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CRITERIA FOR NORMAL AND φ–NORMAL FUNCTIONS

BANARSI LAL*, VIRENDER SINGH AND PRIYANKA

Abstract. We establish the following result: Let $f(z)$ be meromorphic in the unit disk $\mathbb{D}$, and let $S = \{\alpha_1, \alpha_2, \alpha_3\}$ be a set of three distinct complex numbers in $\mathbb{C}$ and $A > 0$. If $f(z) \in S$ implies $|L(f)| < A$, then $f(z)$ is a normal function on $\mathbb{D}$. The result is further extended to φ-normal functions.

1. Introduction

The purpose of this paper is to establish a normality criterion pertaining to differential inequalities for a function defined on unit disk $\mathbb{D}$. A meromorphic function f on the unit disk $\mathbb{D}$ is considered *normal* on $\mathbb{D}$ if and only if the family $\mathscr{F} := \{f o \tau : \tau \in \mathscr{T}\}$ is normal on $\mathbb{D}$, where $\mathscr{T}$ represents the collection of all conformal self-maps of $\mathbb{D}$. Usually, there is a relationship between normal functions and normal families, leading to the expectation that the criteria for normal families will align with the established criteria for normal functions, and vice versa. A classic instance of this is the criterion put forth by Lehto and Virtanen [3], which modifies Marty's criterion for a normal family to provide a criterion for a function to be classified as normal: *A necessary and sufficient condition for a meromorphic function f on unit disc $\mathbb{D}$ to be normal is*

$$\sup_{z \in \mathbb{D}} \{(1 - |z|^2) f^{\#}(z)\} < \infty.$$

In this paper, we examine specific differential inequalities that lead to criteria for determining normal functions. We denote by

$$L(f) := f^{(k)}(z) + \sum_{m=0}^{k-1} a_m(z) f^{(m)}(z),$$

where $k \in \{1, 2, \cdots, p\}$ $(p \in \mathbb{N})$ and $a_0(z), \cdots, a_{k-1}(z)$ are fixed holomorphic functions.

The starting point of this paper is the following result by Chen and Tong [2] concerning normal functions:

THEOREM 1. *Let $f(z)$ be holomorphic in unit disk $\mathbb{D}$, and $\alpha_1, \alpha_2, \alpha_3$ be three distinct complex numbers in $\mathbb{C}$, and $A > 0$. Suppose that $|f'(z)| \leqslant A$ whenever $f(z) = \alpha_i$, $i = 1, 2, 3$, then $f(z)$ is a normal function.*

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* Corresponding author.

This theorem establishes a crucial condition under which a holomorphic function can be considered normal, specifically by bounding the derivative of the function at certain values. In this paper, we prove the following generalization of Theorem 1:

THEOREM 2. *Let $f(z)$ be meromorphic in unit disk $\mathbb{D}$ all of whose poles are of multiplicities at least three, and let $S = \{\alpha_1, \alpha_2, \alpha_3\}$ be set of three distinct complex numbers in $\mathbb{C}$, and $A > 0$. If*

$$f(z) \in S \quad \textit{implies} \quad |L(f)| < A, \tag{1}$$

then $f(z)$ is a normal function on $\mathbb{D}$.

The following example demonstrate the necessity of condition (1) in Theorem 1:

EXAMPLE 1. Consider $f_n(z) = n(e^{\alpha_1 z} - e^{\alpha_2 z})$ $(\alpha_1 \neq \alpha_2)$, for $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Then clearly $f_n(z)$ is not normal on $\mathbb{D}$. Note that whenever $f_n(0) = 0 \in S = \{0, 1, -1\}$, we have

$$|L(f)| = |f_n'(0) + f_n(0)| = |n(\alpha_1 - \alpha_2) + 0| \to \infty \text{ as } n \to \infty.$$

The concept of normal functions has been extended to φ-normal functions by Aulaskari and Rattya [1]. This extension involves a smoothly increasing function $\varphi : [0,1) \to (0,\infty)$ satisfying $\varphi(r)(1-r) \to \infty$ as $r \to 1^-$, and

$$\mathscr{R}_a(z) := \frac{\varphi(|a + z/\varphi(|a|))}{\varphi(|a|)} \to 1, \ \ |a| \to 1^-,$$

uniformly on compact subsets of $\mathbb{C}$. For such function φ, a meromorphic function f on unit disc $\mathbb{D}$ is said to be φ*-normal* if

$$\sup_{z \in \mathbb{D}} \frac{f^{\#}(z)}{\varphi(|z|)} < \infty.$$

We extend Theorem 2 to φ-normal function as:

THEOREM 3. *Let $\varphi : [0,1) \to (0,\infty)$ be a smoothly increasing function. Let $f(z)$ be meromorphic in the unit disk $\mathbb{D}$ all of whose poles are multiplicities at least three, and let $S = \{\alpha_1, \alpha_2, \alpha_3\}$ be set of three distinct complex numbers in $\mathbb{C}$, and $A > 0$. If $f(z) \in S$ implies $|L(f)| < A\varphi^k(|z|)$, then $f(z)$ is φ-normal function on $\mathbb{D}$.*

REMARK 1. We can generalize Theorem 2 and Theorem 3 by considering a set $S = \{1, 2, \cdots, q\}$ consisting q-distinct complex numbers in $\mathbb{C}$, where $(q \geqslant 3)$.

2. Proof of the main results

We require the following results to support the proof of our main results:

LEMMA 1. [2] *If a meromorphic function $f(z)$ is not normal on unit disk $\mathbb{D}$, then there exists a sequence of points $z_n \in \mathbb{D}$ and positive numbers $\rho_n \to 0$ such that $g_n(\zeta) = f(z_n + \rho_n \zeta)$ converges locally uniformly with respect to the spherical metric to $g(\zeta)$, where $g(\zeta)$ is a non-constant meromorphic function on $\mathbb{C}$.*

LEMMA 2. [4] *Let $\varphi : [0,1) \to (0,\infty)$ be a smoothly increasing function, and f be a meromorphic function on unit disk $\mathbb{D}$. Assume that all zeros and all poles of f have multiplicity at least p and q respectively. Let α be a real number satisfying $-p < \alpha < q$. If f is not φ-normal, then there exists*

(i) a sequence $\{a_n\} \subset \mathbb{D}$, $|a_n| \to 1$,

(ii) a sequence $\{z_n\} \subset \mathbb{D}$ with $z_n \to z^* \in \mathbb{D}$ and $w_n = a_n + \frac{z_n}{\varphi(|a_n|)}$,

(iii) a sequence of positive numbers $\rho_n : \rho_n \to 0$,

such that $g_n(\zeta) := {\rho_n}^{\alpha} f(w_n + \frac{\rho_n}{\varphi(|a_n|)}\zeta)$ converges locally uniformly with respect to the spherical metric to $g(\zeta)$, where $g(\zeta)$ is a non-constant meromorphic function on $\mathbb{C}$, all of whose zeros and poles have multiplicity at least p and q respectively.

Proof of Theorem 2. Assume, for the sake of contradiction, that $f(z)$ is not normal at a point $z_0 \in \mathbb{D}$. By applying Lemma 1, we can find sequence of points $z_n \in \mathbb{D}$ that converges to z_0, and positive numbers ρ_n that converges to zero such that $g_n(\zeta) = f(z_n + \rho_n \zeta)$ converges locally uniformly with respect to the spherical metric to $g(\zeta)$, where $g(\zeta)$ is a non-constant meromorphic function on $\mathbb{C}$. Additionally, all the poles of g are of multiplicity at least three.

On differentiating 'k' times, we get

$$g_n^{(k)}(\zeta) = \rho_n^k f^{(k)}(z_n + \rho_n \zeta) \to g^{(k)}(\zeta) \text{ in } \mathbb{C} \setminus P_g,$$

where P_g denotes the set of poles of g. Clearly, g must assume at least one value from the set S; otherwise, by Picard's theorem, g would be constant. Let ζ_0 in $\mathbb{C}$ be such that $g(\zeta_0) - \alpha_i = 0$ $(i = 1,2,3)$. Since $g(\zeta) \not\equiv \alpha_i$, by Hurwitz theorem, there exist a sequence ζ_n converging to ζ_0 such that for sufficiently large n, we have

$$g_n(\zeta_n) = f_n(z_n + \rho_n \zeta_n) = \alpha_i \in S.$$

By hypothesis, $f(z) \in S$ implies $|L(f)| < A$, it follows that

$$\left| f^{(k)}(z_n + \rho_n \zeta_n) + \sum_{m=0}^{k-1} a_m(z_n + \rho_n \zeta_n) f^{(m)}(z_n + \rho_n \zeta_n) \right| < A. \tag{2}$$

