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FRACTIONAL ORDER EULER DIFFERENCE SEQUENCE SPACES BY USING MODULUS FUNCTION

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Abstract. In this paper, we introduces new classes of Euler fractional difference sequence spaces defined by modulus function. These spaces generalize the classical difference sequence spaces by incorporating the fractional difference operator and a modulus function, providing a more flexible and comprehensive framework. The study investigates the topological properties of these spaces and determines their dual spaces, offering significant insights into their structural characteristics.

1. Introduction and preliminaries

Let ω represent the set of all real-valued sequences. The classical spaces ℓ_∞, c and c_0 denotes the Banach spaces of bounded sequences, convergent sequences, and sequences converging to zero, respectively, equipped with the norm $\|x\| = \sup_k |x_k|$. The spaces bs, cs, ℓ_1 and ℓ_p, respectively, the space of all bounded, convergent, absolutely convergent and p-absolutely convergent series $(1 \leqslant p < \infty)$.

The study of difference sequence spaces has gained significant attention due to their applications in summability theory, functional analysis, and matrix transformations. Early contributions to this field include the work of Ahmad and Mursaleen [2], who investigated Köthe-Toeplitz duals of specific sequence spaces and their matrix transformations, highlighting foundational aspects of sequence spaces and duality relationships. Subsequent contributions by Başar [13] provided a comprehensive overview of summability theory and its applications, offering a broad framework for the study of sequence spaces.

In the context of difference sequences, Et and Başarir [20] introduced generalized difference sequence spaces, extending the scope of classical sequence spaces to accommodate generalized difference operators and specific modulus functions. Building on these ideas, Et and Çolak [21] focused on further generalizations, introducing new sequence spaces through innovative formulation of difference operators. Building upon this line of research, Kadak [24] introduced fractional order generalized lacunary statistical difference sequence spaces, thereby bridging the concepts of statistical convergence and fractional calculus.

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Mursaleen and Noman [33] also contributed significantly by studying difference sequence spaces of non-absolute type and their dual spaces in various settings, providing insights into the transformations and relationships within such spaces. Malkowsky et al. [30] examined the dual spaces of higher order sets of difference sequences spaces and their associated matrix transformations, expanding the applicability of difference sequence theory to a broader mathematical context. For additional studies on difference operators and their applications in sequence spaces, see [27, 41, 42] and their references.

This paper seeks to extend these foundational studies, particularly those by Kadak and Baliarsingh [24, 25] investigating new generalizations of difference sequence spaces through the incorporation of fractional-order difference operators. The inclusion of modulus functions enables a more refined analysis of matrix transformations, providing a versatile framework to study the statistical properties and summability behaviors of these spaces. This approach bridges the gap between classical sequence space theory and modern developments in generalized difference operators, offering new perspectives and applications within summability theory. For more details regarding these see [12, 17, 26, 43] and their references. For further details on fundamental theorems in functional analysis, summability theory, sequence spaces, and related topics, readers can refer to the recent textbook [32] and to the articles devoted to the new spaces of difference sequences [7, 15, 16, 36], and [14].

For real number ψ, the Euler gamma function $\Gamma(\psi)$ is defined by an improper integral

$$\Gamma(\psi) = \int_0^\infty e^{-t} t^{\psi-1} dt, \quad \text{for} \quad \psi > 0. \tag{1}$$

Baliarsingh and Dutta [8, 9] (see also [11, 19]) proposed the generalized fractional difference operator $\Delta^{(\psi)}$ for a positive proper fraction ψ, which is defined as follows:

$$\Delta^{(\psi)}(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\psi+1)}{i!\Gamma(\psi-i+1)} x_{k-i}. \tag{2}$$

The series in equation (2) is convergent for $x \in \omega$. Specifically, the difference operator $\Delta^{(\psi)}$ can be expressed in a triangular form as follows

$$(\Delta^{(\psi)})_{mk} = \begin{cases} (-1)^{m-k} \dfrac{\Gamma(\psi+1)}{(m-k)!\Gamma(\psi-m+k+1)}, & 0 \leqslant k \leqslant m, \\ 0, & k > m. \end{cases}$$

The Euler mean matrix $E^s = (e^s_{mk})$ of order s, $(0 < s < 1)$ as defined in [25], as

$$e^s_{mk} = \begin{cases} \dbinom{m}{k} (1-s)^{m-k} s^k, & 0 \leqslant k \leqslant m, \\ 0, & k > m. \end{cases}$$

Alternatively, this can be expressed as

$$E^s = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1-s & s & 0 & 0 & \cdots \\ (1-s)^2 & 2(1-s)s & s^2 & 0 & \cdots \\ (1-s)^3 & 3(1-s)^2 s & 3(1-s)s^2 & s^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Kadak and Baliarsingh [25] introduced the product matrix $E^s(\Delta^{(\psi)})$ by combining the Euler mean matrix of order s with the difference matrix of order (ψ). This product matrix is defined as follows

$$(E^s(\Delta^{(\psi)}))_{mk} = \begin{cases} \sum_{i=k}^{m} (-1)^{i-k} \binom{m}{m-i} \frac{\Gamma(\psi+1)}{(i-k)!\Gamma(\psi-i+k+1)} s^i (1-s)^{m-i}, & 0 \leqslant k \leqslant m, \\ 0, & k > m. \end{cases}$$

