and convex functions defined as follows,

$$\mathscr{S}^* := \left\{ h \in \mathfrak{A} : \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) > 0,\ z \in \mathbb{D} \right\},$$
$$\mathscr{K} := \left\{ h \in \mathfrak{A} : \operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > 0,\ z \in \mathbb{D} \right\}.$$

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In term of subordination Ma and Minda (see [16]) discussed the more general class $\mathcal{S}^*(\varphi)$ by

$$\mathcal{S}^*(\varphi) := \left\{ h \in \mathfrak{A} : \frac{zh'(z)}{h(z)} \prec \varphi(z) \right\},$$

where φ satisfied the condition stated in [16] i.e φ is a regular function with $\operatorname{Re}(\varphi(z)) > 0$ in $\mathbb{D}$ together with gives value one at origin and positive first derivative at origin. In addition, the function φ maps $\mathbb{D}$ onto a star-shaped region with respect to $\varphi(0) = 1$ and is symmetric with respect to the real axis. Taking particular function φ, we can derive different subclasses of $\mathcal{S}^*(\varphi)$, and for details see [6] and reference within.

Motivated by the works of above researchers, we will consider the non-vanishing analytic functions φ and q in $\mathbb{D}$ that satisfy the condition $\varphi(0) = q(0) = 1$. In this paper we define the class of functions $h \in \mathfrak{A}$ that fulfill the following condition

$$\frac{1 + \frac{zh''(z)}{h'(z)}}{q(z)} \prec \varphi(z),$$

where q and φ need to be specific functions. Now, we introduce the following class.

DEFINITION 1. A function $h \in \mathfrak{A}$ given by (1) is belong to the class $\mathcal{C}_{\mathfrak{q}}^*$ defined by

$$\mathcal{C}_{\mathfrak{q}}^* := \left\{ h \in \mathfrak{A} : \frac{1 + \frac{zh''(z)}{h'(z)}}{\mathfrak{q}(z)} \prec 1 + \ln(z + \sqrt{z^2+1}),\ \mathfrak{q}(z) \prec 1 + z - \frac{z^3}{3} \right\}. \tag{3}$$

REMARK 1. (i) It may be noted that the function q defined above should be analytic in $\mathbb{D}$ with $q(0) = 1$ such that the $\dfrac{1 + \frac{zh''(z)}{h'(z)}}{\mathfrak{q}(z)}$ is also analytic in $\mathbb{D}$.

(ii) $\ln(z + \sqrt{z^2+1}) = \sinh^{-1} z$, is a multi-valued function and has the branch cuts along the line segment $(-i\infty, -i) \cup (i, i\infty)$ on the imaginary axis and hence analytic in $\mathbb{D}$.

(iii) If we take $w = \frac{1 + \frac{zh''(z)}{h'(z)}}{q(z)}$ where $h \in \mathcal{C}_{\mathfrak{q}}^*$ where the class $\mathcal{C}_{\mathfrak{q}}^*$ can be written as $e^{w-1} \prec z + \sqrt{z^2+1}$ where $z + \sqrt{z^2+1}$ represents the Crescent shaped domain (see [21]).

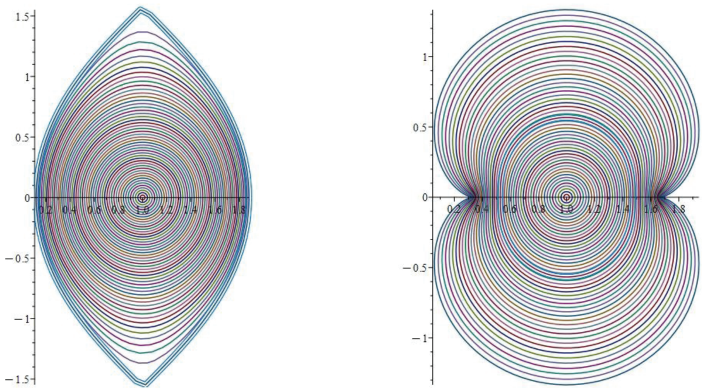
REMARK 2. (i) If we let $\varphi(z) := 1 + \ln(z + \sqrt{z^2+1})$, then $\varphi(0) = 1$ and $\varphi'(z) = \frac{1}{\sqrt{z^2+1}}$, hence $\varphi'(0) = 1 \neq 0$. Further,

$$\frac{z\varphi'(z)}{\varphi(z) - \varphi(0)} = \frac{z\varphi'(z)}{\varphi(z) - 1} = \frac{z\left(\frac{1}{\sqrt{z^2+1}}\right)}{\ln(z + \sqrt{z^2+1})}, \quad z \in \mathbb{D}.$$

We can see in the Figure 1(a) that

$$\mathrm{P}(z) := \mathrm{Re}\left(\frac{z\left(\frac{1}{\sqrt{z^2+1}}\right)}{\log\left(z+\sqrt{z^2+1}\right)}\right) > 0, \quad z \in \mathbb{D},$$

and combined with $\varphi'(0) = 1 \neq 0$ it follows that $\varphi(z) = 1 + \ln\left(z + \sqrt{z^2+1}\right)$ is a starlike (univalent) function in $\mathbb{D}$. Moreover, since $\varphi(\overline{z}) = \overline{\varphi(z)}$, $z \in \mathbb{D}$, the domain $\varphi(\mathbb{D})$ is symmetric with respect to the real axis.



(a) *The visual of* $\mathrm{P}\left(re^{it}\right)$, $r \in [0,1]$, $t \in [0, 2\pi)$

(b) *The visual of* $p(\mathbb{D})$

Figure 1: *Figures for the Remark 2 (i) and (ii)*

(ii) The function $q(z) := 1 + z - \frac{z^3}{3}$ is also correctly choose because $q'(z) = (1 - z^2)$, hence $q'(0) = 1 \neq 0$. Also, we observe that

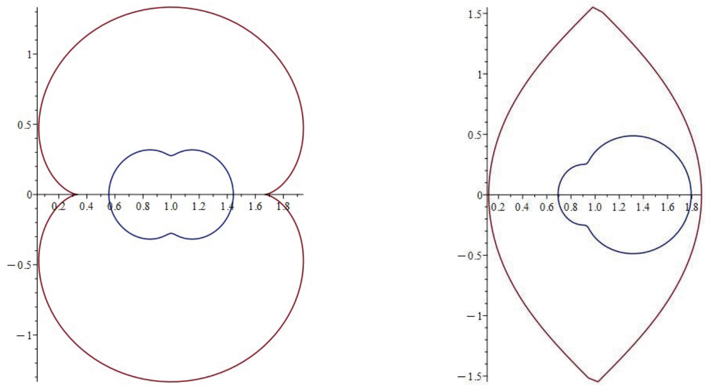
$$\mathrm{Re}\,\frac{zq'(z)}{q(z) - q(0)} = \mathrm{Re}\,\frac{zq'(z)}{q(z) - 1} = \mathrm{Re}\left(\frac{1 - z^2}{1 - z^2/3}\right) > 0, \quad z \in \mathbb{D},$$

and using this fact together with $q'(0) = 1 \neq 0$ it follows that $q(z) = 1 + z - \frac{z^3}{3}$ is also a starlike (univalent) function in $\mathbb{D}$. Moreover, from the Figure 1(b) we can see that $q(z) \neq 0$ for all $z \in \mathbb{D}$.