Since $g(\zeta_0) = \alpha_i$, it follows that ζ_0 is not a pole of g. Consequently, there exists a neighborhood around ζ_0 in which g is analytic. Therefore

$$a_m(z_n + \rho_n\zeta_n)g_n^{(m)}(\zeta_n) \to a_m(z_0)g^{(m)}(\zeta_0) \in \mathbb{C} \text{ as } n \to \infty \text{ for } m = 0,1,\cdots,k-1$$

so that

$$g_n^{(k)}(\zeta_n) + \sum_{m=0}^{k-1} \rho_n^{k-m} a_m(z_n + \rho_n\zeta_n)g_n^{(m)}(\zeta_n) \to g^{(k)}(\zeta_0) \text{ as } n \to \infty. \tag{3}$$

For sufficiently large n, and $k = 1,2,\cdots,p$

$$\begin{aligned}
&\left|g_n^{(k)}(\zeta_n) + \sum_{m=0}^{k-1} \rho_n^{k-m} a_m(z_n + \rho_n\zeta_n)g_n^{(m)}(\zeta_n)\right| \\
&\quad = \left|\rho_n^k f^{(k)}(z_n + \rho_n\zeta_n) + \sum_{m=0}^{k-1} \rho_n^{k-m}.a_m(z_n + \rho_n\zeta_n)\rho_n^m f^{(m)}(z_n + \rho_n\zeta_n)\right| \\
&\quad = \rho_n^k\left|f^{(k)}(z_n + \rho_n\zeta_n) + \sum_{m=0}^{k-1} a_m(z_n + \rho_n\zeta_n)f^{(m)}(z_n + \rho_n\zeta_n)\right| \\
&\quad < \rho_n^k A \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus equation 3 implies that $g^{(k)}(\zeta_0) = 0$ for all $k = 1,2,\cdots,p$. Consequently, ζ_0 is a zero of $g(\zeta) - \alpha_i$ of multiplicity at least $p+1$. By the Second Fundamental Theorem of Nevanlinna, we have

$$\begin{aligned}
2T(r,g) &\leqslant \sum_{i=1}^{3} \overline{N}\left(r, \frac{1}{g-\alpha_i}\right) + \overline{N}(r,g) + S(r,g) \\
&\leqslant \sum_{i=1}^{3} \frac{1}{p+1} N\left(r, \frac{1}{g-\alpha_i}\right) + \frac{1}{3}N(r,g) + S(r,g) \\
&\leqslant \frac{3}{p+1}T(r,g) + \frac{1}{3}T(r,g) + S(r,g) \\
&= \left(\frac{p+10}{3p+3)}\right)T(r,g) + S(r,g).
\end{aligned}$$

This leads to a contradiction, and hence f must be normal at z_0. □

Proof of Theorem 3. Assume, for the sake of contradiction, that $f(z)$ is not φ-normal function in $\mathbb{D}$. By applying Lemma 2, there exist

(i) a sequence $\{a_n\} \subset \mathbb{D}$, $|a_n| \to 1$,

(ii) a sequence $\{z_n\} \subset \mathbb{D}$, $z_n \to z^* \in \mathbb{D}$ and $w_n = a_n + \frac{z_n}{\varphi(|a_n|)} \to w_0$ (say) $\in \mathbb{D}$,

(iii) a sequence $\rho_n^+ \to 0$

such that $g_n(\zeta) = f(w_n + \frac{\rho_n}{\varphi(|a_n|)}\zeta) \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is non-constant meromorphic function in $\mathbb{C}$. Additionally, all the poles of g are of multiplicity at least three. On differentiating 'k' times, we obtain

$$g_n{}^{(k)}(\zeta) = \left(\frac{\rho_n}{\varphi(|a_n|)}\right)^k f^{(k)}\left(w_n + \frac{\rho_n \zeta}{\varphi(|a_n|)}\right) \to g^{(k)}(\zeta) \text{ in } \mathbb{C}.$$

It is evident that g must take on at least one value from S; otherwise, by Picard's theorem, g would be constant. Let ζ_0 be a zero of $g - \alpha_i$ (for $i = 1,2,3$). By applying Hurwitz theorem, we can derive a sequence $\zeta_n \to \zeta_0$ such that

$$f\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right) = g_n(\zeta_n) = \alpha_i \in S$$

From the given assumption, we have $f(z) \in S$ implies $|L(f)| < A\varphi^k(|z|)$.