Furthermore, $(E^s(\Delta^{(\psi)}))_{mk}$ can be written as

$$E^s(\Delta^{(\psi)}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ (1-s)-\psi s & s & 0 & 0 & \cdots \\ (1-s)^2 - 2\psi(1-s)s + \frac{\psi(\psi-1)}{2!}s^2 & 2(1-s)s - \psi s^2 & s^2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $\mathscr{A} = (a_{mk})$ represent an infinite matrix of real numbers a_{mk}, where $m,k \in \mathbb{N}_0$. For sequence spaces X and Y, the matrix mapping $\mathscr{A} : X \to Y$ is defined by

$$(\mathscr{A}x)_m = \sum_k a_{mk} x_k \quad (m \in \mathbb{N}_0),$$

where $x = (x_k) \in X$. The expression $\mathscr{A}x$ is called the $\mathscr{A}$-transform of x, if for each $m \in \mathbb{N}_0$, the series $\sum_k a_{mk} x_k$ converges. The notation (X,Y) represents the set of all infinite matrices $\mathscr{A}$ such that $\mathscr{A}$ maps from X to Y. Specifically, $\mathscr{A} \in (X,Y)$ if and only if the series in the above definition converges for every $m \in \mathbb{N}_0$.

The matrix domain $\Phi_{\mathscr{A}}$ of an infinite matrix $\mathscr{A}$ in a sequence space Φ is defined as

$$\Phi_{\mathscr{A}} = \{x = (x_k) \in \omega : \mathscr{A}x \in \Phi\}.$$

Duality is a important concept in the theory of sequence spaces, particularly in the investigation of their topological structures. Let X and Y be subsets of ω, the space of all sequences. The set $\mathscr{S}(X,Y)$ is formally defined as

$$\mathscr{S}(X,Y) = \{\sigma = (\sigma_k) : x \cdot \sigma = (x_k \sigma_k) \in Y \text{ for all } x \in X\}. \tag{3}$$

The α-, β- and γ- duals of sequence space X, by using the notation (3), are defined as $X^\alpha = \mathscr{S}(X,\ell_1)$, $X^\beta = \mathscr{S}(X,cs)$ and $X^\gamma = \mathscr{S}(X,bs)$. There are some lemmas on the inverse of matrices, as presented by Kadak and Baliarsingh [25], as

LEMMA 1. [10, 18] *The difference matrix $\Delta^{(\psi)}$ has an inverse that can be expressed in triangular form as*

$$(\Delta^{(-\psi)})_{mk} = \begin{cases} (-1)^{(m-k)} \dfrac{\Gamma(-\psi+1)}{(m-k)!\Gamma(-\psi-m+k+1)}, & 0 \leqslant k \leqslant m, \\ 0, & k > m. \end{cases}$$

LEMMA 2. [18] *The Euler mean matrix E^s has an inverse which is given by the triangular form as*

$$(E^{\frac{1}{s}})_{mk} = \begin{cases} (-1)^{(m-k)} \dbinom{m}{k} (1-s)^{m-k} s^{-m}, & 0 \leqslant k \leqslant m, \\ 0, & k > m. \end{cases}$$

LEMMA 3. *The inverse of the Euler mean difference matrix $E^s(\Delta^{(\psi)})$ is represented by a triangular matrix $(\mathfrak{b}_{mk})$ where*

$$\mathfrak{b}_{mk} = \begin{cases} \displaystyle\sum_{j=k}^{m} (-1)^{(m-k)} \binom{j}{k} \frac{\Gamma(-\psi+1)(1-s)^{j-k} s^{-j}}{(m-j)!\Gamma(-\psi-m+j+1)}, & 0 \leqslant k \leqslant m, \\ s^{\frac{1}{m}}, & k = m \\ 0, & k > m. \end{cases}$$

A modulus function is a function $f : [0,\infty) \to [0,\infty)$ (or simply a modulus) if it satisfies the following conditions:

1. $f(u) = 0$ if and only if $u = 0$,
2. f is an increasing function,
3. $f(u+v) \leqslant f(u) + f(v)$ for every $u, v \in [0,\infty)$,
4. f is continuous from the right at 0.

Therefore, the function f must be continuous on the interval $[0,\infty)$. A modulus function can either be bounded or unbounded. Numerous authors have frequently utilized modulus functions in the construction of various difference sequence spaces see [29, 35, 38].

The introduction for generalized difference sequence spaces of fractional order, defined by a modulus function, building on concepts from [39] and [40]. For more information about on these see ([1, 6, 22, 23, 28]) and references therein.

The foundational work of Altay and Başar [3] on Euler sequence spaces of non-absolute type laid a cornerstone for this field, presenting key insights into their structure and functional properties. These studies further extended to include the investigation of Euler sequence spaces that summarize the classical spaces ℓ_p and ℓ_∞, revealing their intricate relationships and applications in mathematical analysis (see [4]).

In the context of difference sequence spaces, the contributions by Altay and Polat [5] introduced new class of Euler difference sequence spaces, emphasizing their importance in understanding the behavior of sequences under difference operators. Specifically, Polat and Başar [34] explored Euler sequence spaces of difference sequences of order m, uncovering new properties and generalizations.