(iii) The class is well choose if we could prove that it isn't empty. First, from the above items (i) and (ii) both of the functions $\varphi(z) = 1 + \ln\left(z + \sqrt{z^2+1}\right)$ and $q(z) = 1 + z - \frac{z^3}{3}$ are univalent in $\mathbb{D}$. To show that $\mathscr{C}^*_{\mathfrak{r}} \neq \emptyset$ for some appropriate choices of $\mathfrak{r}$, let consider the functions

$$h(z) := z + 8 \cdot 10^{-8} z^2 \in \mathfrak{A}, \quad \mathfrak{r}(z) := 1 + 0.36z + 0.0833333333z^3.$$

(a) From the Figure 2(a) we see that $\mathfrak{r}(\mathbb{D}) \subset q(\mathbb{D})$, and using that $\mathfrak{r}(0) = q(0) = 1$ together with the fact that q is univalent in $\mathbb{D}$ which was previously proved, from the equivalence (2) we deduce that $\mathfrak{r}(z) = 1 + 0.36z + 0.0833333333z^3 \prec q(z)$.



(a) *The images of* $\mathfrak{r}\left(e^{it}\right)$ *(blue color) and* $q\left(e^{it}\right)$ *(red color),* $t \in [0, 2\pi)$

(b) *The images of* $R\left(e^{it}\right)$ *(blue color) and* $\varphi\left(e^{it}\right)$ *(red color),* $t \in [0, 2\pi)$

Figure 2: *Figures for the Remark 2 (iii)*

(b) If we denote

$$R(z) := \frac{1+\frac{zh''(z)}{h'(z)}}{\mathfrak{r}(z)} = \frac{1+\frac{1.6*10^{-7}z}{1.6*10^{-7}z+1}}{1+0.36z+0.0833333333z^3},$$

from the Figure 2(b) we get $R(\mathbb{D}) \subset \varphi(\mathbb{D})$, and using that $R(0) = \varphi(0) = 1$ and the above proved fact that φ is univalent in $\mathbb{D}$, from (2) we obtain $\dfrac{1+\frac{zh''(z)}{h'(z)}}{\mathfrak{r}(z)} \prec \varphi(z)$.

Concluding, from (a) and (b) we get that for $h(z) := z + 8\cdot 10^{-8}z^2 \in \mathfrak{A}$ we have

$$\mathfrak{r}(z) = 1+0.36z+0.0833333333z^3 \prec q(z) = 1+z-\frac{z^3}{3},$$

$$\frac{1+\frac{zh''(z)}{h'(z)}}{\mathfrak{r}(z)} \prec \varphi(z) = 1+\ln\left(z+\sqrt{z^2+1}\right),$$

where $\mathfrak{r}(0) = 1$ with $\mathfrak{r}(z) \neq 0$ in $\mathbb{D}$. Therefore, $h(z) := z+8\cdot 10^{-8}z^2 \in \mathfrak{A} \in \mathscr{C}_{\mathfrak{q}}$, hence there exist appropriate choices of the functions $\mathfrak{q}$ and h such that $h \in \mathscr{C}^*_{\mathfrak{q}}$, that is $\mathscr{C}^*_{\mathfrak{q}} \neq \emptyset$.

All the figures of this paper were made by using the MAPLE™ computer software.

Investigating the upper bounds of the moduli of Hankel determinants for different subclasses of analytic univalent functions remains an important topic in Geometric Function Theory. In 1976, Noonan and Thomas [18] presented the expression for the q-th Hankel determinant, for $q \geqslant 1$ and $n \geqslant 1$, for functions $h \in \mathfrak{A}$ of the form (1), as outlined below.

$$H_{q,n}(h) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \dots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1).$$

For example:

$$H_{2,1}(h) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \qquad H_{2,2}(h) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2 \tag{4}$$

$$H_{3,1}(h) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_5\left(a_3 - a_2^2\right) - a_4\left(a_4 - a_2a_3\right) + a_3\left(a_2a_4 - a_3^2\right). \tag{5}$$

The bounds of the third-order Hankel determinant $|H_{3,1}(h)|$ for the families of $\mathscr{S}^*$ of starlike functions, $\mathscr{K}$ of convex functions and $\mathscr{R}$ of functions with bounded turning were investigated for the first time by Babalola [4]. Recently, the sharp bounds of $|H_{3,1}(h)|$ of various subclasses of analytic univalent functions were obtained by several researchers. For details, see [3, 6, 11, 13, 14, 17, 22, 24, 26].

2. Preliminaries

Let $\mathscr{P}$ denote the classical Carathéodory class, which refers to the family of holomorphic functions l defined in $\mathbb{D}$ that satisfy the condition $\operatorname{Re}(l(z)) > 0$ for all $z \in \mathbb{D}$, and can be written in the following form.

$$l(z) = 1 + \sum_{n=1}^{\infty} l_n z^n, \quad z \in \mathbb{D}. \tag{6}$$

We need the following lemmas in order to prove our results.

LEMMA 1. (see [7, 9, 20]) *Let $l \in \mathscr{P}$ be of the form* (6). *Then*

$$|l_n| \leqslant 2 \quad (n \geqslant 1). \tag{7}$$

The inequality is sharp for the function $l(z) = \dfrac{1+\lambda z}{1-\lambda z}$, $|\lambda| = 1$.

LEMMA 2. [3, Lemma 2.2.] *If $l \in \mathscr{P}$ has the form* (6), *then*

$$\left|\alpha l_1^3 - \beta l_1 l_2 + \gamma l_3\right| \leqslant 2\left(|\alpha| + |\beta - 2\alpha| + |\alpha - \beta + \gamma|\right). \tag{8}$$

LEMMA 3. [1, Lemma 3, p. 66] *Let $l \in \mathscr{P}$ and has the expansion of the form* (6). *If $B \in [0,1]$ with $B(2B-1) \leqslant D < B$, then*

$$\left|l_3 - 2Bl_1l_2 + Dl_1^3\right| \leqslant 2. \tag{9}$$

LEMMA 4. [16] *If $l \in \mathscr{P}$ of the form* (6) *then for any $\mu \in \mathbb{C}$, we have*

$$|l_2 - \mu l_1^2| \leqslant 2\max\{1, |2\mu - 1|\}. \tag{10}$$

3. Initial coefficients estimates for the class $\mathscr{C}_{\mathfrak{q}}^{*}$

In this section we explore the upper bounds for the first four initial coefficients and for Fekete-Szegö functional $\left|a_3-\mu a_2^2\right|$ for the new defined function class $\mathscr{C}_{\mathfrak{q}}^{*}$. Our first main result deals with finding the upper bounds of the first four coefficients for the functions from $\mathscr{C}_{\mathfrak{q}}^{*}$.

THEOREM 1. *Let $h\in\mathfrak{A}$ specified by (1) is in the class $\mathscr{C}_{\mathfrak{q}}^{*}$. satisfies,*

$$|a_2|\leqslant 1,\quad |a_3|\leqslant\frac{5}{6},\quad |a_4|\leqslant\frac{2}{3},\quad |a_5|\leqslant\frac{28}{45}. \tag{11}$$

Proof. Assume that $h\in\mathscr{C}_{\mathfrak{q}}^{*}$. Hence by Definition 1 there exists a Schwarz function β with $\beta(0)=0$ and $|\beta(z)|<1$, $z\in\mathbb{D}$, such that

$$\frac{1+\frac{zh''(z)}{h'(z)}}{\mathfrak{q}(z)}=1+\ln\left(\beta(z)+\sqrt{\beta(z)^2+1}\right),\quad (z\in\mathbb{D}) \tag{12}$$

where

$$\mathfrak{q}(z)\prec 1+z-\frac{z^3}{3}. \tag{13}$$

The relation (12) is equivalent to

$$1+\frac{zh''(z)}{h'(z)}=\left(1+\log(\beta(z)+\sqrt{\beta(z)^2+1})\right)\mathfrak{q}(z)\quad (z\in\mathbb{D}), \tag{14}$$

while the subordination (13) implies that there exists another Schwarz function $\gamma(z)$ with $\gamma(0)=0$ and $|\gamma(z)|<1$, $z\in\mathbb{D}$, such that

$$\mathfrak{q}(z)=1+\gamma(z)-\frac{\gamma(z)^3}{3}\quad (z\in\mathbb{D}). \tag{15}$$