Therefore,

$$\left| f^{(k)}\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right) + \sum_{m=0}^{k-1} a_m\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right) f^{(m)}\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right)\right|$$
$$< A\varphi^k\left(\left|w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right|\right). \tag{4}$$

Since α_i is not a pole of g and $g(\zeta_0) = \alpha_i$, there exist a neighborhood of ζ_0 in which g is analytic. Therefore,

$$a_m\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right) g_n^{(m)}(\zeta_n) \to a_m(w_0) g^{(m)}(\zeta_0) \in \mathbb{C} \text{ as } n \to \infty \text{ for } m = 0,1,\cdots,k-1$$

so that

$$g_n^{(k)}(\zeta_n) + \sum_{m=0}^{k-1}\left(\frac{\rho_n}{\varphi(|a_n|)}\right)^{k-m} a_m\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right) g_n^{(m)}(\zeta_n) \to g^{(k)}(\zeta_0) \text{ as } n \to \infty. \tag{5}$$

Thus for sufficiently large n and for $k = 1,2,\cdots,p$, we have

$$\begin{aligned}
&\left| g_n^{(k)}(\zeta_n) + \sum_{m=0}^{k-1}\left(\frac{\rho_n}{\varphi(|a_n|)}\right)^{k-m} a_m\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right) g_n^{(m)}(\zeta_n)\right| \\
&\quad = \left| \left(\frac{\rho_n}{\varphi(|a_n|)}\right)^k f^{(k)}\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right)\right. \\
&\qquad \left. + \sum_{m=0}^{k-1}\left(\frac{\rho_n}{\varphi(|a_n|)}\right)^{k-m} a_m\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right)\left(\frac{\rho_n}{\varphi(|a_n|)}\right)^m f^{(m)}\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right)\right| \\
&\quad = \left(\frac{\rho_n}{\varphi(|a_n|)}\right)^k \left| f^{(k)}\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right)\right. \\
&\qquad \left. + \sum_{m=0}^{k-1} a_m\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right) f^{(m)}\left(w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}\right)\right| \\
&\quad < A\rho_n{}^k \frac{\varphi^k(|w_n + \frac{\rho_n \zeta_n}{\varphi(|a_n|)}|)}{\varphi^k(|a_n|)} \to 0 \text{ as } n \to \infty,
\end{aligned}$$

since $\frac{\varphi(|a+z/\varphi(|a|))}{\varphi(|a|)} \to 1$, $|a| \to 1^-$.

Thus equation 5 implies that $g^{(k)}(\zeta_0) = 0$ for $k = 1,2,\cdots,p$. Consequently, ζ_0 is a zero of $g(\zeta) - \alpha_i$ of multiplicity at least $p+1$. Now, by applying the Second Fundamental Theorem of Nevanlinna, we obtain

$$\begin{aligned}
2T(r,g) &\leqslant \sum_{i=1}^{3} \overline{N}\Big(r, \frac{1}{g-\alpha_i}\Big) + \overline{N}(r,g) + S(r,g) \\
&\leqslant \sum_{i=1}^{3} \frac{1}{p+1} N\Big(r, \frac{1}{g-\alpha_i}\Big) + \frac{1}{3}N(r,g) + S(r,g) \\
&\leqslant \frac{3}{p+1}T(r,g) + \frac{1}{3}T(r,g) + S(r,g) \\
&= \Big(\frac{p+10}{3p+3}\Big)T(r,g) + S(r,g)
\end{aligned}$$

which leads to a contradiction, and hence f must be φ-normal function in $\mathbb{D}$. □

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REFERENCES

[1] A. AULASKARI AND J. RATTYA, *Properties of meromorphic φ-normal function*, J. Michigan. Math. **60** (2011), 93–111.

[2] Q. CHEN AND D. TONG, *Normal functions concerning derivatives and shared sets*, Boletin da la Sociedad Mathematica Mexicana **25** (2019), 589–596.

[3] O. LEHTO, K. L. VIRTANEN, *Boundary behaviour and normal meromorphic functions*, Acta Math. **97** (1957), 47–65.

[4] T. VAN TAN AND N. VAN THIN, *On Lappan's five-point theorem*, Comput. Methods Funct. Theory **17** (2017), 47–63.

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Banarsi Lal
Department of Mathematics, School of Sciences
Cluster University of Jammu
Jammu-180001 India
e-mail: banarsiverma644@gmail.com

Virender Singh
Department of Mathematics
University of Jammu
Jammu-180006, India
e-mail: virendersingh2323@gmail.com

Priyanka
Department of Mathematics, School of Sciences
Cluster University of Jammu
Jammu-180001, India
e-mail: priyanka.schmat@clujammu.ac.in