Building on these foundational works, this paper aims to introduce novel classes of Euler fractional difference sequence spaces defined via a modulus function. These spaces generalize the traditional difference sequence spaces by incorporating the fractional difference operator and a modulus function, thereby enriching the existing theory with additional flexibility and broader applicability. Furthermore, we investigate their topological properties and determine their dual spaces, providing a comprehensive framework for their study.

2. Main results

In this section, we define certain sequence spaces utilizing the Euler mean operator E^s and the fractional difference operator $\Delta^{(\psi)}$, with the construction being based on the application of a modulus function. In addition, the study explores the topological properties of these sequence spaces. Furthermore, their corresponding dual spaces are systematically identified. Let s be a positive real number such that $0 < s < 1$ and $u = (u_k)$ be an arbitrary bounded sequence of positive real numbers. We define the following classes of Euler fractional difference sequence spaces associated with a modulus function as

$$e_p^s(\Delta^{(\psi)}, f, u) =$$

$$\left\{ x = (x_k) : \sum_m \left| \sum_{j=0}^{m} \sum_{i=j}^{m} (-1)^{i-j} f_m \left[\binom{m}{m-i} \frac{\Gamma(\psi+1)}{(i-j)!\Gamma(\psi-i+j+1)} s^i (1-s)^{m-i} x_j \right] \right|^{u_k} < \infty \right\},$$

$$e_0^s(\Delta^{(\psi)}, f, u) =$$

$$\left\{ x = (x_k) : \lim_{m\to\infty} \left| \sum_{j=0}^{m} \sum_{i=j}^{m} (-1)^{i-j} f_m \left[\binom{m}{m-i} \frac{\Gamma(\psi+1)}{(i-j)!\Gamma(\psi-i+j+1)} s^i (1-s)^{m-i} x_j \right] \right|^{u_k} = 0 \right\},$$

$$e_c^s(\Delta^{(\psi)}, f, u) =$$

$$\left\{ x = (x_k) : \lim_{m\to\infty} \left| \sum_{j=0}^{m} \sum_{i=j}^{m} (-1)^{i-j} f_m \left[\binom{m}{m-i} \frac{\Gamma(\psi+1)}{(i-j)!\Gamma(\psi-i+j+1)} s^i (1-s)^{m-i} x_j \right] \right|^{u_k} \text{exists} \right\},$$

$$e_\infty^s(\Delta^{(\psi)}, f, u) =$$

$$\left\{ x = (x_k) : \sup_m \left| \sum_{j=0}^{m} \sum_{i=j}^{m} (-1)^{i-j} f_m \left[\binom{m}{m-i} \frac{\Gamma(\psi+1)}{(i-j)!\Gamma(\psi-i+j+1)} s^i (1-s)^{m-i} x_j \right] \right|^{u_k} < \infty \right\}.$$

It is observed that the spaces $e_p^s(\Delta^{(\psi)}, f, u), e_0^s(\Delta^{(\psi)}, f, u), e_c^s(\Delta^{(\psi)}, f, u)$ and $e_\infty^s(\Delta^{(\psi)}, f, u)$ can be obtained by applying the $E^s(\Delta^{(\psi)}, f, u)$-transform of x in the respective spaces ℓ^p, c_0, c, and ℓ^∞ over modulus function. Specifically,

$$e_p^s(\Delta^{(\psi)}, f, u) = (\ell_p)_{E^s(\Delta^{(\psi)}, f, u)}; \quad e_0^s(\Delta^{(\psi)}, f, u) = (c_0)_{E^s(\Delta^{(\psi)}, f, u)};$$

$$e_c^s(\Delta^{(\psi)}, f, u) = (c)_{E^s(\Delta^{(\psi)}, f, u)}; \quad e_\infty^s(\Delta^{(\psi)}, f, u) = (\ell_\infty)_{E^s(\Delta^{(\psi)}, f, u)}.$$

The sequence $y = (y_k)$ is defined as the $E^s(\Delta^{(\psi)}, f, u)$-transform of a sequence $x = (x_k)$, given by:

$$y_k = \sum_{j=0}^{k}\sum_{i=j}^{k}(-1)^{i-j} f_m\left[\binom{k}{k-i}\frac{\Gamma(\psi+1)}{(i-j)!\Gamma(\psi-i+j+1)}s^i(1-s)^{k-i}x_j\right], \quad k \in \mathbb{N}_0. \quad (4)$$

THEOREM 1. *Let ψ be a positive proper fraction and f be a modulus function. Then the sequence spaces $e_p^s(\Delta^{(\psi)}, f, u), e_0^s(\Delta^{(\psi)}, f, u), e_c^s(\Delta^{(\psi)}, f, u)$ and $e_\infty^s(\Delta^{(\psi)}, f, u)$ are linearly isomorphic to the sequence spaces ℓ_p, c_0, c and ℓ_∞ respectively.*

