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ON UNIFORM CONVERGENCE OF TRIGONOMETRIC INTEGRAL–SERIES

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Abstract. In this paper, we give sufficient conditions for the uniform regular convergence of trigonometric integral-series, which are also necessary if the sequence of functions is non-negative. The new results also bring necessary and sufficient conditions for the uniform regular convergence of trigonometric integral-series in complex form.

1. Introduction

There are a great number of interesting results in Fourier analysis established by assuming the monotonicity of Fourier coefficients. Chaundy and Jolliffe [1] treat the fundamental theorem in the theory of uniform convergence of sine series with coefficients being a monotonic null sequence. Recently, the monotonicity condition has been relaxed by several authors. For instance, many generalizations consider coefficients as a sequence of bounded variation. Uniform convergence or boundedness of sine series with a sequence of coefficients being monotone and null are generalized to a certain subclass of the quasi-monotone decreasing sequence (Rest Bounded Variation Sequences and Head Bounded Variation Sequences, see [15]). Among others, uniform convergence of trigonometric series with coefficients of General Monotone Sequences, Mean Value Bounded Variation Sequences, Supremum Bounded Variation Sequences, (p,β)-General Monotone Sequence are considered in [6,8,16,17,21–23]. Many results on uniform convergence of single trigonometric series are extended to two-dimensional settings as well. For example, necessary and sufficient conditions for uniform convergence of double sine series with coefficients from a new class, Double General Monotone, of double numerical sequences are obtained in [5]. Also, a sufficient condition for uniform convergence of trigonometric series and double sine series with coefficients being a sequence of p-bounded variation are obtained in [9, 10].

Parallel to the study of uniform convergence of trigonometric series, one interesting topic, which we often encounter in literature, is the uniform convergence of integrals. One of the basic results of uniform convergence of sine integrals

$$F(x)=\int_0^\infty f(t)\sin xt\,dt$$

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of f can be found in [18], where $f:\mathbb{R}_+\to\mathbb{C}$ is a measurable function with the property that f is non-increasing and $tf(t)\in L^1_{loc}(\mathbb{R}_+)$, where $\mathbb{R}_+$ is the set of all positive real numbers. In this direction also, many authors studied uniform convergence of sine integrals, in particular, and trigonometric integrals, in general, while considering a bigger class of functions than the class of non-increasing functions. For example, uniform convergence of sine integrals of general monotone functions can be found in [2,12] and their integrability can be found in [4,16,17]. Such results are extended to two dimensions also. For instance, uniform convergence of double sine integrals has been studied by Móricz [19, 20], Kórus and Móricz [14], and Debernardi [3]. Kórus [13] has also studied uniform convergence of double cosine integrals, sine-cosine integrals, and double sine integrals of double general monotone functions extending the results studied earlier by Móricz.

Recently, Kórus, Kransniqi, and Szal [11] studied uniform convergence in the regular sense of the sine integral-series with the general monotone sequence of functions. To state their results and for our results, first, we give some preliminaries.

Let $\{f_k(t)\}_{k=1}^{\infty}$, $\{\phi_k(t)\}_{k=1}^{\infty}$, $\{\psi_k(t)\}_{k=1}^{\infty}$, and $\{g_k(t)\}_{k=1}^{\infty}$ be sequences of functions such that $f_k,\phi_k,\psi_k,g_k:\overline{\mathbb{R}}_+\to\mathbb{C}$ for every $k\in\mathbb{N}$, where $\overline{\mathbb{R}}_+:=[0,\infty)$. Then the integral-series given by

$$\int_0^{\infty}\sum_{k=1}^{\infty} f_k(t)\sin ku\sin tv\,dt,\quad (u,v)\in\overline{\mathbb{R}}_+^2, \tag{1}$$

$$\int_0^{\infty}\sum_{k=1}^{\infty} \phi_k(t)\sin ku\cos tv\,dt,\quad (u,v)\in\overline{\mathbb{R}}_+^2, \tag{2}$$

$$\int_0^{\infty}\sum_{k=1}^{\infty} \psi_k(t)\cos ku\sin tv\,dt,\quad (u,v)\in\overline{\mathbb{R}}_+^2, \tag{3}$$

$$\int_0^{\infty}\sum_{k=1}^{\infty} g_k(t)\cos ku\cos tv\,dt,\quad (u,v)\in\overline{\mathbb{R}}_+^2, \tag{4}$$

are called sine, sine-cosine, cosine-sine, and cosine integral-series, respectively.

We say that

$$\int_0^{\infty}\sum_{k=1}^{\infty} h_k(t)\,dt$$

converges in the regular sense (see [11]), if

$$\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2} h_k(t)\,dt\to 0\quad\text{as}\quad \max\{n_1,x_1\}\to\infty,\ n_2\geqslant n_1,\ x_2>x_1.$$

Throughout the paper, we build upon some assumptions. We assume that f_k, ϕ_k, ψ_k, and g_k (or h_k if we do not distinguish between them) $(k\in\mathbb{N})$ are of locally bounded variation on $\mathbb{R}_+$ (that is, of bounded variation on every closed interval of $\overline{\mathbb{R}}_+$), shortly, $h_k(t) \in BV_{loc}(\mathbb{R}_+)$, and to ensure local integrability, $tf_k(t) \in L^1_{loc}(\overline{\mathbb{R}}_+)$, $\phi_k(t) \in L^1_{loc}(\overline{\mathbb{R}}_+)$, $t\psi_k(t) \in L^1_{loc}(\overline{\mathbb{R}}_+)$, and $g_k(t) \in L^1_{loc}(\overline{\mathbb{R}}_+)$. We call such functions admissible (for (1), (2), (3), or (4), respectively).

In [11] Kórus et. al. defined the following class $GMSF(\alpha,\beta,\gamma)$ of General Monotone Sequence of Functions with majorants α, β, and γ. Also, they defined four classes based on the choices of majorants α, β, and γ, and gave relations of them.

DEFINITION 1. Let $\alpha,\beta,\gamma : \overline{\mathbb{R}}^2_+ \to \overline{\mathbb{R}}_+$ be functions. Then a sequence of admissible functions $\{h_k(t)\}_{k=1}^{\infty}$, where $h_k : \overline{\mathbb{R}}_+ \to \mathbb{C}$ for every $k \in \mathbb{N}$, is said to belong to the class $GMSF(\alpha,\beta,\gamma)$ (called the class of General Monotone Sequence of Functions with majorants α,β,γ) if there exists a positive constant C which depends only on $\{h_k(x)\}_{k=1}^{\infty}$ and satisfy the conditions

$$\sum_{k=n}^{2n-1} |\Delta h_k(t)| \leqslant C\alpha(n,t) \quad \text{for all} \quad n,t>0,$$
$$\int_x^{2x} |d_t h_k(t)| \leqslant C\beta(k,x) \quad \text{for all} \quad k,x>0,$$
$$\int_x^{2x} \sum_{k=n}^{2n-1} |d_t \Delta h_k(t)| \leqslant C\gamma(n,x) \quad \text{for all} \quad n,x>0,$$

where $\Delta h_k = h_k - h_{k+1}$.

If we choose majorants

$$\alpha_0(k,t) = \beta_0(k,t) = \gamma_0(k,t) = |h_k(t)|,$$

then the class $GMSF(\alpha_0,\beta_0,\gamma_0)$ can be considered as the class of Rest Bounded Variation Sequence of Functions $(RBVSF)$ (see [11]).

Other notable mojorants of General Monotone Sequence of Functions for a real number $\lambda \geqslant 2$ are:

$$\alpha_1(n,t) = \frac{1}{n} \sum_{k=[n/\lambda]}^{[\lambda n]} |h_k(t)|,$$
$$\beta_1(k,x) = \frac{1}{x} \int_{x/\lambda}^{\lambda x} |h_k(t)|\, dt,$$
$$\gamma_1(n,x) = \frac{1}{nx} \int_{x/\lambda}^{\lambda x} \sum_{[n/\lambda]}^{[\lambda n]} |h_k(t)|\, dt,$$

and the class $GMSF(\alpha_1,\beta_1,\gamma_1)$ can be called the class of Mean Value Bounded Variation Sequence of Functions $(MVBVSF)$ (see [11]).