Expressing the Schwarz function β in terms of $d\in\mathscr{P}$, that is

$$d(z)=\frac{1+\beta(z)}{1-\beta(z)}=1+d_1z+d_2z^2+\ldots,\ \ z\in\mathbb{D},$$

which equality is equivalent to

$$\begin{aligned}\beta(z)=&\frac{1}{2}d_1z+\left(\frac{1}{2}d_2-\frac{1}{4}d_1^2\right)z^2+\left(\frac{1}{8}d_1^3-\frac{1}{2}d_1d_2+\frac{1}{2}d_3\right)z^3\\&+\left(\frac{1}{2}d_4-\frac{1}{2}d_1d_3-\frac{1}{4}d_2^2-\frac{1}{16}d_1^4+\frac{3}{8}d_1^2d_2\right)z^4+\ldots,\ \ z\in\mathbb{D}.\end{aligned}$$

Similarly, we can write the Schwarz function γ in terms of $e\in\mathscr{P}$, that is

$$e(z)=\frac{1+\gamma(z)}{1-\gamma(z)}=1+e_1z+e_2z^2+\ldots,\ \ z\in\mathbb{D},$$

or equivalently

$$\gamma(z)=\frac{1}{2}e_1z+\left(\frac{1}{2}e_2-\frac{1}{4}e_1^2\right)z^2+\left(\frac{1}{8}e_1^3-\frac{1}{2}e_1e_2+\frac{1}{2}e_3\right)z^3$$
$$+\left(\frac{1}{2}e_4-\frac{1}{2}e_1e_3-\frac{1}{4}e_2^2-\frac{1}{16}e_1^4+\frac{3}{8}e_1^2e_2\right)z^4+\ldots,\ \ z\in\mathbb{D}.$$

It follows that

$$\begin{aligned}&\left(1+\ln\left(\beta(z)+\sqrt{\beta(z)^2+1}\right)\right)\\&=1+\frac{1}{2}d_1z+\left(\frac{d_2}{2}-\frac{d_1^2}{4}\right)z^2+\left(\frac{1}{2}d_3+\frac{7}{48}d_1^3-\frac{1}{2}d_1d_2\right)z^3\\&\quad+\left(\frac{1}{2}d_4+\frac{7}{16}d_2d_1^2-\frac{1}{2}d_1d_3-\frac{1}{4}d_2^2-\frac{3}{32}d_1^4\right)z^4+\ldots,\quad(z\in\mathbb{D}),\end{aligned}\tag{16}$$

and

$$\begin{aligned}1+\gamma(z)-\frac{\gamma(z)^3}{3}&=1+\frac{1}{2}e_1z+\left(\frac{1}{2}e_2-\frac{1}{4}e_1^2\right)z^2+\left(\frac{e_3}{2}-\frac{1}{2}e_1e_2+\frac{e_1^3}{12}\right)z^3\\&\quad+\left(\frac{1}{2}e_4-\frac{1}{2}e_1e_3-\frac{1}{4}e_2^2+\frac{1}{4}e_2e_1^2\right)z^4+\ldots,\quad(z\in\mathbb{D}).\end{aligned}\tag{17}$$

Then multiplying the relations (16) and (17) we get

$$\begin{aligned}&\left(1+\ln(\beta(z)+\sqrt{\beta(z)^2+1})\right)\left(1+\gamma(z)-\frac{\gamma(z)^3}{3}\right)\\&=1+\left(\frac{e_1}{2}+\frac{d_1}{2}\right)z+\left(\frac{e_2}{2}+\frac{1}{4}d_1e_1+\frac{d_2}{2}-\frac{1}{4}d_1^2-\frac{1}{4}e_1^2\right)z^2\\&\quad+\left(\frac{7}{48}d_1^3+\frac{e_3}{2}+\frac{d_1e_2}{4}+\frac{e_1d_2}{4}-\frac{e_1d_1^2}{8}-\frac{d_1e_1^2}{8}+\frac{d_3}{2}+\frac{e_1^3}{12}-\frac{d_1d_2}{2}-\frac{e_1e_2}{2}\right)z^3\\&\quad+\left(-\frac{3}{32}d_1^4-\frac{e_1d_1d_2}{4}-\frac{d_1e_1e_2}{4}+\frac{e_2e_1^2}{4}+\frac{e_4}{2}+\frac{d_4}{2}-\frac{d_2^2}{4}-\frac{e_2^2}{4}\right.\\&\quad-\frac{d_1d_3}{2}-\frac{e_1e_3}{2}+\frac{7}{16}d_2d_1^2-\frac{d_1e_1^3}{24}+\frac{d_1e_3}{4}+\frac{e_2d_2}{4}\\&\quad\left.+\frac{d_1^2e_1^2}{16}-\frac{e_2d_1^2}{8}-\frac{d_2e_1^2}{8}+\frac{e_1d_3}{4}+\frac{7e_1d_1^3}{96}\right)z^4+\ldots,\ \ z\in\mathbb{D}.\end{aligned}\tag{18}$$

As function h defined in (1), it gives that

$$\begin{aligned}1+\frac{zh''(z)}{h'(z)}&=1+2a_2z+(6a_3-4a_2^2)z^2+(8a_2^3-18a_2a_3+12a_4)z^3\\&\quad+(20a_5-16a_2^4+48a_2^2a_3-32a_2a_4-18a_3^2)z^4+\ldots,\ \ z\in\mathbb{D},\end{aligned}\tag{19}$$

and equating the coefficients of z, z^2, z^3 and z^4 from the relations (18) and (19) we obtain

$$a_2 = \frac{e_1}{4} + \frac{d_1}{4}, \tag{20}$$

$$a_3 = \frac{e_2}{12} + \frac{1}{8}d_1e_1 + \frac{d_2}{12}, \tag{21}$$

$$\begin{aligned} a_4 = &\frac{1}{576}d_1^3 + \frac{1}{192}e_1d_1^2 + \frac{1}{192}d_1e_1^2 - \frac{1}{96}d_1d_2 + \frac{5}{96}d_1e_2 - \frac{1}{288}e_1^3 + \frac{5}{96}e_1d_2 \\ &- \frac{1}{96}e_1e_2 + \frac{1}{24}d_3 + \frac{1}{24}e_3 \end{aligned} \tag{22}$$

and

$$\begin{aligned} a_5 = &\frac{1}{576}e_1^4 - \frac{1}{1152}d_1^4 + \frac{1}{5760}e_1d_1^3 + \frac{1}{384}d_1^2e_1^2 + \frac{1}{192}d_1^2d_2 + \frac{1}{480}e_2d_1^2 - \frac{1}{288}d_1e_1^3 \\ &- \frac{1}{480}e_1d_1d_2 - \frac{1}{480}e_1d_1d_2 - \frac{1}{120}d_1d_3 + \frac{7}{240}d_1e_3 + \frac{1}{480}e_1^2d_2 - \frac{1}{240}e_1^2e_2 \\ &+ \frac{7}{240}e_1d_3 - \frac{1}{120}e_1e_3 - \frac{1}{160}d_2^2 - \frac{1}{160}e_2^2 + \frac{1}{40}e_2d_2 + \frac{1}{40}e_4 + \frac{1}{40}d_4. \end{aligned} \tag{23}$$

Taking the modulus on the both sides of (20), then applying the triangle inequality and using the inequality (7) of Lemma 1 we obtain the first desired estimate of (11).