Proof. We establish the proof for the space $e_\infty^s(\Delta^{(\psi)}, f, u)$ by showing the spaces $e_\infty^s(\Delta^{(\psi)}, f, u)$ and ℓ_∞ are linearly bijective. Define a mapping, by using the equation (4) as $T : e_\infty^s(\Delta^{(\psi)}, f, u) \to \ell_\infty$ by $x \mapsto Tx = y$. Clearly T is linear. Next, we demonstrate injectivity. If $Tx = \theta = (o, o, o, \cdots)$, then $x = \theta$. Therefore, T is injective. For surjective, consider $y \in \ell_\infty$. Using Lemma 3, we define the sequence $x = (x_k)$ by

$$x_k = \sum_{i=0}^{k}\sum_{j=i}^{k}(-1)^{k-i} f_m\left[\binom{j}{i}\frac{\Gamma(-\psi+1)(1-s)^{j-i}s^{-j}}{(k-j)!\Gamma(-\psi-k+j+1)}y_i\right], \quad k \in \mathbb{N}_0. \quad (5)$$

Then we have

$$\sup_m \left|\sum_{j=0}^{m}\sum_{i=j}^{m}(-1)^{i-j} f_m\left[\binom{m}{m-i}\frac{\Gamma(\psi+1)}{(i-j)!\Gamma(\psi-i+j+1)}s^i(1-s)^{m-i}x_j\right]\right|^{u_k}$$

$$= \sup_m |y_m| = \|y\|_\infty < \infty.$$

Thus, we have $x \in e_\infty^s(\Delta^{(\psi)}, f, u)$ so T is surjective. This establish the proof of the theorem. □

THEOREM 2. *Let f be a modulus function. Then the spaces $e_p^s(\Delta^{(\psi)}, f, u)$, $e_0^s(\Delta^{(\psi)}, f, u)$, $e_c^s(\Delta^{(\psi)}, f, u)$ and $e_\infty^s(\Delta^{(\psi)}, f, u)$ are paranormed spaces with the paranorm*

$$\mathfrak{g}(x) = \sup_m \left|\sum_{j=0}^{m}\sum_{i=j}^{m}(-1)^{i-j} f_m\left[\binom{m}{m-i}\frac{\Gamma(\psi+1)}{(i-j)!\Gamma(\psi-i+j+1)}s^i(1-s)^{m-i}x_j\right]\right|^{\frac{u_k}{H}}$$

where $H = \max\{1, \sup_k u_k\}$.

Proof. Consider the space $e_0^s(\Delta^{(\psi)}, f, u)$. Clearly, $\mathfrak{g}(\theta) = 0$ where $\theta = (0,0,0,\cdots)$ and $\mathfrak{g}(-x) = \mathfrak{g}(x)$ for all $x \in e_0^s(\Delta^{(\psi)}, f, u)$. For linearity, suppose that the sequences $x = (x_k)$, $y = (y_k) \in e_0^s(\Delta^{(\psi)}, f, u)$. By using the definition of modulus function and Minkowski's inequality, we have

$$\begin{aligned}
&\mathfrak{g}(x_k + y_k) \\
&= \sup_m \left| \sum_{j=0}^{m} \sum_{i=j}^{m} (-1)^{i-j} f_m \left[\binom{m}{m-i} \frac{\Gamma(\psi+1)}{(i-j)!\Gamma(\psi-i+j+1)} s^i (1-s)^{m-i} (x_k + y_k) \right] \right|^{\frac{u_k}{H}} \\
&\leqslant \sup_m \left| \sum_{j=0}^{m} \sum_{i=j}^{m} (-1)^{i-j} f_m \left[\binom{m}{m-i} \frac{\Gamma(\psi+1)}{(i-j)!\Gamma(\psi-i+j+1)} s^i (1-s)^{m-i} (x_k) \right] \right|^{\frac{u_k}{H}} \\
&\quad + \sup_m \left| \sum_{j=0}^{m} \sum_{i=j}^{m} (-1)^{i-j} f_m \left[\binom{m}{m-i} \frac{\Gamma(\psi+1)}{(i-j)!\Gamma(\psi-i+j+1)} s^i (1-s)^{m-i} (y_k) \right] \right|^{\frac{u_k}{H}} \\
&= \mathfrak{g}(x) + \mathfrak{g}(y).
\end{aligned}$$

Hence $\mathfrak{g}$ is subadditive,

$$\mathfrak{g}(x+y) \leqslant \mathfrak{g}(x) + \mathfrak{g}(y), \quad \text{for all } x, y \in e_0^s(\Delta^{(\psi)}, f, u).$$

Next, consider scalar multiplication, let λ be any scalar. According to the definition of the modulus function, we have

$$\begin{aligned}
\mathfrak{g}(\lambda x) &= \sup_m \left| \sum_{j=0}^{m} \sum_{i=j}^{m} (-1)^{i-j} f_m \left[\binom{m}{m-i} \frac{\Gamma(\psi+1)}{(i-j)!\Gamma(\psi-i+j+1)} s^i (1-s)^{m-i} (\lambda x_j) \right] \right|^{\frac{u_k}{H}} \\
&\leqslant K^{\frac{Q}{H}} \mathfrak{g}(x),
\end{aligned}$$

where K represents positive integer such that $K = 1 + |\lambda|$. By modulus function, we have $x \to 0$ implies $\mathfrak{g}(x)$. Similarly $x \to 0$ and $\lambda \to 0$ implies $\mathfrak{g}(\lambda x) \to 0$. Finally, we fixed x and $\lambda \to 0$ implies $\mathfrak{g}(\lambda x) \to 0$. Therefore, $e_p^s(\Delta^{(\psi)}, f, u)$ is a paranorm space. Similarly, the proof for other sequence spaces can be established. □