In the case of functions

$$\alpha_2(n,t) = \frac{1}{n}\left[\max_{B_1(n)\leqslant m\leqslant cB_1(n)} \sum_{k=m}^{2m} |h_k(t)|\right],$$

$$\beta_2(k,x) = \frac{1}{x}\left[\max_{B_2(x)\leqslant y\leqslant cB_2(x)} \int_y^{2y} |h_k(t)|\,dt\right],$$

$$\gamma_2(n,x) = \frac{1}{nx}\left[\sup_{m+y\geqslant B_3(n+x)} \int_y^{2y} \sum_{k=m}^{2m} |h_k(t)|\,dt\right],$$

where B_1 is a positive sequence which tends to infinity, $B_2, B_3 : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ such that $\lim_{x\to\infty} B_2(x) = \lim_{x\to\infty} B_3(x) = \infty$ and $c \geqslant 2$, the class $GMSF(\alpha_2,\beta_2,\gamma_2)$ can be called the class of Supremum Bounded Variation Sequence of Functions of first type $(SBVSF_1)$ (see [11]), while for

$$\alpha_3(n,x) = \frac{1}{n}\left[\sup_{m\geqslant B(n)} \sum_{k=m}^{2m} |h_k(x)|\right],$$

$$\beta_3(k,x) = \frac{1}{x}\left[\sup_{y\geqslant B(x)} \int_y^{2y} |h_k(t)|\,dt\right],$$

$$\gamma_3(n,x) = \frac{1}{nx}\left[\sup_{m+y\geqslant B(n+x)} \int_y^{2y} \sum_{k=m}^{2m} |h_k(t)|\,dt\right],$$

where $B : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ such that $\lim_{x\to\infty} B(x) = \infty$, the class $GMSF(\alpha_3,\beta_3,\gamma_3)$ can be called the class of Supremum Bounded Variation Sequence of Functions of second type $(SBVSF_2)$ (see [11]).

The inclusion relations

$$GMSF(\alpha_0,\beta_0,\gamma_0) \subsetneq GMSF(\alpha_1,\beta_1,\gamma_1) \subsetneq GMSF(\alpha_2,\beta_2,\gamma_2) \subsetneq GMSF(\alpha_3,\beta_3,\gamma_3),$$

hold.

Concerning these classes they proved the following two theorems.

THEOREM 1. *If* $\{f_k(t)\}_{k=1}^{\infty}$ *belongs to the class* $GMSF(\alpha_3,\beta_3,\gamma_3)$, *where* $f_k : \overline{\mathbb{R}}_+ \to \mathbb{C}$ *for every* $k \in \mathbb{N}$ *and*

$$kt\, f_k(t) \to 0 \quad \text{as} \quad \max\{k,t\} \to \infty, \; k,t > 0, \tag{5}$$

then the sine integral-series (1) *converges in the regular sense uniformly in* $(u,v) \in \overline{\mathbb{R}}_+^2$.

THEOREM 2. *If* $\{f_k(t)\}_{k=1}^{\infty}$ *belongs to the class* $GMSF(\alpha_2,\beta_2,\gamma_2)$, *where* $f_k : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ *for every* $k \in \mathbb{N}$ *and the regular convergence of* (1) *is uniform in* (u,v), *then condition* (5) *is satisfied.*

In the present paper, we have studied the uniform regular convergence of integral-series (2)–(4) with a sequence of admissible functions $\{h_k(t)\}_{k=1}^{\infty}$ belonging to the class $GMSF(\alpha,\beta,\gamma)$.

2. Main results

We prove three results for trigonometric integral-series with the sequence of functions from the class $SBVSF_2$ (or one of its subclasses). The conditions we give for uniform convergence are sufficient for the sequence of complex-valued functions and are necessary for the sequence of non-negative functions.

THEOREM 3. *(i) If* $\{\phi_k(t)\}_{k=1}^{\infty} \in SBVSF_2$, *where* $\phi_k : \overline{\mathbb{R}}_+ \to \mathbb{C}$ *for every* $k \in \mathbb{N}$ *and for* $x_1 > 0,\ 0 < x < x_1$,

$$kx\phi_k(x) \to 0 \quad \text{and} \quad k\int_x^{x_1} |\phi_k(t)|\,dt \to 0 \quad \text{as} \quad \max\{k,x\} \to \infty, \tag{6}$$

then sine-cosine integral-series (2) *converges in the regular sense uniformly in* $(u,v) \in \overline{\mathbb{R}}_+^2$.

(ii) Conversely, if $\{\phi_k(t)\}_{k=1}^{\infty} \in SBVSF_1$, *where* $\phi_k : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ *for every* $k \in \mathbb{N}$ *and the regular convergence of* (2) *is uniform in* (u,v), *then condition* (6) *is satisfied.*

THEOREM 4. *(i) If* $\{\psi_k(t)\}_{k=1}^{\infty} \in SBVSF_2$, *where* $\psi_k : \overline{\mathbb{R}}_+ \to \mathbb{C}$ *for every* $k \in \mathbb{N}$ *and for* $n_1 \in \mathbb{N},\ 1 \leqslant n \leqslant n_1$,

$$nt\psi_n(t) \to 0 \quad \text{and} \quad t\sum_{k=n}^{n_1} |\psi_k(t)| \to 0 \quad \text{as} \quad \max\{n,t\} \to \infty, \tag{7}$$

then cosine-sine integral-series (3) *converges in the regular sense uniformly in* $(u,v) \in \overline{\mathbb{R}}_+^2$.

(ii) Coversely, if $\{\psi_k(t)\}_{k=1}^{\infty} \in SBVSF_1$, *where* $\psi_k : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ *for every* $k \in \mathbb{N}$ *and the regular convergence of* (3) *is uniform in* (u,v), *then* (7) *holds.*

THEOREM 5. *(i) If* $\{g_k(t)\}_{k=1}^{\infty} \in SBVSF_2$, *where* $g_k : \overline{\mathbb{R}}_+ \to \mathbb{C}$ *for every* $k \in \mathbb{N}$ *and for* $x_1 > 0,\ n_1 \in \mathbb{N},\ 0 < x < x_1$ *and* $1 \leqslant n \leqslant n_1$,

$$nxg_n(x) \to 0, \quad n\int_x^{x_1} |g_n(t)|\,dt \to 0, \quad x\sum_{k=n}^{n_1} |g_k(x)| \to 0,$$

$$\int_x^{x_1} \sum_{k=n}^{n_1} g_k(t) \to 0 \quad \text{as} \quad \max\{n,x\} \to \infty, \tag{8}$$

then cosine integral-series (4) *converges in the regular sense uniformly in* $(u,v) \in \overline{\mathbb{R}}_+^2$.

(ii) Coversely, if $\{g_k(t)\}_{k=1}^{\infty} \in SBVSF_1$, *where* $g_k : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ *for every* $k \in \mathbb{N}$ *and the regular convergence of* (4) *is uniform in* (u,v), *then the conditions in* (8) *are satisfied.*

COROLLARY 1. *The results of Theorem* 5 *also hold for the integral-series*

$$\int_0^{\infty} \sum_{k=1}^{\infty} h_k(t)e^{iku}e^{itv}\,dt, \quad (u,v) \in \overline{\mathbb{R}}_+^2. \tag{9}$$

3. Auxiliary results

To prove our main results we need the following lemmas, first two of them are used in the investigations of the uniform regular convergence of sine integral-series as well.

LEMMA 1. [11, Lemma 3.1] *Let* $\{h_k(t)\}_{k=1}^{\infty}$ *be a sequence of functions belonging to the class* $SBVSF_2$*, where* $h_k : \overline{\mathbb{R}}_+ \to \mathbb{C}$ *for every* $k \in \mathbb{N}$*. If*

$$kth_k(t) \to 0 \quad \text{as} \quad \max\{k,t\} \to \infty,$$

then

$$kx\int_x^{\infty} |d_t h_k(t)| \to 0 \quad \text{as} \quad \max\{k,x\} \to \infty;$$

$$nt\sum_{k=n}^{\infty} |\Delta h_k(t)| \to 0 \quad \text{as} \quad \max\{n,t\} \to \infty;$$

$$nx\int_x^{\infty}\sum_{k=n}^{\infty} |d_t \Delta h_k(t)| \to 0 \quad \text{as} \quad \max\{n,x\} \to \infty.$$

LEMMA 2. [11, Lemma 3.2] *Let* $\{h_k(t)\}_{k=1}^{\infty}$ *be a sequence of functions belonging to the class* $SBVSF_1$*, where* $h_k : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ *for every* $k \in \mathbb{N}$*. Then for all* $n,x > 0$*, we have*

$$\begin{aligned} nxh_n(x) \leqslant C\left[\sup_{m+y\geqslant B_3(n+x)}\int_y^{2y}\sum_{k=m}^{2m} h_k(t)dt\right] \\ + \int_x^{2x}\sum_{k=n}^{2n} h_k(t)dt + C\int_{B_2(x)}^{2cB_2(x)}\sum_{k=n}^{2n} h_k(t)dt + C\int_x^{2x}\sum_{k=B_1(n)}^{2cB_1(n)} h_k(t)dt, \end{aligned}$$

where $C > 0$ *and* $c \geqslant 2$*.*

LEMMA 3. *Let* $\{h_k(t)\}_{k=1}^{\infty}$ *be a sequence of functions belonging to the class* $SBVSF_2$*, where* $h_k : \overline{\mathbb{R}}_+ \to \mathbb{C}$ *for every* $k \in \mathbb{N}$*. If*

$$k\int_x^{x_1} |h_k(t)|\,dt \to 0 \quad \text{as} \quad \max\{k,x\} \to \infty, \tag{10}$$

then

$$n\int_x^{x_1}\sum_{k=n}^{\infty} |\Delta h_k(t)|\,dt \to 0 \quad \text{as} \quad \max\{n,x\} \to \infty,$$

and if

$$t\sum_{k=n}^{n_1} |h_k(t)| \to 0 \quad \text{as} \quad \max\{n,t\} \to \infty, \tag{11}$$

then

$$x\sum_{k=n}^{n_1}\int_x^{\infty}|d_t h_k(t)|\to 0 \quad \text{as} \quad \max\{n,x\}\to\infty.$$