Taking modulus on the both sides of (21) and applying triangle inequality we get

$$|a_3| = \left|\frac{e_2}{12} + \frac{1}{8}d_1e_1 + \frac{d_2}{12}\right| \leqslant \frac{|e_2|}{12} + \frac{|d_1||e_1|}{8} + \frac{|d_2|}{12}.$$

Application of (7) of Lemma 1 in the above inequality gives

$$|a_3| \leqslant \frac{1}{6} + \frac{1}{2} + \frac{1}{6} = \frac{5}{6}.$$

Using the triangle inequality on the both the sides of (22) we obtain

$$\begin{aligned} |a_4| = &\left|\frac{1}{576}d_1^3 + \frac{1}{192}e_1d_1^2 + \frac{1}{192}d_1e_1^2 - \frac{1}{96}d_1d_2 + \frac{5}{96}d_1e_2 - \frac{1}{288}e_1^3 + \frac{5}{96}e_1d_2\right. \\ &\left.-\frac{1}{96}e_1e_2 + \frac{1}{24}d_3 + \frac{1}{24}e_3\right| \\ &\leqslant \frac{1}{24}\left|d_3 - 2.\frac{1}{8}d_1d_2 + \frac{1}{24}d_1^3\right| + \frac{1}{24}\left|e_3 - 2.\frac{1}{8}e_1e_2 - \frac{1}{12}e_1^3\right| \\ &\quad + \frac{5}{96}|e_1|\left|d_2 - \left(-\frac{1}{10}\right)d_1^2\right| + \frac{5}{96}|d_1|\left|e_2 - \left(-\frac{1}{10}\right)e_1^2\right|. \end{aligned} \tag{24}$$

The expression $\left|d_3 - 2.\frac{1}{8}d_1d_2 + \frac{1}{24}d_1^3\right|$ and $\left|e_3 - 2.\frac{1}{8}e_1e_2 - \frac{1}{12}e_1^3\right|$ satisfies the conditions (9) of Lemma 3. Applications of (7) of Lemma 1, (9) of Lemma 3 and (10) of Lemma 4 give

$$|a_4| \leqslant \frac{2}{24} + \frac{2}{24} + \frac{5}{24}.\frac{6}{5} + \frac{5}{24}.\frac{6}{5} = \frac{2}{3}.$$

To determine the upper bound of $|a_5|$, rearranging and simplifying the terms of the relation (23) we get

$$\begin{aligned} a_5 = &\left(\frac{-d_1}{120}\right)\left(d_3 - 2.\frac{5}{16}d_1d_2 + \frac{5}{48}d_1^3\right) + \left(\frac{e_1}{120}\right)\left(\frac{5}{24}e_1^3 - \frac{1}{2}e_1e_2 - e_3\right) \\ &+ \left(\frac{7.e_1}{240}\right)\left(d_3 - 2.\frac{1}{28}d_1d_2 + \frac{1}{168}d_1^3\right) + \left(\frac{7.d_1}{240}\right)\left(\frac{-5}{42}e_1^3 - \frac{1}{14}e_1e_2 + e_3\right) \\ &+ \frac{1}{40}\left(e_4 - \frac{1}{4}e_2^2\right) + \frac{1}{40}\left(d_4 - \frac{1}{4}d_2^2\right) + \frac{1}{480}e_1^2\left(d_2 + \frac{5}{4}d_1^2\right) + \frac{e_2}{40}\left(d_2 + \frac{1}{12}d_1^2\right). \end{aligned} \tag{25}$$

Taking modules on both sides of (25) and then applying triangle inequality we obtain

$$\begin{aligned} |a_5| \leqslant &\frac{|d_1|}{120}\left|d_3 - 2.\frac{5}{16}d_1d_2 + \frac{5}{48}d_1^3\right| + \frac{|e_1|}{120}\left|\frac{5}{24}e_1^3 - \frac{1}{2}e_1e_2 - e_3\right| \\ &+ \frac{7.|e_1|}{240}\left|d_3 - 2.\frac{1}{28}d_1d_2 + \frac{1}{168}d_1^3\right| + \frac{7.|d_1|}{240}\left|\frac{-5}{42}e_1^3 - \frac{1}{14}e_1e_2 + e_3\right| \\ &+ \frac{1}{40}\left|e_4 - \frac{1}{4}e_2^2\right| + \frac{1}{40}\left|d_4 - \frac{1}{4}d_2^2\right| \\ &+ \frac{1}{480}|e_1|^2\left|d_2 + \frac{5}{4}d_1^2\right| + \frac{1}{40}|e_2|\left|d_2 + \frac{1}{12}d_1^2\right|. \end{aligned} \tag{26}$$

Using (7) of Lemma 1, (8) of Lemma 2, (9) of Lemma 3 and (10) of Lemma 4 in the inequality (26) we conclude that

$$\begin{aligned} |a_5| \leqslant &\frac{4}{120} + \frac{4}{120}\left(\frac{5}{24} + \frac{1}{12} + \frac{31}{24}\right) + \frac{28}{240} + \frac{28}{240}\left(\frac{5}{42} + \frac{13}{42} + \frac{17}{21}\right) \\ &+ \frac{2}{40} + \frac{2}{40} + \frac{7}{120} + \frac{7}{60} = \frac{28}{45}. \end{aligned}$$

This complete the proof of Theorem 1. □

For the Fekete-Szegö functional we obtained the following estimation.

THEOREM 2. *If $h \in \mathscr{C}_q^*$ follows the structure in* (1), *then for any complex number μ we have*

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{1}{3}\max\left\{1; \frac{|3\mu - 2|}{2}\right\} + \frac{|1-\mu|}{2}. \tag{27}$$

Proof. If $h \in \mathscr{C}_q^*$, making use of (20) and (21) we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= \left(\frac{e_2}{12} + \frac{d_1e_1}{8} + \frac{d_2}{12}\right) - \mu\left(\frac{e_1}{4} + \frac{d_1}{4}\right)^2 \\ &= \frac{1}{12}\left(d_2 - \frac{3\mu}{4}d_1^2\right) + \frac{1}{12}\left(e_2 - \frac{3\mu}{4}e_1^2\right) + \frac{1-\mu}{8}d_1e_1. \end{aligned} \tag{28}$$

Taking modules on the both sides of (28) and then applying triangle inequality it follows

$$|a_3-\mu a_2^2| \leqslant \frac{1}{12}\left|d_2-\frac{3\mu}{4}d_1^2\right|+\frac{1}{12}\left|e_2-\frac{3\mu}{4}e_1^2\right|+\left|\frac{1-\mu}{8}\right||d_1||e_1|.$$

Application of (7) of Lemma 1 and (10) of Lemma 4 in the above inequality leads to

$$|a_3-\mu a_2^2| \leqslant \frac{1}{12}\cdot 2\cdot \max\left\{1;\left|1-2\cdot\frac{3\mu}{4}\right|\right\}+\frac{1}{12}\cdot 2\cdot \max\left\{1;\left|1-2\cdot\frac{3\mu}{4}\right|\right\}+4\left|\frac{1-\mu}{8}\right|,$$

that proves our result. □

For $\mu=1$ in the Theorem 2 reduces to the following special case that we will use in the following parts of the article.