THEOREM 3. *Let f be a modulus function and ψ be a positive proper fraction. Then the sequence space $e_p^s(\Delta^{(\psi)}, f, u)$ is a complete normed linear space, with respect to the co-ordinatewise addition and scalar multiplication of sequences. It is a BK-space with the norm which is defined as*

$$\|x\|_{e_p^s(\Delta^{(\psi)}, f, u)} = \|E^s(\Delta^{(\psi)}, f, u)\|_p, \quad 1 \leqslant p < \infty.$$

Also, the sequence spaces $e_0^s(\Delta^{(\psi)}, f, u), e_c^s(\Delta^{(\psi)}, f, u)$ and $e_\infty^s(\Delta^{(\psi)}, f, u)$ are complete normed linear spaces, with operations of addition and scalar multiplication defined coordinate-wise. These spaces are also BK-spaces, equipped with the norm

$$\|x\|_{e_c^s(\Delta^{(\psi)}, f, u)} = \|x\|_{e_\infty^s(\Delta^{(\psi)}, f, u)} = \|E^s(\Delta^{(\psi)}, f, u)x\|_\infty.$$

Proof. The proof is straightforward, so we omit details. □

THEOREM 4. *Let f be a modulus function and for all $k \in \mathbb{N}_0$, $\Lambda_k = (E^s(\Delta^{(\psi)}, f, u)x)_k$. For fixed $k \in \mathbb{N}_0$, the sequence $\mathfrak{b}_m^{(k)} = \{\mathfrak{b}_m^{(k)}\}_{n \in \mathbb{N}_0}$ defined as*

$$\mathfrak{b}_m^{(k)} = \begin{cases} \sum_{j=k}^{m} (-1)^{(m-k)} f_m \left[\binom{j}{k} \frac{\Gamma(-\psi+1)(1-s)^{j-k} s^{-j}}{(m-j)!\Gamma(-\psi-m+j+1)} \right], & 0 \leqslant k \leqslant m, \\ s^{\frac{1}{m}}, & k = m \\ 0, & k > m. \end{cases}$$

Then

1. *The sequence $\{\mathfrak{b}_m^{(k)}\}_{n \in \mathbb{N}_0}$ forms a basis for the space $e_0^s(\Delta^{(\psi)}, f, u)$ and any $x \in e_0^s(\Delta^{(\psi)}, f, u)$ can be uniquely expressed as*

$$x = \sum_k \Lambda_k \mathfrak{b}_m^{(k)}.$$

2. *The set $\{\nu, \beta^{(k)}\}$ is a basis for the space $e_c^s(\Delta^{(\psi)}, f, u)$ and any $x \in e_c^s(\Delta^{(\psi)}, f, u)$ has a unique representation of the form*

$$x = l\nu + \sum_k (\Lambda_k - l)\mathfrak{b}^{(k)},$$

where $l = \lim_{k \to \infty} \Lambda_k$ and $\nu = (\nu_k)$, defined by

$$\nu_k = \sum_{i=0}^{k} \sum_{j=i}^{k} (-1)^{(k-i)} f_m \left[\binom{j}{i} \frac{\Gamma(-\psi+1)(1-s)^{j-i} s^{-j}}{(k-j)!\Gamma(-\psi-k+j+1)} \right].$$

Proof. (1) From the definition of $E^s(\Delta^{(\psi)}, f, u)$ and $\mathfrak{b}_m^{(k)}(t)$, it follows that

$$E^s(\Delta^{(\psi)}, f, u)\mathfrak{b}_m^{(k)}(t) = e^{(k)},$$

where $e^{(k)}$ represents the standard basis element in the space $e_0^s(\Delta^{(\psi)}, f, u)$.

Let $x \in e_0^s(\Delta^{(\psi)}, f, u)$. Then

$$x^{[v]} = \sum_{k=0}^{v} \Lambda_k \mathfrak{b}_m^{(k)}(t), \quad \text{for } v \geqslant 0.$$

By applying $E^s(\Delta^{(\psi)}, f, u)$, we have

$$\begin{aligned} E^s(\Delta^{(\psi)}, f, u)x^{[v]} &= \sum_{k=0}^{v} \Lambda_k E^s(\Delta^{(\psi)}, f, u)\mathfrak{b}_m^{(k)}(t) \\ &= \sum_{k=0}^{v} \Lambda_k e^{(k)} \\ &= ((E^s(\Delta^{(\psi)}, f, u))x)_k e^{(k)} \end{aligned}$$

and

$$E^s(\Delta^{(\psi)},f,u)(x-x^{[v]})_s = \begin{cases} 0, & \text{if } 0 \leqslant s \leqslant v \\ ((E^s(\Delta^{(\psi)},f,u)x)_k, & \text{if } s > v; \end{cases}$$

where $s,v \in \mathbb{N}_0$. For any $\varepsilon > 0$, there exists an integer q_0 such that

$$\sup_{s \geqslant v} \left|((E^s(\Delta^{(\psi)},f,u)x)_s\right|^{\frac{u_k}{H}} < \frac{\varepsilon}{2}, \text{ for all } v \geqslant q_0.$$