Proof. Let $\{h_k\}_{k=1}^{\infty}\in SBVSF_2$ be such that (10) is satisfied. Then we have

$$\begin{aligned}
n\int_x^{x_1}\sum_{k=n}^{\infty}|\Delta h_k(t)|\,dt &= \int_x^{x_1} n\left(\sum_{j=0}^{\infty}\sum_{2^j n}^{2^{j+1}n-1}|\Delta h_k(t)|\right)dt\\
&\leqslant \int_x^{x_1} n\left\{\sum_{j=0}^{\infty}\left(\frac{C}{2^j n}\sup_{m\geqslant B(2^j n)}\sum_{k=m}^{2m}|h_k(t)|\right)\right\}dt\\
&= C\sum_{j=0}^{\infty}\frac{1}{2^j}\sup_{m\geqslant B(2^j n)}\sum_{k=m}^{2m}\int_x^{x_1}|h_k(t)|\,dt\\
&= C\sum_{j=0}^{\infty}\frac{1}{2^j}\sup_{m\geqslant B(2^j n)}\sum_{k=m}^{2m}\frac{1}{k}\left\{k\int_x^{x_1}|h_k(t)|\,dt\right\}\to 0
\end{aligned}$$

as $\max\{n,x\}\to\infty$.

Now, let $\{h_k\}_{k=1}^{\infty}\in SBVSF_2$ be such that (11) is satisfied. Then we have

$$\begin{aligned}
x\sum_{k=n}^{n_1}\int_x^{\infty}|d_t h_k(t)| &= x\sum_{k=n}^{n_1}\sum_{l=0}^{\infty}\int_{2^l x}^{2^{l+1}x}|d_t h_k(t)|\\
&\leqslant x\sum_{k=n}^{n_1}\sum_{l=0}^{\infty}\left(\frac{C}{2^l x}\sup_{y\geqslant B(2^l x)}\int_y^{2y}|h_k(t)|\,dt\right)\\
&= C\sum_{k=n}^{n_1}\sum_{l=0}^{\infty}\left(\frac{1}{2^l}\sup_{y\geqslant B(2^l x)}\int_y^{2y}|h_k(t)|\,dt\right)\\
&= C\sum_{l=0}^{\infty}\frac{1}{2^l}\sup_{y\geqslant B(2^l x)}\int_y^{2y}\sum_{k=n}^{n_1}|h_k(t)|\,dt\\
&= C\sum_{l=0}^{\infty}\frac{1}{2^l}\sup_{y\geqslant B(2^l x)}\int_y^{2y}\frac{1}{t}\left\{t\sum_{k=n}^{n_1}|h_k(t)|\right\}dt\to 0
\end{aligned}$$

as $\max\{n,x\}\to\infty$. □

LEMMA 4. *Let $h_k:\overline{\mathbb{R}}_+\to\overline{\mathbb{R}}_+$ for every $k\in\mathbb{N}$ belong to $SBVSF_1$. Then for all $0<x<x_1$ and $1\leqslant n\leqslant n_1$, we have*

$$n\int_x^{x_1}h_n(t)dt\leqslant C\int_x^{x_1}\sum_{k=B_1(n)}^{2cB_1(n)}h_k(t)dt+\int_x^{x_1}\sum_{k=n}^{2n}h_k(t)dt,$$

$$x\sum_{k=n}^{n_1}h_k(x)\leqslant C\sum_{k=n}^{n_1}\int_{B_2(x)}^{2cB_2(x)}h_k(t)dt+\sum_{k=n}^{n_1}\int_x^{2x}h_k(t)dt.$$

Proof. For all $n \leqslant j \leqslant 2n$, we calculate

$$h_n(t) = \sum_{k=n}^{j-1} \Delta h_k(t) + h_j(t) \leqslant \sum_{k=n}^{2n-1} |\Delta h_k(t)| + h_j(t).$$

Since $\{h_k\}_{k=1}^{\infty} \in SBVSF_1$ and $h_k : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ for every $k \in \mathbb{N}$, we have

$$h_n(t) \leqslant \frac{C}{n} \left(\max_{B_1(n) \leqslant m \leqslant cB_1(n)} \sum_{k=m}^{2m} h_k(t) \right) + h_j(t) \leqslant \frac{C}{n} \sum_{k=B_1(n)}^{2cB_1(n)} h_k(t) + h_j(t).$$

By summing both sides with respect to $j \in [n, 2n]$, we have

$$nh_n(t) \leqslant C \sum_{k=B_1(n)}^{2cB_1(n)} h_k(t) + \sum_{j=n}^{2n} h_j(t).$$

Now, integrating both sides with respect to $t \in [x, x_1]$, we get

$$n \int_x^{x_1} h_n(t)dt \leqslant C \int_x^{x_1} \sum_{k=B_1(n)}^{2cB_1(n)} h_k(t)dt + \int_x^{x_1} \sum_{k=n}^{2n} h_k(t)dt.$$

Next, for all $x < l < 2x$, we calculate

$$h_k(x) = -\int_x^l d_t h_k(t) + h_k(l) \leqslant \int_x^{2x} |d_t h_k(t)| + h_k(l).$$

Again, using $\{h_k\}_{k=1}^{\infty} \in SBVSF_1$ and $h_k : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ for every $k \in \mathbb{N}$, we have

$$h_k(x) \leqslant \frac{C}{x} \left(\max_{B_2(x) \leqslant y \leqslant cB_2(x)} \int_y^{2y} h_k(t)dt \right) + h_k(l) \leqslant \frac{C}{x} \int_{B_2(x)}^{2cB_2(x)} h_k(t)dt + h_k(l).$$

By summing both sides with respect to $k \in [n, n_1]$, we have

$$\sum_{k=n}^{n_1} h_k(x) \leqslant \frac{C}{x} \int_{B_2(x)}^{2cB_2(x)} \sum_{k=n}^{n_1} h_k(t)dt + \sum_{k=n}^{n_1} h_k(l).$$

Now, integrating both sides with respect to $l \in [x, 2x]$, we get

$$x \sum_{k=n}^{n_1} h_k(x) \leqslant C \int_{B_2(x)}^{2cB_2(x)} \sum_{k=n}^{n_1} h_k(t)dt + \sum_{k=n}^{n_1} \int_x^{2x} h_k(t)dt. \quad \square$$

4. Proofs of the main results

Proof of Theorem 3. To shorten the proof, for $0 \leqslant x_1 < x_2$ and $1 \leqslant n_1 < n_2$, let

$$I_{u,v}(\phi_k; x_1, x_2; n_1, n_2) = \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \phi_k(t) \sin ku \cos tv dt.$$

Part (i): Let $\varepsilon > 0$ be given. Then by (6), Lemma 1, and Lemma 3, there exists $N_0 = N_0(\varepsilon)$ so that

$$|my\phi_m(y)| < \varepsilon, \tag{12}$$

$$my \sum_{k=m}^{\infty} |\Delta\phi_k(y)| < \varepsilon, \tag{13}$$

$$k \int_{y}^{y_1} |\phi_k(t)| \, dt < \varepsilon, \tag{14}$$

$$m \int_{y}^{y_1} \sum_{k=m}^{\infty} |\Delta\phi_k(t)| \, dt < \varepsilon, \tag{15}$$

$$my \int_{y}^{\infty} |d_t \phi_m(t)| < \varepsilon, \tag{16}$$

$$my \int_{y}^{\infty} \sum_{k=m}^{\infty} |d_t \Delta\phi_k(t)| < \varepsilon, \tag{17}$$

for $0 < y < y_1$ with $\max\{m, y\} > N_0$. From now on, we always suppose that $0 \leqslant x_1 < x_2$, $1 \leqslant n_1 \leqslant n_2$ and $\max\{n_1, x_1\} > N_0$. We will prove that for any $u, v \in \mathbb{R}_+$,

$$|I_{u,v}(\phi_k; x_1, x_2; n_1, n_2)| < (5 + 15\pi)\varepsilon, \tag{18}$$

which is equivalent to the required uniform regular convergence of (2).