COROLLARY 1. *if $h\in\mathscr{C}_{\mathrm{q}}^*$ has the form (1), then*

$$|a_3-a_2^2| = |H_{2,1}(h)| \leqslant \frac{1}{3}.$$

4. Hankel determinants bounds for the class $\mathscr{C}_{\mathrm{q}}^*$

In this section we investigate the upper bounds of Hankel determinants of order two and three for the functions that belong to the class $\mathscr{C}_{\mathrm{q}}^*$.

THEOREM 3. *If $h\in\mathfrak{A}$ belongs to the class $\mathscr{C}_{\mathrm{q}}^*$, then*

$$|a_2a_4-a_3^2| = |H_{2,2}(h)| \leqslant \frac{55}{144}. \tag{29}$$

Proof. From (20), (21) and (22) it follows that

$$\begin{aligned}
H_{2,2}(h) = a_2a_4-a_3^2 &= \frac{1}{576}e_1d_1^3-\frac{5}{384}e_1^2d_1^2+\frac{1}{2304}d_1e_1^3-\frac{1}{96}d_1d_2e_1-\frac{1}{96}d_1e_1e_2\\
&\quad -\frac{1}{1152}e_1^4+\frac{5}{384}e_1^2d_2-\frac{1}{384}e_1^2e_2+\frac{1}{96}e_1d_3+\frac{1}{96}e_1e_3+\frac{1}{2304}d_1^4\\
&\quad -\frac{1}{384}d_1^2d_2+\frac{5}{384}d_1^2e_2+\frac{1}{96}d_1d_3+\frac{1}{96}d_1e_3-\frac{1}{144}e_2^2-\frac{1}{72}d_2e_2-\frac{1}{144}d_2^2,\\
&= \frac{e_1}{96}\left(e_3-2.\frac{1}{8}e_1e_2-\frac{1}{12}e_1^3\right)+\frac{d_1}{96}\left(d_3-2.\frac{1}{8}d_1d_2+\frac{1}{24}d_1^3\right)\\
&\quad +\frac{d_1}{96}\left(e_3-2.\frac{1}{2}e_1e_2+\frac{1}{24}e_1^3\right)+\frac{e_1}{96}\left(d_3-2.\frac{1}{2}d_1d_2+\frac{1}{6}d_1^3\right)\\
&\quad +\frac{5}{384}d_1^2\left(e_2-e_1^2\right)-\frac{d_2}{72}\left(e_2-\frac{15}{16}e_1^2\right)-\frac{1}{144}e_2^2-\frac{1}{144}d_2^2
\end{aligned} \tag{30}$$

and using the triangle inequality, the relation (30) implies

$$
\begin{aligned}
|H_{2,2}(h)| \leqslant & \frac{|e_1|}{96}\left|e_3 - 2.\frac{1}{8}e_1e_2 - \frac{1}{12}e_1^3\right| + \frac{|d_1|}{96}\left|d_3 - 2.\frac{1}{8}d_1d_2 + \frac{1}{24}d_1^3\right| \\
& + \frac{|d_1|}{96}\left|e_3 - 2.\frac{1}{2}e_1e_2 + \frac{1}{24}e_1^3\right| + \frac{|e_1|}{96}\left|d_3 - 2.\frac{1}{2}d_1d_2 + \frac{1}{6}d_1^3\right| \\
& + \frac{5}{384}|d_1|^2\left|e_2 - e_1^2\right| + \frac{|d_2|}{72}\left|e_2 - \frac{15}{16}e_1^2\right| \\
& + \frac{1}{144}|e_2|^2 + \frac{1}{144}|d_2|^2. \qquad (31)
\end{aligned}
$$

Making use of Lemmas 1, 3 and 4 in the right hand side of (31) we finally get

$$
|H_{2,2}(h)| \leqslant \frac{4}{96} + \frac{4}{96} + \frac{4}{96} + \frac{4}{96} + \frac{5}{48} + \frac{1}{18} + \frac{1}{36} + \frac{1}{36} = \frac{55}{144}.
$$

That completes the proof of Theorem 3. □

THEOREM 4. *If $h \in \mathfrak{A}$ expressed by (1) belongs to the class $\mathscr{C}_{\mathfrak{q}}^*$, then*

$$
|a_4 - a_2a_3| \leqslant \frac{31}{72}.
$$

Proof. If $h \in \mathscr{C}_{\mathfrak{q}}^*$ has the form (1), using the relations (20), (21) and (22) we get

$$
\begin{aligned}
a_4 - a_2a_3 = & -\frac{1}{32}e_1e_2 - \frac{5}{192}d_1e_1^2 + \frac{1}{32}e_1d_2 + \frac{1}{32}d_1e_2 - \frac{5}{192}e_1d_1^2 - \frac{1}{32}d_1d_2 \\
& + \frac{1}{576}d_1^3 - \frac{1}{288}e_1^3 + \frac{1}{24}d_3 + \frac{1}{24}e_3 \\
= & \frac{1}{24}\left(-\frac{1}{12}e_1^3 - \frac{3}{4}e_1e_2 + e_3\right) + \frac{1}{24}\left(\frac{1}{24}d_1^3 - \frac{3}{4}d_1d_2 + d_3\right) \\
& + \frac{d_1}{32}\left(e_2 - \frac{5}{6}e_1^2\right) + \frac{e_1}{32}\left(d_2 - \frac{5}{6}d_1^2\right). \qquad (32)
\end{aligned}
$$

Considering the modules on the both sides of (32) followed by the triangle inequality one may get

$$
\begin{aligned}
|a_4 - a_2a_3| \leqslant & \frac{1}{24}\left|-\frac{1}{12}e_1^3 - \frac{3}{4}e_1e_2 + e_3\right| + \frac{1}{24}\left|\frac{1}{24}d_1^3 - \frac{3}{4}d_1d_2 + d_3\right| \\
& + \frac{|d_1|}{32}\left|e_2 - \frac{5}{6}e_1^2\right| + \frac{|e_1|}{32}\left|d_2 - \frac{5}{6}d_1^2\right|. \qquad (33)
\end{aligned}
$$

Using the inequalities (7) of Lemma 1, (8) of Lemma 2 and (10) of Lemma 4 in the right hand side of (33) we obtain

$$
|a_4 - a_2a_3| \leqslant \frac{2}{24}\left(\frac{1}{12} + \frac{11}{12} + \frac{1}{6}\right) + \frac{2}{24}\left(\frac{1}{24} + \frac{2}{3} + \frac{7}{24}\right) + \frac{4}{32} + \frac{4}{32} = \frac{31}{72}.
$$

This proof the result of Theorem 4. □

Using the previous results we could easily find an upper bound for $|H_{3,1}(h)|$ if $h \in \mathscr{C}_{\mathfrak{q}}^{*}$, as follows.

THEOREM 5. *Let* $h \in \mathfrak{A}$ *expressed by* (1) *be classified in the class* $\mathscr{C}_{\mathfrak{q}}^{*}$. *Then,*

$$|H_{3,1}(h)| \leqslant \frac{3511}{4320}.$$

Proof. Application of triangle inequality to the relation (5) yields

$$|H_{3,1}(h)| \leqslant |a_5|\left|a_3 - a_2^2\right| + |a_4|\,|a_4 - a_2 a_3| + |a_3|\left|a_2 a_4 - a_3^2\right|. \tag{34}$$

Making use of the results of Theorems 1, 3, 4 and Corollary 1 to (34), we conclude that

$$|H_{3,1}(h)| \leqslant \frac{28}{45}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{31}{72} + \frac{5}{6}\cdot\frac{55}{144} = \frac{3511}{4320}. \quad \square$$

5. The Zalcman functional for the class $\mathscr{C}_{\mathfrak{q}}^{*}$

Zalcman conjectured that the coefficients of every univalent functions $h \in \mathscr{S}$ given by (1) satisfy the inequality

$$\left|a_n^2 - a_{2n-1}\right| \leqslant (n-1)^2, \quad n \geqslant 2, \tag{35}$$

and the equality holds only for the *Koebe function* $k(z) := \dfrac{z}{(1-z)^2}$ and its rotations. The Zalcman functional mentioned above has been extensively examined by various researchers; for instance, refer to [5, 10, 15].