Thus

$$\mathfrak{g}\left(x-x^{[v]}\right) = \sup_{s \geqslant v} \left|((E^s(\Delta^{(\psi)},f,u))x)_s\right|^{\frac{u_k}{H}} < \frac{\varepsilon}{2}, \text{ for all } v \geqslant q_0.$$

Now, suppose that $x = \sum_k c_k(t)\mathfrak{b}_m^{(k)}(t)$. Since the linear mapping T from $e_0^s(\Delta^{(\psi)},f,u)$ to $c_0(p)$ is continuous, we have

$$\begin{aligned} ((E^s(\Delta^{(\psi)},f,u))x)_k &= \sum_k c_k(t)\left(E^s(\Delta^{(\psi)},f,u)\mathfrak{b}_m^{(k)}(t)\right) \\ &= \sum_k c_k(t)e^{(k)} = c_m(t). \end{aligned}$$

This is contrary to our assumption that $((E^s(\Delta^{(\psi)},f,u)x)_k = \Lambda_k(t)$ for each $k \in \mathbb{N}_0$. Therefore, the representation is unique.

(2) It directly follows from part (1). □

3. Duals of the spaces $e_p^s(\Delta^{(\psi)},f,u), e_0^s(\Delta^{(\psi)},f,u)$, $e_c^s(\Delta^{(\psi)},f,u)$ and $e_\infty^s(\Delta^{(\psi)},f,u)$

In this section, we aim to establish and prove the theorems that precisely characterize the α-, β- and γ-duals of spaces of non-absolute type.

For $1 \leqslant p < \infty$ with $\frac{1}{p}+\frac{1}{q} = 1$, it is well-known that the β-dual $\{\ell_\infty\}^\beta$ of the space ℓ_∞ is ℓ_1 and the β-dual $\{\ell_p\}^\beta$ of the space ℓ_p is ℓ_q. Throughout this paper, by $\mathscr{F}$ we denote the set of all finite subsets of $\mathbb{N}$. Additionally, we will revisit several lemmas from Stieglitz and Tietz [37], which play a crucial role in establishing the theorems that follow.

LEMMA 4.

1. *Let f be a modulus function. Then the matrix $\mathscr{A} = (a_{mk}) \in (c_0,\ell_1) = (c,\ell_1)$ if and only if*

$$\sup_{k \in \mathscr{F}} \sum_m \left| \sum_{k \in \mathscr{K}} f_k a_{mk} \right| < \infty. \tag{6}$$

2. *Let f be a modulus function. Then the matrix $\mathscr{A} = (a_{mk}) \in (c_0,c)$ if and only if*

$$\lim_{m \to \infty} a_{mk} = \ell_k \text{ for all } k, \tag{7}$$

and

$$\sup_{m \in \mathbb{N}} \sum_k f_k |a_{mk}| < \infty. \tag{8}$$

3. *Let f be a modulus function. Then the matrix $\mathscr{A}=(a_{mk})\in(c_0,\ell_\infty)$ if and only if (8) holds.*

THEOREM 5. *Let f be a modulus function. Define the sets $\mathfrak{h}_\infty^\psi(s),\mathfrak{h}^\psi(s)$ and $\mathfrak{h}_p^\psi(s)$ as follows*

$$\mathfrak{h}_\infty^\psi(s)=\Bigg\{x=(x_k)\in z:\sup_{k\in\mathbb{N}}\sum_m\Bigg|\sum_{j=k}^m(-1)^{m-k}f_m\Bigg[\frac{\Gamma(1-\psi)}{(m-j)!\Gamma(1-\psi-m+j)}\times\binom{j}{k}(1-s)^{j-k}s^{-j}x_j\Bigg]\Bigg|^{u_k}<\infty\Bigg\},$$

$$\mathfrak{h}^\psi(s)=\Bigg\{x=(x_k)\in z:\sup_{\mathscr{K}\in\mathscr{F}}\sum_m\Bigg|\sum_{k\in\mathscr{K}}\sum_{j=k}^m(-1)^{m-k}f_m\Bigg[\frac{\Gamma(1-\psi)}{(m-j)!\Gamma(1-\psi-m+j)}\times\binom{j}{k}(1-s)^{j-k}s^{-j}x_j\Bigg]\Bigg|^{u_k}<\infty\Bigg\},$$

$$\mathfrak{h}_p^\psi(s)=\Bigg\{x=(x_k)\in z:\sup_{\mathscr{K}\in\mathscr{F}}\sum_k\Bigg|\sum_{m\in\mathscr{K}}\sum_{j=k}^m(-1)^{m-k}f_m\Bigg[\frac{\Gamma(1-\psi)}{(m-j)!\Gamma(1-\psi-m+j)}\times\binom{j}{k}(1-s)^{j-k}s^{-j}x_j\Bigg]\Bigg|^{u_k}<\infty\Bigg\}.$$

Then $\{e_1^s(\Delta^{(\psi)},f,u)\}^\alpha=\mathfrak{h}_\infty^\psi(s)$, $\{e_0^s(\Delta^{(\psi)},f,u)\}^\alpha=\{e_c^s(\Delta^{(\psi)},f,u)\}^\alpha=\mathfrak{h}^\psi(s)$ and $\{e_p^s(\Delta^{(\psi)},f,u)\}^\alpha=\mathfrak{h}_p^\psi(s)$.