By Fatou's lemma, we may assume that $x_1 > 0$. Since $\sin ku$ is an odd function and 2π-periodic with respect to u we should prove (18) only for $u \in [0, \pi]$. For $u = 0$ and arbitrary v, (18) is trivial. Now, let $0 < u \leqslant \pi$ and $v = 0$. Denote $\mu = \mu(u) = [\frac{1}{u}]$, the integral part of $\frac{1}{u}$. For $v = 0$ and $0 < u \leqslant \pi$, we distinguish two basic cases.

Case (i): $n_1 \leqslant n_2 \leqslant \mu$. Then by $\sin ku \leqslant ku$ and (14), we have

$$\begin{aligned} |I_{u,0}(\phi_k; x_1, x_2; n_1, n_2)| &= \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \phi_k(t) \sin ku dt \right| \\ &\leqslant \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} ku |\phi_k(t)| \, dt \\ &\leqslant \frac{1}{\mu} \sum_{k=n_1}^{\mu} \int_{x_1}^{x_2} k |\phi_k(t)| \, dt < \varepsilon. \end{aligned}$$

Case (ii): $\mu < n_1 \leqslant n_2$. Using summation by parts (see [7, (7.1.12)])

$$\sum_{k=n_1}^{n_2} \phi_k(t)\sin ku = \sum_{k=n_1}^{n_2-1} \Delta\phi_k(t)\tilde{D}_k(u) + \phi_{n_2}(t)\tilde{D}_{n_2}(u) - \phi_{n_1}(t)\tilde{D}_{n_1-1}(u) \tag{19}$$

where

$$\tilde{D}_k(u) = \sum_{j=1}^{k} \sin ju,$$

is the conjugate Dirichlet kernel. Using well-known estimate $\left|\tilde{D}_k(u)\right| \leqslant \frac{\pi}{u}$ for $u \in (0,\pi]$, we get

$$\left|\sum_{k=n_1}^{n_2} \phi_k(t)\sin ku\right| \leqslant \frac{\pi}{u}\left(\sum_{k=n_1}^{n_2-1} |\Delta\phi_k(t)| + |\phi_{n_2}(t)| + |\phi_{n_1}(t)|\right). \tag{20}$$

Using (20) and then (14) and (15), we obtain

$$\begin{aligned}
|I_{u,0}(\phi_k;x_1,x_2;n_1,n_2)| &= \left|\int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \phi_k(t)\sin ku dt\right| \\
&\leqslant \int_{x_1}^{x_2} \left|\sum_{k=n_1}^{n_2} \phi_k(t)\sin ku\right| dt \\
&\leqslant \frac{\pi}{u}\int_{x_1}^{x_2} \left(\sum_{k=n_1}^{n_2-1} |\Delta\phi_k(t)| + |\phi_{n_2}(t)| + |\phi_{n_1}(t)|\right) dt \\
&\leqslant \pi n_1 \int_{x_1}^{x_2} \sum_{k=n_1}^{\infty} |\Delta\phi_k(t)|\, dt + \pi n_2 \int_{x_1}^{x_2} |\phi_{n_2}(t)|\, dt \\
&\qquad + \pi n_1 \int_{x_1}^{x_2} |\phi_{n_1}(t)|\, dt \\
&< 3\pi\varepsilon.
\end{aligned}$$

For arbitrary $u,v > 0$, we distinguish four basic cases, which together guarantee (18).

Case (a): $n_1 \leqslant n_2 \leqslant \mu$ and $x_2 < x_2 \leqslant 1/v$. Then, by Case (i) and (12), we have

$$\begin{aligned}
|I_{u,v}(\phi_k;x_1,x_2;n_1,n_2)| &= \left|\int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \phi_k(t)\sin ku\left(1 - 2\sin^2\frac{tv}{2}\right) dt\right| \\
&\leqslant \left|\int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \phi_k(t)\sin ku dt\right| + \left|\int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} 2\phi_k(t)\sin ku \sin^2\frac{tv}{2}\right| dt \\
&\leqslant |I_{u,0}(\phi_k;x_1,x_2;n_1,n_2)| + 2\int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} |\phi_k(t)|\, ku\frac{tv}{2} dt \\
&\leqslant \varepsilon + \frac{v}{\mu}\int_{x_1}^{1/v} \sum_{k=n_1}^{\mu} kt\,|\phi_k(t)|\, dt < 2\varepsilon.
\end{aligned}$$

Case (b): $\mu < n_1 \leqslant n_2$ and $x_1 < x_2 \leqslant 1/v$. Then, proceeding as in Case (a), we get

$$|I_{u,v}(\phi_k;x_1,x_2;n_1,n_2)| = \left|\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2}\phi_k(t)\sin ku\left(1-2\sin^2\frac{tv}{2}\right)dt\right|$$
$$\leqslant |I_{u,0}(\phi_k;x_1,x_2;n_1,n_2)| + v\int_{x_1}^{x_2} t\left|\sum_{k=n_1}^{n_2}\phi_k(t)\sin ku\right|dt.$$

Now, using Case (ii), (20), (12), and (13), we get

$$\begin{aligned}
&|I_{u,v}(\phi_k;x_1,x_2;n_1,n_2)| \\
&\leqslant 3\pi\varepsilon + \frac{\pi v}{u}\int_{x_1}^{x_2} t\left(\sum_{k=n_1}^{n_2-1}|\Delta\phi_k(t)| + |\phi_{n_2}(t)| + |\phi_{n_1}(t)|\right)dt \\
&\leqslant 3\pi\varepsilon + \pi v\int_{x_1}^{1/v}\left(tn_1\sum_{k=n_1}^{\infty}|\Delta\phi_k(t)| + tn_2|\phi_{n_2}(t)| + tn_1|\phi_{n_1}(t)|\right)dt \\
&< 3\pi\varepsilon + 3\pi\varepsilon = 6\pi\varepsilon.
\end{aligned}$$

Case (c): $n_1 \leqslant n_2 \leqslant \mu$ and $1/v \leqslant x_1 < x_2$. Integrating by parts we get

$$\begin{aligned}
\int_{x_1}^{x_2}\phi_k(t)\cos tv\,dt &= \left[\phi_k(t)\frac{\sin tv}{v}\right]_{x_1}^{x_2} - \frac{1}{v}\int_{x_1}^{x_2}\sin tv\,d_t\phi_k(t) \\
&= \frac{1}{v}\left(\phi_k(x_2)\sin x_2v - \phi_k(x_1)\sin x_1v - \int_{x_1}^{x_2}\sin tv\,d_t\phi_k(t)\right).
\end{aligned} \tag{21}$$

Using $\sin ku \leqslant ku$, $|\sin tv| \leqslant 1$, (21), (12), and (16), we get

$$\begin{aligned}
|I_{u,v}(\phi_k;x_1,x_2;n_1,n_2)| &= \left|\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2}\phi_k(t)\sin ku\cos tv\,dt\right| \\
&= \left|\sum_{k=n_1}^{n_2}\sin ku\int_{x_1}^{x_2}\phi_k(t)\cos tv\,dt\right| \\
&\leqslant u\sum_{k=n_1}^{n_2}k\left|\int_{x_1}^{x_2}\phi_k(t)\cos tv\,dt\right| \\
&\leqslant \frac{1}{\mu}\sum_{k=n_1}^{\mu}k\left(x_2|\phi_k(x_2)| + x_1|\phi_k(x_1)| + \frac{1}{v}\int_{x_1}^{x_2}|d_t\phi_k(t)|\right) \\
&\leqslant \frac{1}{\mu}\sum_{k=n_1}^{\mu}\left(kx_2|\phi_k(x_2)| + kx_1|\phi_k(x_1)| + kx_1\int_{x_1}^{\infty}|d_t\phi_k(t)|\right) \\
&< 3\varepsilon.
\end{aligned}$$

Case (d): $\mu \leqslant n_1 \leqslant n_2$ and $1/v \leqslant x_1 < x_2$. Using (19), we have

$$\int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \phi_k(t) \sin ku \cos tv dt$$
$$= \int_{x_1}^{x_2} \left(\sum_{k=n_1}^{n_2-1} \Delta\phi_k(t)\tilde{D}_k(u) + \phi_{n_2}(t)\tilde{D}_{n_2}(u) - \phi_{n_1}(t)\tilde{D}_{n_1-1}(u) \right) \cos tv dt.$$