From the Corollary 1 we can see that the above inequality (35) holds for the class $\mathscr{C}_{\mathfrak{q}}^{*}$ and $n = 2$. Our next result proves that the Zalcman conjecture (35) is applicable for the class $\mathscr{C}_{\mathfrak{q}}^{*}$ when $n = 3$.

THEOREM 6. *If* $h \in \mathscr{C}_{\mathfrak{q}}^{*}$ *has the form* (1), *then*

$$\left|a_3^2 - a_5\right| \leqslant \frac{7}{10}.$$

Proof. If $h \in \mathscr{C}^*_{\mathrm{q}}$ then from (21) and (23) we have

$$\begin{aligned} a_3^2 - a_5 &= \frac{19}{1440}e_2^2 - \frac{1}{90}d_2e_2 + \frac{5}{384}e_1^2d_1^2 + \frac{19}{1440}d_2^2 - \frac{1}{40}e_4 - \frac{1}{576}e_1^4 - \frac{1}{40}d_4 + \frac{1}{1152}d_1^4 \\ &\quad + \frac{11}{480}e_1d_1d_2 - \frac{1}{5760}e_1d_1^3 + \frac{1}{120}d_1d_3 - \frac{7}{240}d_1e_3 - \frac{1}{192}d_2d_1^2 - \frac{1}{460}d_1^2e_2 \\ &\quad - \frac{7}{240}e_1d_3 + \frac{1}{120}e_1e_3 - \frac{1}{480}e_1^2d_2 + \frac{1}{240}e_1^2e_2 + \frac{11}{480}d_1e_1e_2 + \frac{1}{288}d_1e_1^3 \\ &= \left(\frac{1}{120}e_1e_3 + \frac{1}{240}e_1^2e_2 - \frac{1}{576}e_1^4\right) + \left(\frac{1}{1152}d_1^4 - \frac{1}{192}d_1^2d_2 + \frac{1}{120}d_1d_3\right) \\ &\quad + \left(\frac{-7}{240}e_1d_3 + \frac{11}{480}e_1d_1d_2 - \frac{1}{5760}e_1d_1^3\right) \\ &\quad + \left(\frac{-7}{240}d_1e_3 + \frac{11}{480}d_1e_1e_2 + \frac{1}{288}d_1e_1^3\right) + \left(\frac{-1}{40}e_4 + \frac{19}{1440}e_2^2\right) \\ &\quad + \left(\frac{-1}{40}d_4 + \frac{19}{1440}d_2^2\right) + \left(\frac{-1}{480}e_1^2d_2 + \frac{5}{384}d_1^2e_1^2\right) \\ &\quad + \left(\frac{-1}{90}e_2d_2\right) + \left(\frac{-1}{480}d_1^2e_2\right). \end{aligned} \tag{36}$$

Taking modulus on both sides of the relation (36) and followed by application of triangle inequality yield

$$\begin{aligned} \left|a_3^2 - a_5\right| &\leqslant \left|\frac{-e_1}{24}\right| \left|\frac{1}{24}e_1^3 - \frac{1}{10}e_1e_2 - \frac{1}{5}e_3 +\right| + \left|\frac{d_1}{120}\right| \left|d_3 - 2\cdot\frac{5}{16}d_1d_2 + \frac{5}{48}d_1^3\right| \\ &\quad + \left|\frac{-7e_1}{240}\right| \left|d_3 - 2\cdot\frac{11}{28}d_1d_2 + \frac{1}{168}d_1^3\right| + \left|\frac{-7d_1}{240}\right| \left|\frac{-5}{42}e_1^3 - \frac{11}{14}e_1e_2 + e_3\right| \\ &\quad + \left|\frac{-1}{40}\right| \left|e_4 - \frac{19}{36}e_2^2\right| + \left|\frac{-1}{40}\right| \left|d_4 - \frac{19}{36}d_2^2\right| + \left|\frac{-e_1^2}{480}\right| \left|d_2 - \frac{25}{4}d_1^2\right| \\ &\quad + \frac{1}{90}|e_2| \left|d_2 + \frac{3}{16}d_1^2\right|. \end{aligned} \tag{37}$$

First, if we consider the term $\left|d_3 - 2\cdot\frac{5}{16}d_1d_2 + \frac{5}{48}d_1^3\right|$, comparing with the left hand side of (9) we get $B = \frac{5}{16} = 0.3125\ldots$, $D = \frac{5}{48} = 0.10416\ldots$. Clearly, $B \in [0,1]$ and

$$B(2B-1) = \frac{-15}{128} = -0.117187\ldots \leqslant D = 0.10416\ldots < B = 0.3125\ldots.$$

Hence by virtue of Lemma 3 we obtain

$$\left|d_3 - 2\cdot\frac{5}{16}d_1d_2 + \frac{5}{48}d_1^3\right| \leqslant 2. \tag{38}$$

In similar manner, we obtained

$$\left| d_3 - 2 \cdot \frac{11}{28} d_1 d_2 + \frac{1}{168} d_1^3 \right| \leqslant 2. \tag{39}$$

Making the use of the inequalities (7) of Lemma 1, (8) of Lemma 2, (10) of Lemma 4 and the relations (38), (39) to the inequality (37) lead to

$$\left| a_3^2 - a_5 \right| \leqslant \frac{19}{360} + \frac{1}{30} + \frac{7}{60} + \frac{13}{90} + \frac{1}{20} + \frac{1}{20} + \frac{23}{120} + \frac{11}{180} = \frac{7}{10},$$

which proofs our result. □

6. Logarithmic coefficients bounds for the class $\mathscr{C}_{\mathrm{q}}^*$

For each function $h \in \mathscr{S}$ the logarithmic coefficients p_n, $n \in \mathbb{N}$, are given by

$$\log \frac{h(z)}{z} = 2 \sum_{n=1}^{\infty} p_n z^n, \quad (z \in \mathbb{D}). \tag{40}$$

The logarithmic coefficients play a very important role in the theory of univalent functions. The bounds on logarithmic coefficients of h can be transferred to the Taylor coefficients of univalent functions or to their powers via the Lebedev-Milin inequalities. Little information are known about the logarithmic coefficients of h when $h \in \mathscr{S}$, but recently upper bounds of logarithmic coefficients of functions h in some subclasses of $\mathscr{S}$ have been obtained by various authors (see, for example [2, 8, 12, 19, 23, 25]).

In this section, we provide upper bound estimates for the first four logarithmic coefficients of functions that are members of the class $\mathscr{C}_{\mathrm{q}}^*$.