Proof. Since the proofs for the spaces $e_1^s(\Delta^{(\psi)},f,u)$ and $e_p^s(\Delta^{(\psi)},f,u)$ are similar in structure. Now, we establish the proof for the spaces $e_0^s(\Delta^{(\psi)},f,u)$ and $e_c^s(\Delta^{(\psi)},f,u)$.

Consider $x=(x_m)$, define the matrix $(\mathfrak{b}_{mk}^\psi(s))$ as follows

$$\mathfrak{b}_{mk}^\psi(s)=\begin{cases}\sum_{j=k}^m(-1)^{(m-k)}f_m\left[\frac{\Gamma(1-\psi)}{(m-j)!\Gamma(1-\psi-m+j)}\binom{j}{k}(1-s)^{j-k}s^{-j}\right], & k<m,\\ \frac{1}{s^m}, & k=m\\ 0, & k>m.\end{cases}\tag{9}$$

Consider the equality (5), it follows that

$$\begin{aligned}x_mz_m&=\sum_{k=0}^m\left[\sum_{j=k}^m(-1)^{m-k}f_m\left[\frac{\Gamma(1-\psi)}{(m-j)!\Gamma(1-\psi-m+j)}\binom{j}{k}(1-s)^{j-k}s^{-j}x_j\right]\right]y_k\\&=(\mathfrak{b}_{mk}^\psi(s)y)_m,\quad m\in\mathbb{N}.\end{aligned}\tag{10}$$

Thus, from the equality (10), we observe that $xz=(x_mz_m)\in\ell_1$ for $z\in e_0^s(\Delta^{(\psi)},f,u)$ or $e_c^s(\Delta^{(\psi)},f,u)$ if and only if $\mathfrak{b}_{mk}^\psi(s)y\in\ell_1$ whenever $y\in c_0$ or c. By Lemma 4(1), we have

$$\sup_{k\in\mathscr{F}}\sum_m\left|\sum_{k\in\mathscr{K}}\sum_{j=k}^m(-1)^{m-k}f_m\left[\frac{\Gamma(1-\psi)}{(m-j)!\Gamma(1-\psi-m+j)}\binom{j}{k}(1-s)^{j-k}s^{-j}x_j\right]\right|^{u_k}<\infty.$$

Therefore $\{e_0^s(\Delta^{(\psi)}, f, u)\}^{\alpha} = \{e_c^s(\Delta^{(\psi)}, f, u)\}^{\alpha} = \mathfrak{h}^{\psi}(s)$. □

THEOREM 6. *Let f be a modulus function. Define the sets $\mathfrak{d}_1^{\psi}(s)$, $\mathfrak{d}_2^{\psi}(s)$ and $\mathfrak{d}_3^{\psi}(s)$ by*

$$\mathfrak{d}_1^{\psi}(s) = \Bigg\{x = (x_k) \in z : \sup_{m\in\mathbb{N}} \sum_{k=0}^{m} \Bigg| \sum_{j=k}^{m} (-1)^{m-k} f_m \Bigg[\frac{\Gamma(1-\psi)}{(m-j)!\Gamma(1-\psi-m+j)} \times \binom{j}{k} (1-s)^{j-k} s^{-j} x_j\Bigg]\Bigg|^{u_k} < \infty\Bigg\},$$

$$\mathfrak{d}_2^{\psi}(s) = \Bigg\{x = (x_k) \in z : \lim_{m\to\infty} \Bigg| \sum_{j=k}^{m} (-1)^{m-k} f_m \Bigg[\frac{\Gamma(1-\psi)}{(m-j)!\Gamma(1-\psi-m+j)} \times \binom{j}{k} (1-s)^{j-k} s^{-j} x_j\Bigg]\Bigg|^{u_k} \text{ exists}\Bigg\},$$

$$\mathfrak{d}_3^{\psi}(s) = \Bigg\{x = (x_k) \in z : \lim_{m\to\infty} \Bigg| \sum_{k=0}^{m} \sum_{j=k}^{m} (-1)^{m-k} f_m \Bigg[\frac{\Gamma(1-\psi)}{(m-j)!\Gamma(1-\psi-m+j)} \times \binom{j}{k} (1-s)^{j-k} s^{-j} x_j\Bigg]\Bigg|^{u_k} \text{ exists}\Bigg\}.$$

Then $\{e_0^s(\Delta^{(\psi)}, f, u)\}^{\beta} = \mathfrak{d}_1^{\psi}(s) \cap \mathfrak{d}_2^{\psi}(s)$ and $\{e_c^s(\Delta^{(\psi)}, f, u)\}^{\beta} = \mathfrak{d}_1^{\psi}(s) \cap \mathfrak{d}_2^{\psi}(s) \cap \mathfrak{d}_3^{\psi}(s)$.