Now, integrating by parts, we have

$$\int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \phi_k(t) \sin ku \cos tv dt$$
$$= \left[\left(\sum_{k=n_1}^{n_2-1} \Delta\phi_k(t)\tilde{D}_k(u) + \phi_{n_2}(t)\tilde{D}_{n_2}(u) - \phi_{n_1}(t)\tilde{D}_{n_1-1}(u) \right) \frac{\sin tv}{v} \right]_{x_1}^{x_2}$$
$$- \frac{1}{v} \int_{x_1}^{x_2} \sin tv \left(\sum_{k=n_1}^{n_2-1} d_t\Delta\phi_k(t)\tilde{D}_k(u) + d_t\phi_{n_2}(t)\tilde{D}_{n_2}(u) - d_t\phi_{n_1}(t)\tilde{D}_{n_1-1}(u) \right).$$

Therefore, by using (12), (13), (16), and (17), we get

$$|I_{u,v}(\phi_k; x_1, x_2; n_1, n_2)|$$
$$\leqslant \frac{\pi}{uv} \left(\sum_{k=n_1}^{n_2-1} |\Delta\phi_k(x_2)| + |\phi_{n_2}(x_2)| + |\phi_{n_1}(x_2)| + \sum_{k=n_1}^{n_2-1} |\Delta\phi_k(x_1)| + |\phi_{n_2}(x_1)| + |\phi_{n_1}(x_1)| \right)$$
$$+ \frac{\pi}{uv} \int_{x_1}^{x_2} \left(\sum_{k=n_1}^{n_2-1} |d_t\Delta\phi_k(t)| + |d_t\phi_{n_2}(t)| + |d_t\phi_{n_1}(t)| \right) dt$$
$$\leqslant \pi \left(n_1 x_2 \sum_{k=n_1}^{\infty} |\Delta\phi_k(x_2)| + n_2 x_2 |\phi_{n_2}(x_2)| + n_1 x_2 |\phi_{n_1}(x_2)| \right)$$
$$+ \pi \left(n_1 x_1 \sum_{k=n_1}^{n_2-1} |\Delta\phi_k(x_1)| + n_2 x_1 |\phi_{n_2}(x_1)| + n_1 x_1 |\phi_{n_1}(x_1)| \right)$$
$$+ \pi \left(n_1 x_1 \int_{x_1}^{\infty} \sum_{k=n_1}^{n_2-1} |d_t\Delta\phi_k(t)| dt + n_2 x_1 \int_{x_1}^{\infty} |d_t\phi_{n_2}(t)| dt + n_1 x_1 \int_{x_1}^{\infty} |d_t\phi_{n_1}(t)| dt \right)$$
$$< 9\pi\varepsilon.$$

The following cases: $n_1 \leqslant n_2 \leqslant \mu$ and $x_1 < 1/v < x_2$; $\mu < n_1 \leqslant n_2$ and $x_1 < 1/v < x_2$; $n_1 \leqslant \mu \leqslant n_2$ and $x_1 < x_2 \leqslant 1/v$; $n_1 \leqslant \mu < n_2$ and $1/v \leqslant x_1 < x_2$; $n_1 \leqslant \mu < n_2$ and $x_1 < 1/v < x_2$ can be split into two or four cases which fall under cases (a)–(d).

Part (ii): Let $\varepsilon > 0$ be arbitrary. Suppose $\{\phi_k(t)\}_{k=1}^{\infty} \in SBVSF_1$, where $\phi_k : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ for every $k \in \mathbb{N}$ and the regular convergence of (2) is uniform in (u, v). There exists $N' = N'(\varepsilon)$ such that

$$I_{u,0}(\phi_k; x, 2cx; n, 2cn) = \int_{x}^{2cx} \sum_{k=n}^{2cn} \phi_k(t) \sin ku dt < \varepsilon, \tag{22}$$

for $\max\{n,x\} > N'$ and $c \geqslant 2$. For $n \geqslant 1$, set $u = \frac{\pi}{4cn}$. Clearly, for all $n \leqslant k \leqslant 2cn$, $\sin ku \geqslant \sin \frac{\pi}{4c}$ and for all $n \leqslant k \leqslant 2n$, $\sin ku \geqslant \sin \frac{\pi}{4c}$. Since $B_1(n) \to \infty$ as $n \to \infty$ and $B_2(y), B_3(y) \to \infty$ as $y \to \infty$, there exists $N'' > 0$ such that for $\max\{n,x\} > N''$ implies $n + x > N'$, $B_1(n) + x > N'$, $n + B_2(x) > N'$ and $B_3(n+x) > N'$. Then by (22) and Lemma 2, for $\max\{n,x\} > N''$ and $C > 0$ we get

$$\begin{aligned}
\sin \frac{\pi}{4c} nx\phi_n(x) &\leqslant C \sin \frac{\pi}{4c} \sup_{m+y \geqslant B_3(n+x)} \int_y^{2y} \sum_{k=n}^{2cn} \phi_k(t)dt + \sin \frac{\pi}{4c} \int_x^{2cx} \sum_{k=n}^{2n} \phi_k(t)dt \\
&\quad + C \sin \frac{\pi}{4c} \int_{B_2(x)}^{2cB_2(x)} \sum_{k=n}^{2n} \phi_k(t)dt + C \sin \frac{\pi}{4c} \int_x^{2cx} \sum_{k=B_1(n)}^{2cB_1(n)} \phi_k(t)dt \\
&\leqslant C \sup_{m+y \geqslant B_3(n+x)} I_{u,0}(\phi_k; y, 2y; n, 2cn) + I_{u,0}(\phi_k; x, 2x; n, 2n) \\
&\quad + CI_{u,0}(\phi_k; B_2(x), 2cB_2(x); n, 2n) + CI_{u,0}(\phi_k; x, 2x; B_1(n), 2cB_1(n)) \\
&\leqslant (3C+1)\varepsilon.
\end{aligned}$$

That is, $nx\phi_n(x) \to 0$ as $\max\{n,x\} \to \infty$. Also by (22) and Lemma 4, for $\max\{n,x\} > N''$, $0 < x < x_1$, and $C > 0$ we get

$$\begin{aligned}
\sin \frac{\pi}{4c} \left(n \int_x^{x_1} \phi_n(t)dt \right) &\leqslant C \sin \frac{\pi}{4c} \int_x^{x_1} \sum_{k=B_1(n)}^{2cB_1(n)} \phi_k(t)dt + \sin \frac{\pi}{4c} \int_x^{x_1} \sum_{k=n}^{2n} \phi_k(t)dt \\
&\leqslant CI_{u,0}(\phi_k; x, x_1; B_1(n), 2cB_1(n)) + I_{u,0}(\phi_k; x, x_1; n, 2n) \\
&\leqslant (C+1)\varepsilon.
\end{aligned}$$

Hence, $n \int_x^{x_1} \phi_n(t)dt \to 0$ as $\max\{n,x\} \to \infty$. And hence we have obtained (6). $\square$

Proof of Theorem 4. To shorten the proof, for $0 \leqslant x_1 < x_2$ and $1 \leqslant n_1 < n_2$, let

$$J_{u,v}(\psi_k; x_1, x_2; n_1, n_2) = \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \psi_k(t) \cos ku \sin tv dt.$$

Part (i): Let $\varepsilon > 0$ be given. Then by (7), Lemma 1, and Lemma 3, there exists $M_0 = M_0(\varepsilon)$ so that

$$|my\psi_m(y)| < \varepsilon, \tag{23}$$

$$my \sum_{k=m}^{\infty} |\Delta\psi_k(y)| < \varepsilon, \tag{24}$$

$$y \sum_{k=m}^{m_1} |\psi_k(y)| < \varepsilon, \tag{25}$$

$$y\sum_{k=m}^{m_1}\int_y^\infty |d_t\psi_k(t)|<\varepsilon \tag{26}$$

$$my\int_y^\infty |d_t\psi_m(t)|<\varepsilon, \tag{27}$$

$$my\int_y^\infty \sum_{k=m}^{\infty} |d_t\Delta\psi_k(t)|<\varepsilon, \tag{28}$$

for $0<m<m_1$ with $\max\{m,y\}>M_0$. From now on, we always suppose that $0\leqslant x_1<x_2$, $1\leqslant n_1\leqslant n_2$ and $\max\{n_1,x_1\}>M_0$. We will prove that for any $u,v\in\mathbb{R}_+$,

$$|J_{u,v}(\psi_k;x_1,x_2;n_1,n_2)|<(8+12\pi)\varepsilon, \tag{29}$$

which is equivalent to the required uniform regular convergence of (3).

By Fatou's lemma, we may assume that $x_1>0$. Since $\cos ku$ is an even function and 2π-periodic with respect to u we should prove (29) only for $u\in[0,\pi]$. For $v=0$ and arbitrary u, (29) is trivial. Now, for $v>0$ and $u=0$, we have two basic cases.