THEOREM 7. *If $h \in \mathscr{C}_{\mathrm{q}}^*$ given by (1), then*

$$|p_1| \leqslant \frac{1}{2}, \quad |p_2| \leqslant \frac{7}{24}, \quad |p_3| \leqslant \frac{5}{24}, \quad |p_4| \leqslant \frac{3059}{11520}. \tag{41}$$

Proof. If the function $h \in \mathfrak{A}$ given by (1) belongs to the function class $\mathscr{C}_{\mathrm{q}}^*$, it follows that

$$\begin{aligned} \log \frac{h(z)}{z} &= a_2 z + \left(a_3 - \frac{a_2^2}{2} \right) z^2 + \left(a_4 - a_2 a_3 + \frac{a_2^3}{3} \right) z^3 \\ &+ \left(a_5 - a_2 a_4 + a_2^2 a_3 - \frac{a_3^2}{2} - \frac{a_2^4}{4} \right) z^4 \\ &+ \left(a_6 - a_2 a_5 + a_2^2 a_4 - a_3 a_4 - a_2^3 a_3 + a_2 a_3^2 + \frac{a_2^5}{5} \right) z^5 + \ldots, \quad z \in \mathbb{D}. \end{aligned} \tag{42}$$

Equating the first four coefficients of the relations (40) and (42) we get

$$p_1 = \frac{a_2}{2}, \tag{43}$$

$$p_2 = \frac{1}{4}\left(2a_3 - a_2^2\right), \tag{44}$$

$$p_3 = \frac{1}{6}\left(a_2^3 - 3a_2a_3 + 3a_4\right), \tag{45}$$

$$p_4 = \frac{1}{8}\left(-a_2^4 + 4a_2^2a_3 - 4a_2a_4 - 2a_3^2 + 4a_5\right). \tag{46}$$

Using the relations (20)–(23) we replace the values of a_2, a_3, a_4 and a_5 into (43)–(46) and we obtain

$$p_1 = \frac{d_1}{8} + \frac{e_1}{8}, \tag{47}$$

$$\begin{aligned} p_2 &= \frac{1}{24}e_2 + \frac{1}{32}d_1e_1 + \frac{1}{24}d_2 - \frac{1}{64}d_1^2 - \frac{1}{64}e_1^2 \\ &= \frac{1}{24}\left(d_2 - \frac{3}{8}d_1^2\right) + \frac{1}{24}\left(e_2 - \frac{3}{8}e_1^2\right) + \frac{1}{32}d_1e_1, \end{aligned} \tag{48}$$

$$\begin{aligned} p_3 =& -\frac{1}{64}e_1e_2 - \frac{1}{192}d_1e_1^2 + \frac{1}{64}e_1d_2 + \frac{1}{64}d_1e_2 - \frac{1}{64}d_1d_2 + \frac{1}{288}d_1^3 + \frac{1}{1152}e_1^3 \\ & - \frac{1}{192}e_1d_1^2 + \frac{1}{48}e_3 + \frac{1}{48}d_3 \\ =& \frac{1}{48}\left(d_3 - \frac{3}{4}d_1d_2 + \frac{1}{6}d_1^3\right) + \frac{1}{48}\left(e_3 - \frac{3}{4}e_1e_2 + \frac{1}{24}e_1^3\right) \\ & + \frac{e_1}{64}\left(d_2 - \frac{64}{192}d_1^2\right) + \frac{d_1}{64}\left(e_2 - \frac{64}{192}e_1^2\right), \end{aligned} \tag{49}$$

and

$$\begin{aligned} p_4 =& \frac{3}{2560}e_1d_1^3 - \frac{1}{160}e_1d_1d_2 + \frac{3}{320}d_1e_3 + \frac{13}{1440}e_2d_2 - \frac{11}{3840}e_2d_1^2 + \frac{3}{320}e_1d_3 \\ & + \frac{5}{768}d_2d_1^2 - \frac{3}{320}d_1d_3 + \frac{7}{3840}e_2e_1^2 + \frac{1}{1024}e_1^2d_1^2 - \frac{11}{3840}e_1^2d_2 - \frac{7}{1440}e_2^2 \\ & - \frac{3}{320}e_1e_3 - \frac{1}{160}d_1e_1e_2 - \frac{7}{6144}d_1^4 - \frac{7}{1440}d_2^2 + \frac{1}{80}d_4 + \frac{1}{80}e_4 + \frac{5}{6144}e_1^4 \\ =& \left(-\frac{3}{320}d_1\right)\left(d_3 - \frac{25}{36}d_1d_2 + \frac{35}{288}d_1^3\right) + \left(\frac{3e_1}{320}\right)\left(d_3 - \frac{2}{3}d_1d_2 + \frac{1}{8}d_1^3\right) \\ & + (-e_1)\left(\frac{-5}{6144}e_1^3 - \frac{7}{3840}e_1e_2 + \frac{3}{320}e_3\right) + \left(\frac{3d_1}{320}\right)\left(e_3 - \frac{2}{3}e_1e_2\right) \\ & + \left(\frac{-11}{3840}e_1^2\right)\left(d_2 - \frac{15}{44}d_1^2\right) + \left(\frac{13}{1440}e_2\right)\left(d_2 - \frac{33}{104}d_1^2\right) \\ & + \frac{1}{80}\left(d_4 - \frac{7}{18}d_2^2\right) + \frac{1}{80}\left(e_4 - \frac{7}{18}e_2^2\right). \end{aligned} \tag{50}$$

Using (7) of Lemma 1 in (47) yields

$$|p_1| = \left|\frac{d_1}{8} + \frac{e_1}{8}\right| \leqslant \frac{|d_1|}{8} + \frac{|e_1|}{8} \leqslant \frac{1}{2}.$$

Further, by virtue of Lemma 1 and Lemma 4 give

$$|p_2| \leqslant \frac{1}{24}\left|d_2 - \frac{3}{8}d_1^2\right| + \frac{1}{24}\left|e_2 - \frac{3}{8}e_1^2\right| + \frac{1}{32}|d_1||e_1| = \frac{7}{24}.$$

Taking the modules of the both sides of (49) and applying the triangle inequality we get

$$|p_3| \leqslant \frac{1}{48}\left|d_3 - 2.\frac{3}{8}d_1d_2 + \frac{1}{6}\right| + \frac{1}{48}\left|e_3 - 2.\frac{3}{8}e_1e_2 + \frac{1}{24}\right| + \frac{e_1}{64}\left|d_2 - \frac{64}{192}d_1^2\right| + \frac{d_1}{64}\left|e_2 - \frac{64}{192}e_1^2\right|. \tag{51}$$

Applying (7) of Lemma 1, (9) of Lemma 3 and (10) of Lemma 4 in (51) we deduce that

$$|p_3| \leqslant \frac{1}{24} + \frac{1}{24} + \frac{1}{16} + \frac{1}{16} = \frac{5}{24}.$$

Finally, taking modules on the both sides of (50) and applying triangle inequality we get

$$\begin{aligned} |p_4| \leqslant & \left|\frac{3}{320}d_1\right|\left|d_3 - 2\cdot\frac{72}{36}d_1d_2 + \frac{35}{288}d_1^3\right| + \left|\frac{3e_1}{320}\right|\left|d_3 - 2\cdot\frac{2}{6}d_1d_2 + \frac{1}{8}d_1^3\right| \\ & + |e_1|\left|\frac{-5}{6144}e_1^3 - \frac{7}{3840}e_1e_2 + \frac{3}{320}e_3\right| + \left|\frac{3d_1}{320}\right|\left|e_3 - \frac{2}{3}e_1e_2\right| \\ & + \left|\frac{-11}{3840}e_1^2\right|\left|d_2 - \frac{15}{44}d_1^2\right| + \left|\frac{13}{1440}e_2\right|\left|d_2 - \frac{33}{104}d_1^2\right| \\ & + \frac{1}{80}\left|d_4 - \frac{7}{18}d_2^2\right| + \frac{1}{80}\left|e_4 - \frac{7}{18}e_2^2\right|. \end{aligned} \tag{52}$$

Making the use of (7) of Lemma 1, (8) of Lemma 2, (9) of Lemma 3 and (10) of Lemma 4 to the inequality (52) we conclude that

$$|p_4| \leqslant \frac{3}{80} + \frac{3}{80} + \frac{11}{480} + \frac{169}{3840} + \frac{1}{40} + \frac{1}{40} + \frac{3}{80} + \frac{13}{360} = \frac{3059}{11520}.$$

Hence, the proof of Theorem 7 is now concluded. □

7. Inverse coefficient bounds for the class $\mathscr{C}_q^*$

For each function $h \in \mathscr{S}$ defined on $\mathbb{D}$ the famous *one-quarter theorem of Koebe* (see [9]) ensures that its inverse h^{-1} exists at least on a disk of radius $\frac{1}{4}$ having the Taylor's series of the form

$$h^{-1}(w) = w + \sum_{n=2}^{\infty} h_n w^n, \quad (|w| < \frac{1}{4}). \tag{53}$$

In this section we are going to find the first four upper bounds of inverse function coefficients that belongs to the class $\mathscr{C}_q^*$.