Proof. We provide the proof for the space $e_0^s(\Delta^{(\psi)}, f, u)$. Let us consider the equality

$$\begin{aligned}\sum_{k=0}^{m} x_k z_k &= \sum_{k=0}^{m} \Bigg[\sum_{j=0}^{k} (-1)^{m-k} f_m \Bigg[\frac{\Gamma(1-\psi)}{(m-j)!\Gamma(1-\psi-m+j)} \binom{k}{j} (1-s)^{k-j} s^{-k} y_j\Bigg]\Bigg] x_k \\ &= \sum_{k=0}^{m} \Bigg[\sum_{j=k}^{m} (-1)^{m-k} f_m \Bigg[\frac{\Gamma(1-\psi)}{(m-j)!\Gamma(1-\psi-m+j)} \binom{j}{k} (1-s)^{j-k} s^{-j} x_j\Bigg]\Bigg] y_k \\ &= (\mathfrak{b}_{mk}^{\psi}(s) y)_m \end{aligned} \tag{11}$$

where $\mathfrak{b}_{mk}^{\psi}(s)$ is defined in (9). Therefore, by applying Lemma 4(2) in conjunction with (11) it follows that $(x_k z_k) \in cs$ whenever $z \in e_0^s(\Delta^{(\psi)}, f, u)$ if and only if $\mathfrak{b}_{mk}^{\psi}(s) \in c$ whenever $y = (y_k) \in c_0$. Hence, from (7) and (8), we deduce that

$$\lim_m \mathfrak{b}_{mk}^{\psi}(s) \text{ exists for each } k \in \mathbb{N} \text{ and } \sup_{m\in\mathbb{N}} \sum_{k=0}^{m} |\mathfrak{b}_{mk}^{\psi}(s)| < \infty.$$

Hence $\{e_0^s(\Delta^{(\psi)}, f, u)\}^{\beta} = \mathfrak{d}_1^{\psi}(s) \cap \mathfrak{d}_2^{\psi}(s)$. In similar way we can prove that $\{e_c^s(\Delta^{(\psi)}, f, u)\}^{\beta} = \mathfrak{d}_1^{\psi}(s) \cap \mathfrak{d}_2^{\psi}(s) \cap \mathfrak{d}_3^{\psi}(s)$. □

THEOREM 7. *Let f be a modulus function. Then the γ-duals of the spaces $e_0^s(\Delta^{(\psi)}, f, u)$, $e_c^s(\Delta^{(\psi)}, f, u)$ and $e_\infty^s(\Delta^{(\psi)}, f, u)$ is $\mathfrak{d}_1^{\psi}(s)$.*

Proof. Consider the space $e_0^s(\Delta^{(\psi)}, f, u)$. Suppose that $x = (x_k) \in \mathfrak{d}_1^{\psi}(s)$ and $z = (z_k) \in \{e_0^s(\Delta^{(\psi)}, f, u)\}^{\gamma}$. Consider the equality

$$\begin{aligned}\left|\sum_{k=0}^{m} x_k z_k\right| &= \left|\sum_{k=0}^{m}\left[\sum_{j=0}^{k}(-1)^{m-k} f_m\left[\frac{\Gamma(1-\psi)}{(m-j)!\Gamma(1-\psi-m+j)}\binom{k}{j}(1-s)^{k-j}s^{-k}y_j\right]\right]x_k\right| \\ &= \left|\sum_{k=0}^{m} \mathfrak{b}_{mk}^{\psi}(s) y_k\right| \\ &\leqslant \sum_{k=0}^{m} |\mathfrak{b}_{mk}^{\psi}(s)|\ |y_k|.\end{aligned}$$

By taking supremum over $m \in \mathbb{N}$, this leads to

$$\sup_{m\in\mathbb{N}}\left|\sum_{k=0}^{m} x_k z_k\right| \leqslant \sup_{m\in\mathbb{N}}\sum_{k=0}^{m} |\mathfrak{b}_{mk}^{\psi}(s)|\ |y_k| \leqslant \|y\|_{\infty} \sup_{m\in\mathbb{N}} |\mathfrak{b}_{mk}^{\psi}(s)| \leqslant \infty.$$

Thus $x = (x_k) \in \{e_0^s(\Delta^{(\psi)}, f, u)\}^{\gamma}$. Therefore, $\mathfrak{d}_1^{\psi}(s) \subset \{e_0^s(\Delta^{(\psi)}, f, u)\}^{\gamma}$.

Conversely, suppose that $x = (x_k) \in \{e_0^s(\Delta^{(\psi)}, f, u)\}^{\gamma}$ and $z \in \mathfrak{d}_1^{\psi}(s)$. Thus we may deduce that the sequence $\left(\sum_{k=0}^{m} \mathfrak{b}_{mk}^{\psi}(s) y_k\right)_{m\in\mathbb{N}} \in \ell_{\infty}$ whenever $(x_k z_k) \in bs$. This implies the matrix $(\mathfrak{b}_{mk}^{\psi}(s))$ is in (c_0, ℓ_{∞}). Hence

$$\sup_{m\in\mathbb{N}}\sum_{k=0}^{m} |\mathfrak{b}_{mk}^{\psi}(s)| < \infty$$

implies that $x = (x_k) \in \mathfrak{d}_1^{\psi}(s)$. Therefore, $\{e_0^s(\Delta^{(\psi)}, f, u)\}^{\gamma} \subset \mathfrak{d}_1^{\psi}(s)$. This proves that the γ-dual of the space $e_0^s(\Delta^{(\psi)}, f, u)$ is $\mathfrak{d}_1^{\psi}(s)$. $\square$

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