Case (i): $x_1<x_2\leqslant 1/v$. Then by $\sin tv\leqslant tv$ and (25), we have

$$\begin{aligned}|J_{0,v}(\psi_k;x_1,x_2;n_1,n_2)| &= \left|\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2}\psi_k(t)\sin tv dt.\right|\\ &\leqslant v\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2} t\,|\psi_k(t)|\,dt\\ &\leqslant v\int_{x_1}^{1/v} t\sum_{k=n_1}^{n_2}|\psi_k(t)|\,dt<\varepsilon.\end{aligned}$$

Case (ii): $1/v\leqslant x_1<x_2$. Then integrating by parts and by using (25), (26), and $|\cos tv|\leqslant 1$, we get

$$\begin{aligned}|J_{0,v}(\psi_k;x_1,x_2;n_1,n_2)| &= \left|\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2}\psi_k(t)\sin tv dt\right|\\ &= \left|\sum_{k=n_1}^{n_2}\left(\left[-\psi_k(t)\frac{\cos tv}{v}\right]_{x_1}^{x_2}+\frac{1}{v}\int_{x_1}^{x_2}\cos tv d_t\psi_k(t)\right)\right|\\ &\leqslant \frac{1}{v}\sum_{k=n_1}^{n_2}|\psi_k(x_1)|+\frac{1}{v}\sum_{k=n_1}^{n_2}|\psi_k(x_2)|+\frac{1}{v}\sum_{k=n_1}^{n_2}\int_{x_1}^{\infty}|d_t\psi_k(t)|\\ &\leqslant x_1\sum_{k=n_1}^{n_2}|\psi_k(x_1)|+x_2\sum_{k=n_1}^{n_2}|\psi_k(x_2)|+x_1\sum_{k=n_1}^{n_2}\int_{x_1}^{\infty}|d_t\psi_k(t)|\\ &< 3\varepsilon.\end{aligned}$$

For arbitrary $v>0$ and $0<u\leqslant\pi$, we distinguish four basic cases, which together guarantee (29).

Case (a): $n_1 \leqslant n_2 \leqslant \mu$ and $x_2 < x_2 \leqslant 1/v$, where $\mu = \mu(u) = [\frac{1}{u}]$, the integral part of $\frac{1}{u}$, $0 < u \leqslant \pi$. Then proceeding as in Case (a) of proof of Theorem 3 and using Case (i) above and (23), we have

$$\begin{aligned}
&|J_{u,v}(\psi_k;x_1,x_2;n_1,n_2)| \\
&= \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \psi_k(t) \left(1 - 2\sin^2 \frac{ku}{2}\right) \sin tv dt \right| \\
&\leqslant \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \psi_k(t) \sin tv dt \right| + \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} 2\psi_k(t) \sin^2 \frac{ku}{2} \sin tv dt \right| \\
&\leqslant |J_{0,v}(\psi_k;x_1,x_2;n_1,n_2)| + 2\int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} |\psi_k(t)| \frac{ku}{2} tv dt \\
&\leqslant \varepsilon + \frac{v}{\mu} \int_{x_1}^{1/v} \sum_{k=n_1}^{\mu} kt\, |\psi_k(t)|\, dt \leqslant 2\varepsilon.
\end{aligned}$$

Case (b): $\mu \leqslant n_1 \leqslant n_2$ and $x_1 < x_2 \leqslant 1/v$. Then proceeding as in Case (i) of proof of Theorem 3, except for replacing each conjugate Dirichlet kernel with the appropriate Dirichlet kernel $D_k(u)$, and using $|D_k(u)| \leqslant \frac{\pi}{u}$ for $u \in (0,\pi]$ and by using (23) and (24) we get

$$\begin{aligned}
&|J_{u,v}(\psi_k;x_1,x_2;n_1,n_2)| \\
&= \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \psi_k(t) \cos ku \sin tv dt \right| \\
&\leqslant v \int_{x_1}^{x_2} t \left| \sum_{k=n_1}^{n_2} \psi_k(t) \cos ku \right| dt \\
&\leqslant v \int_{x_1}^{x_2} t \left| \sum_{k=n_1}^{n_2-1} \Delta\psi_k(t) D_k(u) + \psi_{n_2}(t) D_{n_2}(u) - \psi_{n_1}(t) D_{n_1-1}(u) \right| dt \\
&\leqslant \pi v \int_{x_1}^{1/v} \left(tn_1 \sum_{k=n_1}^{\infty} |\Delta\psi_k(t)| + tn_2 |\psi_{n_2}(t)| + tn_1 |\psi_{n_1}(t)| \right) dt \\
&< 3\pi\varepsilon.
\end{aligned}$$

Case (c): $n_1 \leqslant n_2 \leqslant \mu$ and $1/v \leqslant x_1 < x_2$. Then by Case (ii) above and integrating by parts and then by using (23) and (27), we get

$$\begin{aligned}
&|J_{u,v}(\psi_k;x_1,x_2;n_1,n_2)| \\
&= \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \psi_k(t) \left(1 - 2\sin^2 \frac{ku}{2}\right) \sin tv dt \right| \\
&\leqslant \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} \psi_k(t) \sin tv dt \right| + \sum_{k=n_1}^{n_2} ku \left| \int_{x_1}^{x_2} \psi_k(t) \sin tv dt \right|
\end{aligned}$$

$$\leqslant |J_{0,v}(\psi_k;x_1,x_2;n_1,n_2)| + \frac{1}{\mu}\sum_{k=n_1}^{\mu} k\left|\left[-\psi_k(t)\frac{\cos tv}{v}\right]_{x_1}^{x_2} + \frac{1}{v}\int_{x_1}^{x_2}\cos tv\,d_t\psi_k(t)\right|$$

$$\leqslant 3\varepsilon + \frac{1}{\mu}\sum_{k=n_1}^{\mu}\left(|kx_1\psi_k(x_1)| + |kx_2\psi_k(x_2)| + kx_1\int_{x_1}^{\infty}|d_t\psi_k(t)|\right)$$

$$< 6\varepsilon.$$

Case (d): $\mu \leqslant n_1 \leqslant n_2$ and $1/v \leqslant x_1 < x_2$. Proceeding as in Case (d) of proof of Theorem 3, first, summation by parts, we have

$$\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2}\psi_k(t)\cos ku\sin tv\,dt$$
$$= \int_{x_1}^{x_2}\left(\sum_{k=n_1}^{n_2-1}\Delta\psi_k(t)D_k(u) + \psi_{n_2}(t)D_{n_2}(u) - \psi_{n_1}(t)D_{n_1-1}(u)\right)\sin tv\,dt.$$

Next, integrating by parts, we have

$$\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2}\psi_k(t)\cos ku\sin tv\,dt$$
$$= \left[-\left(\sum_{k=n_1}^{n_2-1}\Delta\psi_k(t)D_k(u) + \psi_{n_2}(t)D_{n_2}(u) - \psi_{n_1}(t)D_{n_1-1}(u)\right)\frac{\cos tv}{v}\right]_{x_2}^{x_1}$$
$$+ \int_{x_1}^{x_2}\frac{\cos tv}{v}\left(\sum_{k=n_1}^{n_2-1}d_t\Delta\psi_k(t)D_k(u) + d_t\psi_{n_2}(t)D_{n_2}(u) - d_t\psi_{n_1}(t)D_{n_1-1}(u)\right).$$

Therefore by using (23), (24), (27) and (28), we get

$$|J_{u,v}(\psi_k;x_1,x_2;n_1,n_2)|$$
$$\leqslant \frac{\pi}{uv}\left(\sum_{k=n_1}^{n_2-1}|\Delta\psi_k(x_2)| + |\psi_{n_2}(x_2)| + |\psi_{n_1}(x_2)| + \sum_{k=n_1}^{n_2-1}|\Delta\psi_k(x_1)| + |\psi_{n_2}(x_1)| + |\psi_{n_1}(x_1)|\right)$$
$$+ \frac{\pi}{uv}\int_{x_1}^{x_2}\left(\sum_{k=n_1}^{n_2-1}|d_t\Delta\psi_k(t)| + |d_t\psi_{n_2}(t)| + |d_t\psi_{n_1}(t)|\right)$$
$$\leqslant \pi\left(n_1x_2\sum_{k=n_1}^{\infty}|\Delta\psi_k(x_2)| + n_2x_2|\psi_{n_2}(x_2)| + n_1x_2|\psi_{n_1}(x_2)|\right)$$
$$+ \pi\left(n_1x_1\sum_{k=n_1}^{n_2-1}|\Delta\psi_k(x_1)| + n_2x_1|\psi_{n_2}(x_1)| + n_1x_1|\psi_{n_1}(x_1)|\right)$$
$$+ \pi\left(n_1x_1\int_{x_1}^{\infty}\sum_{k=n_1}^{n_2-1}|d_t\Delta\psi_k(t)| + n_2x_1\int_{x_1}^{\infty}|d_t\psi_{n_2}(t)| + n_1x_1\int_{x_1}^{\infty}|d_t\psi_{n_1}(t)|\right)$$
$$\leqslant 9\pi\varepsilon.$$

The following cases: $n_1 \leqslant n_2 \leqslant \mu$ and $x_1 < 1/v < x_2$; $\mu < n_1 \leqslant n_2$ and $x_1 < 1/v < x_2$; $n_1 \leqslant \mu \leqslant n_2$ and $x_1 < x_2 \leqslant 1/v$; $n_1 \leqslant \mu < n_2$ and $1/v \leqslant x_1 < x_2$; $n_1 \leqslant \mu < n_2$ and $x_1 < 1/v < x_2$ can be split into two or four cases which fall under cases (a)–(d).