THEOREM 8. *If $h \in \mathscr{C}_q^*$ is given by (1) and its inverse h^{-1} has the form (53), then*

$$|h_2| \leqslant 1, \quad |h_3| \leqslant \frac{7}{6}, \quad |h_4| \leqslant \frac{101}{72}, \quad |h_5| \leqslant \frac{1741}{720}. \tag{54}$$

Proof. Since $h\left(h^{-1}(w)\right) = w,\ |w| < \frac{1}{4}$, we get

$$h_2 = -a_2, \tag{55}$$
$$h_3 = -a_3 + 2a_2^2, \tag{56}$$
$$h_4 = -a_4 + 5a_2a_3 - 5a_2^3, \tag{57}$$
$$h_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \tag{58}$$

Using the relations (20)–(23) into (55)–(58) a simple computation shows that

$$h_2 = -\frac{e_1}{4} - \frac{d_1}{4}, \tag{59}$$

$$\begin{aligned} h_3 =& \frac{1}{8}e_1^2 + \frac{1}{8}d_1^2 + \frac{1}{8}e_1d_1 - \frac{1}{12}e_2 - \frac{1}{12}d_2 \\ =& -\frac{1}{12}\left(e_2 - \frac{3}{2}e_1^2\right) + \frac{-1}{12}\left(d_2 - \frac{3}{2}d_1^2\right) + \frac{1}{8}d_1e_1, \end{aligned} \tag{60}$$

$$\begin{aligned} h_4 =& -\frac{43}{576}e_1^3 - \frac{1}{24}e_3 + \frac{11}{96}e_1e_2 - \frac{1}{24}d_3 + \frac{11}{96}d_1d_2 - \frac{23}{288}d_1^3 + \frac{5}{96}e_1d_2 \\ & - \frac{1}{12}e_1d_1^2 + \frac{5}{96}d_1e_2 - \frac{1}{12}d_1e_1^2, \end{aligned} \tag{61}$$

and

$$\begin{aligned} h_5 =& \frac{-11}{120}d_1e_1e_2 - \frac{11}{120}e_1d_1d_2 - \frac{1}{30}e_1^2d_2 - \frac{29}{240}e_1^2e_2 - \frac{1}{30}d_1^2e_2 + \frac{35}{576}d_1e_1^3 + \frac{1}{30}e_1d_3 \\ & + \frac{17}{240}e_1e_3 + \frac{187}{2880}e_1d_1^3 + \frac{17}{240}d_1d_3 + \frac{1}{30}d_1e_3 - \frac{25}{192}d_1^2d_2 + \frac{1}{60}e_2d_2 + \frac{23}{384}d_1^2e_1^2 \\ & + \frac{67}{1152}d_1^4 + \frac{55}{1152}e_1^4 - \frac{1}{40}d_4 + \frac{13}{480}d_2^2 - \frac{1}{40}e_4 + \frac{13}{480}e_2^2. \end{aligned} \tag{62}$$

The inequality (7) of Lemma 1 together with the triangle inequality in (59) gives the estimates for $|h_2|$.

Taking the modules in the both the sides of (60) and applying triangle inequality we get

$$|h_3| \leqslant \frac{1}{12}\left|e_2 - \frac{3}{2}e_1^2\right| + \frac{1}{12}\left|d_2 - \frac{3}{2}d_1^2\right| + \frac{1}{8}|d_1||e_1|, \tag{63}$$

and applying Lemma 1 and Lemma 4 to the relation (63) we obtain

$$|h_3| \leqslant \frac{1}{3} + \frac{1}{3} + \frac{1}{2} = \frac{7}{6}.$$

Next, applying the triangle inequality to the relation (61) we get

$$\begin{aligned} |h_4| \leqslant & \frac{1}{24}\left|e_3 - \frac{11}{4}e_1e_2 + \frac{43}{24}e_1^3\right| + \frac{1}{24}\left|d_3 - \frac{11}{4}d_1d_2 + \frac{23}{12}d_1^3\right| \\ & + \frac{5}{96}|e_1|\left|d_2 - \frac{8}{5}d_1^2\right| + \frac{5}{96}|d_1|\left|e_2 - \frac{8}{5}e_1^2\right|. \end{aligned} \tag{64}$$

Making use of Lemmas 1, 2 and 4 in (64) we deduce that

$$|h_4| \leqslant \frac{1}{12}\cdot\frac{8}{3} + \frac{1}{12}\cdot\frac{19}{6} + \frac{20}{96}\cdot\frac{11}{5} + \frac{20}{96}\cdot\frac{11}{5} = \frac{101}{72}.$$

Now taking modulus and arranging the terms of (62), we get

$$\begin{aligned} |h_5| \leqslant & \left|\frac{17e_1}{240}\right|\left|\frac{275}{408}e_1^3 - \frac{29}{17}e_1e_2 + e_3\right| + \left|\frac{17d_1}{240}\right|\left|\frac{335}{408}d_1^3 - \frac{125}{68}d_1d_2 + d_3\right| \\ & + \left|\frac{d_1}{30}\right|\left|\frac{175}{96}e_1^3 - \frac{11}{4}e_1e_2 + e_3\right| + \left|\frac{e_1}{30}\right|\left|\frac{187}{96}d_1^3 - \frac{11}{4}d_1d_2 + d_3\right| + \left|\frac{-1}{40}\right|\left|d_4 \right.\\ & \left. - \frac{13}{12}d_2^2\right| + \left|\frac{-1}{40}\right|\left|e_4 - \frac{13}{12}e_2^2\right| + \left|\frac{-d_2}{60}\right|\left|e_2 - 2e_1^2\right| + \left|\frac{-d_1^2}{30}\right|\left|e_2 - \frac{115}{64}e_1^2\right| \end{aligned} \tag{65}$$

Making the use of (7) of Lemma 1, (8) of Lemma 2 and (10) of Lemma 4 to the inequality (65) we conclude that

$$|h_5| \leqslant \frac{217}{720} + \frac{107}{360} + \frac{67}{180} + \frac{79}{180} + \frac{7}{120} + \frac{7}{120} + \frac{1}{5} + \frac{83}{120} = \frac{1741}{720}.$$

This completes the proof of Theorem 8. □

Concluding remarks

In the present paper, the authors mainly investigated the upper bounds of the first initial four coefficients, Fekete-Szegö functional, Hankel determinant of order two and three for the class $\mathscr{C}_q^*$. Further, Zalcman conjecture for $n = 3$ holds for the class $\mathscr{C}_q^*$. We determined the upper bounds estimates for the first four logarithmic coefficients and for the coefficients of the inverse function h^{-1} for the class $\mathscr{C}_q^*$. We mention here that the results in these paper are not sharp. Finding the sharp upper bounds remains future work for the researchers.

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