Part (ii). Let $\varepsilon > 0$ be arbitrary. Suppose $\{\psi_k(t)\}_{k=1}^{\infty} \in SBVSF_1$, where $\psi_k : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ for every $k \in \mathbb{N}$ and the regular convergence of (3) is uniform in (u,v), there exists $M' = M'(\varepsilon)$ such that

$$J_{0,v}(\psi_k;x,2cx;n,2cn) = \int_x^{2cx} \sum_{k=n}^{2cn} \psi_k(t) \sin tv dt < \varepsilon, \tag{30}$$

for $\max\{n,x\} > M'$ and $c \geqslant 2$. For $x > 0$, set $v = \frac{\pi}{4cx}$. Clearly, for all $x \leqslant t \leqslant 2cx$, $\sin tv \geqslant \sin \frac{\pi}{4c}$ and for all $x \leqslant t \leqslant 2x$, $\sin tv \geqslant \sin \frac{\pi}{4c}$. Since $B_1(n) \to \infty$ as $n \to \infty$ and $B_2(y), B_3(y) \to \infty$ as $y \to \infty$, there exists $M'' > 0$ such that for $\max\{n,x\} > M''$ implies $n + x > N'$, $B_1(n) + x > M'$, $n + B_2(x) > M'$ and $B_3(n+x) > M'$. Then by (30) and Lemma 2, for $\max\{n,x\} > M''$ and $C > 0$ we get

$$\begin{aligned}
\sin \frac{\pi}{4c} nx\psi_n(x) &\leqslant C \sin \frac{\pi}{4c} \sup_{m+y \geqslant B_3(n+x)} \int_y^{2y} \sum_{k=n}^{2cn} \psi_k(t) dt + \sin \frac{\pi}{4c} \int_x^{2cx} \sum_{k=n}^{2n} \psi_k(t) dt \\
&\quad + C \sin \frac{\pi}{4c} \int_{B_2(x)}^{2cB_2(x)} \sum_{k=n}^{2n} \psi_k(t) dt + C \sin \frac{\pi}{4c} \int_x^{2cx} \sum_{k=B_1(n)}^{2cB_1(n)} \psi_k(t) dt \\
&\leqslant C \sup_{m+y \geqslant B_3(n+x)} J_{0,v}(\psi_k;y,2y;n,2cn) + J_{0,v}(\psi_k;x,2x;n,2n) \\
&\quad + CJ_{0,v}(\psi_k;B_2(x),2cB_2(x);n,2n) + CJ_{0,v}(\psi_k;x,2x;B_1(n),2cB_1(n)) \\
&\leqslant (3C+1)\varepsilon.
\end{aligned}$$

That is, $nx\psi_n(x) \to 0$ as $\max\{n,x\} \to \infty$. Also by (30) and Lemma 4, for $\max\{n,x\} > M''$, $0 < x < x_1$, and $C > 0$, we get

$$\begin{aligned}
\sin \frac{\pi}{4c} \left(n \int_x^{x_1} \psi_n(t) dt \right) &\leqslant C \sin \frac{\pi}{4c} \int_x^{x_1} \sum_{k=B_1(n)}^{2cB_1(n)} \psi_k(t) dt + \sin \frac{\pi}{4c} \int_x^{x_1} \sum_{k=n}^{2n} \psi_k(t) dt \\
&\leqslant CJ_{0,v}(\psi_k;x,x_1;B_1(n),2cB_1(n)) + J_{0,v}(\psi_k;x,x_1;n,2n \\
&\leqslant (C+1)\varepsilon.
\end{aligned}$$

Hence, $n \int_x^{x_1} \psi_n(t) dt \to 0$ as $\max\{n,x\} \to \infty$. And hence we have obtained (7). $\square$

Proof of Theorem 5. To shorten the proof, for $0 \leqslant x_1 < x_2$ and $1 \leqslant n_1 \leqslant n_2$, let

$$T_{u,v}(g_k;x_1,x_2;n_1,n_2) = \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t) \cos ku \cos tv dt.$$

Part (i): Let $\varepsilon > 0$ be given. By (8), Lemma 1, and Lemma 3, there exists $L_0 = L_0(\varepsilon)$, so that

$$|myg_m(y)| < \varepsilon, \tag{31}$$

$$k\int_y^{y_1} |g_k(t)|\,dt < \varepsilon, \tag{32}$$

$$y\sum_{k=m}^{m_1} |g_k(y)| < \varepsilon, \tag{33}$$

$$my\int_y^{\infty} |d_t g_m(t)|\,dt < \varepsilon, \tag{34}$$

$$my\sum_{k=m}^{\infty} |\Delta g_k(y)| < \varepsilon, \tag{35}$$

$$y\int_y^{\infty}\sum_{k=m}^{m_1} |d_t g_k(t)| < \varepsilon, \tag{36}$$

$$m\int_y^{y_1}\sum_{k=m}^{\infty} |\Delta g_k(t)|\,dt < \varepsilon, \tag{37}$$

$$my\int_y^{\infty}\sum_{k=m}^{\infty} |d_t \Delta g_k(t)| < \varepsilon, \tag{38}$$

$$\left|\int_y^{y_1}\sum_{k=m}^{m_1} g_k(t)\right| < \varepsilon, \tag{39}$$

for all $0 < y < y_1$ and $1 \leqslant m \leqslant m_1$ with $\max\{m, y\} > L_0$. From now on, we always assume that $0 \leqslant x_1 < x_2$, $1 \leqslant n_1 \leqslant n_2$ and $\max\{n_1, x_1\} > L_0$. We will prove that for all u, v,

$$|T_{u,v}(g_k; x_1, x_2; n_1, n_2)| < (9 + 12\pi)\varepsilon. \tag{40}$$

By Fatou's lemma, we may assume that $x_1 > 0$. Since $\cos ku$ is an even function and 2π-periodic with respect to u we should prove (40) only for $u \in [0, \pi]$. If $u = v = 0$, then the inequality in (39) immediately implies (40). Now onwards, we use inequalities in (31) several times. For $u = 0$ and $v > 0$, we have two basic cases:

Case (i): $x_1 < x_2 < 1/v$. Then by using (33) and (39), we have

$$\begin{aligned}
|T_{0,v}(g_k; x_1, x_2; n_1, n_2)| &= \left|\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2} g_k(t)\left(1 - 2\sin^2\frac{tv}{2}\right)dt\right| \\
&\leqslant \left|\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2} g_k(t)dt\right| + v\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2} t\,|g_k(t)|\,dt \\
&\leqslant \left|\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2} g_k(t)dt\right| + v\int_0^{1/v} t\sum_{k=n_1}^{n_2} |g_k(t)|\,dt < 2\varepsilon.
\end{aligned}$$

Case (ii): $1/v \leqslant x_1 \leqslant x_2$. Then integrating by parts and then by using (34) and (36), we have

$$
\begin{aligned}
|T_{0,v}(g_k;x_1,x_2;n_1,n_2)| &= \left| \sum_{k=n_1}^{n_2} \int_{x_1}^{x_2} g_k(t)\cos tv\,dt \right| \\
&= \left| \sum_{k=n_1}^{n_2} \left(\left[g_k(t)\frac{\sin tv}{v} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{\sin tv}{v} d_t g_k(t) \right) \right| \\
&\leqslant x_2 \sum_{k=n_1}^{n_2} |g_k(x_2)| + x_1 \sum_{k=n_1}^{n_2} |g_k(x_1)| + x_1 \int_{x_1}^{\infty} \sum_{k=n_1}^{n_2} |d_t g_k(t)| < 3\varepsilon.
\end{aligned}
$$

Now, for $u > 0$ and $v = 0$, also we have two basic cases:

Case (I): $n_1 \leqslant n_2 < \mu$. Then by using (32) and (39), we have

$$
\begin{aligned}
|T_{u,0}(g_k;x_1,x_2;n_1,n_2)| &= \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t)\cos ku\,dt \right| \\
&= \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t) \left(1 - 2\sin^2\frac{ku}{2} \right) dt \right| \\
&\leqslant \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t)\,dt \right| + u \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} k\,|g_k(t)|\,dt \\
&\leqslant \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t)\,dt \right| + \frac{1}{\mu} \sum_{k=n_1}^{\mu} k \int_{x_1}^{x_2} |g_k(t)|\,dt < 2\varepsilon.
\end{aligned}
$$

Case (II): $\mu < n_1 \leqslant n_2$. Then summation by parts and then by using (37) and (33) we get

$$
\begin{aligned}
&|T_{u,0}(g_k;x_1,x_2;n_1,n_2)| \\
&\quad = \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t)\cos ku\,dt \right| \\
&\quad = \left| \int_{x_1}^{x_2} \left(\sum_{k=n_1}^{n_2-1} \Delta g_k(t) D_k(u) + g_{n_2}(t) D_{n_2}(u) - g_{n_1}(t) D_{n_1-1}(u) \right) dt \right| \\
&\quad \leqslant \pi \left(n_1 \int_{x_1}^{x_2} \sum_{k=n_1}^{\infty} |\Delta g_k(t)|\,dt + n_2 \int_{x_1}^{x_2} |g_{n_2}(t)|\,dt + n_1 \int_{x_1}^{x_2} |g_{n_1}(t)|\,dt \right) \\
&\quad < 3\pi\varepsilon.
\end{aligned}
$$

For any $v > 0$ and $0 < u \leqslant \pi$, we distinguish four basic cases, which together guarantee (40).

Case (a): $n_1 \leqslant n_2 \leqslant \mu$ and $x_2 < x_2 \leqslant 1/v$. Then by using (31), (32), (33), and (39), we have

$$\begin{aligned}
&|T_{u,v}(g_k;x_1,x_2;n_1,n_2)| \\
&\quad = \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t) \left(1 - 2\sin^2 \frac{ku}{2}\right) \left(1 - 2\sin^2 \frac{tv}{2}\right) dt \right| \\
&\quad \leqslant \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t) dt \right| + u \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} k\,|g_k(t)|\,dt + v \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} t\,|g_k(t)|\,dt \\
&\qquad + uv \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} kt\,|g_k(t)|\,dt \\
&\quad \leqslant \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t) dt \right| + \frac{1}{\mu} \sum_{k=n_1}^{\mu} k \int_{x_1}^{x_2} |g_k(t)|\,dt + v \int_{x_1}^{1/v} t \sum_{k=n_1}^{n_2} |g_k(t)|\,dt \\
&\qquad + \frac{v}{\mu} \int_{x_1}^{1/v} \sum_{k=1}^{\mu} kt\,|g_k(t)|\,dt \\
&\quad < 4\varepsilon.
\end{aligned}$$

Case (b): $\mu \leqslant n_1 \leqslant n_2$ and $x_1 < x_2 \leqslant 1/v$. Then by using Case (II) above and by using (33), we get

$$\begin{aligned}
|T_{u,v}(g_k;x_1,x_2;n_1,n_2)| &= \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t) \cos ku \left(1 - 2\sin^2 \frac{tv}{2}\right) dt \right| \\
&\leqslant \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t) \cos ku dt \right| + v \int_{x_1}^{x_2} t \sum_{k=n_1}^{n_2} |g_k(t)|\,dt \\
&\leqslant |T_{u,0}(g_k;x_1,x_2;n_1,n_2)| + v \int_{x_1}^{1/v} t \sum_{k=n_1}^{n_2} |g_k(t)|\,dt \\
&< (3\pi + 1)\varepsilon.
\end{aligned}$$

Case (c): $n_1 \leqslant n_2 \leqslant \mu$ and $1/v \leqslant x_1 < x_2$. Then by using Case (ii) above and by (32), we get

$$\begin{aligned}
|T_{u,v}(g_k;x_1,x_2;n_1,n_2)| &= \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t) \left(1 - 2\sin^2 \frac{ku}{2}\right) \cos tv dt \right| \\
&\leqslant \left| \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t) \cos tv dt \right| + u \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} k\,|g_k(t)|\,dt \\
&\leqslant |T_{0,v}(g_k;x_1,x_2;n_1,n_2)| + \frac{1}{\mu} \sum_{k=1}^{\mu} k \int_{x_1}^{x_2} |g_k(t)|\,dt \\
&< 4\varepsilon.
\end{aligned}$$

Case (d): $\mu \leqslant n_1 \leqslant n_2$ and $1/v \leqslant x_1 < x_2$. Applying summation by parts and integrating by parts, we obtain

$$\begin{aligned}
&|T_{u,v}(g_k;x_1,x_2;n_1,n_2)| \\
&= \left| \int_{x_1}^{x_2} \left(\sum_{k=n_1}^{n_2-1} \Delta g_k(t) D_k(u) + g_{n_2}(t) D_{n_2}(u) - g_{n_1}(t) D_{n_1-1}(u) \right) \cos tv dt \right| \\
&= \left| \left[\left(\sum_{k=n_1}^{n_2-1} \Delta g_k(t) D_k(u) + g_{n_2}(t) D_{n_2}(u) - g_{n_1}(t) D_{n_1-1}(u) \right) \frac{\sin tv}{v} \right]_{x_1}^{x_2} \right. \\
&\qquad \left. - \frac{1}{v} \int_{x_1}^{x_2} \sin tv \left(\sum_{k=n_1}^{n_2-1} d_t \Delta g_k(t) D_k(u) + d_t g_{n_2}(t) D_{n_2}(u) - d_t g_{n_1}(t) D_{n_1-1}(u) \right) \right| \\
&\leqslant \frac{\pi}{uv} \left\{ \sum_{k=n_1}^{n_2-1} |\Delta g_k(x_2)| + \sum_{k=n_1}^{n_2-1} |\Delta g_k(x_1)| + |g_{n_2}(x_2)| + |g_{n_1}(x_2)| + |g_{n_2}(x_1)| + |g_{n_1}(x_1)| \right. \\
&\qquad \left. + \int_{x_1}^{x_2} \left(\sum_{k=n_1}^{n_2-1} |d_t \Delta g_k(t)| + |d_t g_{n_2}(t)| + |d_t g_{n_1}(t)| \right) \right\}.
\end{aligned}$$

Then by using (31), (34), (35), and (38), we have

$$\begin{aligned}
&|T_{u,v}(g_k;x_1,x_2;n_1,n_2)| \\
&\leqslant \pi \left(n_1 x_2 \sum_{k=n_1}^{\infty} |\Delta g_k(x_2)| + n_1 x_1 \sum_{k=n_1}^{\infty} |\Delta g_k(x_1)| \right. \\
&\qquad + n_2 x_2 |g_{n_2}(x_2)| + n_1 x_2 |g_{n_1}(x_2)| + n_2 x_1 |g_{n_2}(x_1)| + n_1 x_1 |g_{n_1}(x_1)| \\
&\qquad \left. + n_1 x_1 \int_{x_1}^{\infty} \sum_{k=n_1}^{\infty} |d_t \Delta g_k(t)| dt + n_2 x_1 \int_{x_1}^{\infty} |d_t g_{n_2}(t)| dt + n_1 x_1 \int_{x_1}^{x_2} |d_t g_{n_1}(t)| dt \right) \\
&< 9\pi\varepsilon.
\end{aligned}$$

Part (ii): Conversely, from the regular uniform convergence of (4) at $(0,0)$ we deduce that

$$T_{0,0}(g_k;x_1,x_2;n_1,n_2) = \int_{x_1}^{x_2} \sum_{k=n_1}^{n_2} g_k(t) dt \to 0 \quad \text{as} \quad \max\{n_1,x_1\} \to \infty,$$

and hence from Lemma 2 and Lemma 4 we obtain (8). $\square$

Proof of Corollary 1. Obviously,

$$\begin{aligned}
&\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2} h_k(t)e^{iku}e^{itv}dt \\
&\quad= \int_{x_1}^{x_2}\sum_{k=n_1}^{n_2} h_k(t)\,(\cos ku + i\sin ku)\,(\cos tv + i\sin tv)\,dt \\
&\quad= \int_{x_1}^{x_2}\sum_{k=n_1}^{n_2} h_k(t)\cos ku\cos tv dt + i\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2} h_k(t)\cos ku\sin tv dt \\
&\qquad + i\int_{x_1}^{x_2}\sum_{k=n_1}^{n_2} h_k(t)\sin ku\cos tv dt - \int_{x_1}^{x_2}\sum_{k=n_1}^{n_2} h_k(t)\sin ku\sin tv dt,
\end{aligned}$$

and then we can apply Theorem 1 and Theorems 3–5 to conclude the uniform regular convergence of (9). For the converse part, we just need to use the regular convergence of (9) at $(0,0)$, and using the converse part of the proof of Theorem 5, we get our conclusions. □

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