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EXPLORING GENERALIZED IDEAL CONVERGENCE WITH ORLICZ FUNCTIONS IN NEUTROSOPHIC n–NORMED SPACES

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Abstract. This paper introduces the concept of $\mathscr{I}_\lambda$-convergence within the framework of neutrosophic n-normed spaces. Leveraging this notion alongside Orlicz functions, we construct novel sequence spaces that extend the theoretical landscape of neutrosophic n-normed linear spaces. Our investigation delves into the structural properties of these spaces, exploring fundamental attributes such as linearity and Hausdorffness while establishing significant results. This study deepens the understanding of neutrosophic n-normed spaces and also contributes to the broader mathematical framework by unveiling new perspectives in sequence space theory.

1. Introduction and basic fundamentals

Zadeh [54] is widely celebrated as the visionary pioneer who revolutionized classical set theory through the groundbreaking introduction of fuzzy set theory. This paradigm-shifting concept has since evolved into a cornerstone of modern mathematics, driving innovations and applications across diverse domains of science and engineering, including chaos control [15], nonlinear dynamical systems [21], and beyond. An intriguing advancement in the realm of fuzzy sets is the introduction of intuitionistic fuzzy sets by Atanassov [1]. These sets enrich the classical fuzzy framework by integrating a non-membership function alongside the membership function, offering a more nuanced perspective. This innovation has catalyzed the development of numerous groundbreaking concepts in mathematical analysis, further broadening the scope of fuzzy set theory.

Smarandache [51] introduced the revolutionary concept of neutrosophic sets, offering a comprehensive generalization of intuitionistic fuzzy sets. By incorporating an indeterminacy function, neutrosophic sets characterize each element with a triplet: truth-membership, indeterminacy-membership, and falsity-membership functions. This innovative framework enables a more nuanced representation, where every element in the universe is precisely defined by its unique degrees of these three notions. The evolution of fuzzy normed spaces, first introduced by Felbin [14] in 1992, has unfolded into a dynamic field of study. Saadati and Park [52] extended this framework to intuitionistic fuzzy normed spaces in 2006, paving the way for Karakus et al.'s [29] exploration of statistical convergence in 2008 and Kumar et al.'s [30] subsequent generalization to

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ideal convergence in 2009. More recently, Kirişci and Şimşek [31] introduced neutrosophic normed linear spaces, advancing the field by investigating statistical convergence and inspiring further research into diverse types of sequence convergence within these spaces. For deeper exploration, see [4, 33, 34, 36]. The concept of 2-normed linear spaces, later extended to n-normed linear spaces, was pioneered by Gähler [16, 17], igniting significant interest among researchers. Murtaza et al. [45] introduced the groundbreaking concept of neutrosophic 2-normed linear spaces in 2023-a profound extension of neutrosophic normed spaces-while exploring their statistical convergence and completeness. More recently, Kumar et al. [40] advanced the field further with the innovative notion of neutrosophic n-normed linear spaces, delving into their convergence structures and defining Cauchy sequences within this novel framework. However, early studies reveal intriguing parallels in the behavior of sequence convergence within these spaces. Building on these observations, we define and explore the concept of newly introduced spaces such as $\mathscr{C}^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$, $\mathscr{C}_0^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$. We nurture some important properties including linearity and topological of the spaces within the frameork of neutrosophic n-normed linear space.

Now, we present an overview of key definitions and terminology essential for describing our main results. Throughout the study, $\mathbb{N}$ denotes the set of all natural numbers.

Ideals

A family $\mathscr{I}$ of subsets of a non-empty set $\mathscr{X}$ is called an ideal [27] if the following conditions hold:

1. $\emptyset \in \mathscr{I}$;
2. $\mathscr{A}, \mathscr{B} \in \mathscr{I} \implies \mathscr{A} \cup \mathscr{B} \in \mathscr{I}$;
3. $\mathscr{A} \in \mathscr{I}$ and $\mathscr{B} \subseteq \mathscr{A} \implies \mathscr{B} \in \mathscr{I}$.

An ideal $\mathscr{I}$ is called non-trivial if $\mathscr{X} \notin \mathscr{I}$ and $\mathscr{I} \neq \emptyset$. Moreover, a non-trivial ideal $\mathscr{I} \subset 2^{\mathscr{X}}$ is termed admissible if $\{\{x\} : x \in \mathscr{X}\} \subset \mathscr{I}$. For instance, the class $\mathscr{I}_f$ of all finite subsets of $\mathbb{N}$ forms an admissible ideal on $\mathbb{N}$. Unless otherwise stated, $\mathscr{I}$ will denote a non-trivial admissible ideal throughout this paper.

Filters

A non-empty family $\mathscr{F}$ of subsets of a non-empty set $\mathscr{X}$ is called a filter [27] if:

1. $\emptyset \notin \mathscr{F}$;
2. $\mathscr{A}, \mathscr{B} \in \mathscr{F} \implies \mathscr{A} \cap \mathscr{B} \in \mathscr{F}$;
3. $\mathscr{A} \in \mathscr{F}$ and $\mathscr{A} \subseteq \mathscr{B} \implies \mathscr{B} \in \mathscr{F}$.

If $\mathscr{I} \subset 2^{\mathscr{X}}$ is a non-trivial ideal, then the class $\mathscr{F}(\mathscr{I}) = \{\mathscr{X} \setminus \mathscr{A} : \mathscr{A} \in \mathscr{I}\}$ forms a filter on $\mathscr{X}$, known as the filter associated with the ideal $\mathscr{I}$ [27]. Ideal and statistical convergence are generalizations of the classical notion of sequence convergence in mathematical analysis. These concepts are particularly useful in summability theory, real analysis, and functional analysis. In recent years, the study and application of these concepts has been growing rapidly in various settings [37, 38, 39].

Generalized de la Vallée-Poussin mean and summability

Let $\lambda = \lambda_n$ be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \leqslant \lambda_n + 1$ and $\lambda_1 = 1$. Throughout this discussion, Δ denotes such a sequence λ_n. The generalized de la Vallée-Poussin mean of a sequence $w = \{w_k\}$ is given by: $\sigma_n(w) := \frac{1}{\lambda_n} \sum_{k \in J_n} w_k$, where $J_n = [n - \lambda_n + 1, n]$. A sequence $w = \{w_k\}$ is referred to as (V,λ)-summable [41] to a number ξ if $\sigma_n(w) \to \xi$ as $n \to \infty$.

When $\lambda_n = n$, (V,λ)-summability reduces to $(C,1)$-summability. For the sets of sequences $w = \{w_k\}$ strongly (V,λ)-summable and strongly $(C,1)$-summable to ξ, we define:

$$[V,\lambda] = \left\{ w = \{w_k\} : \exists\, \xi \in \mathbb{R},\ \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k \in J_n} |w_k - \xi| = 0 \right\},$$

$$[C,1] = \left\{ w = \{w_k\} : \exists\, \xi \in \mathbb{R},\ \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |w_k - \xi| = 0 \right\}.$$

λ-density

Let $\mathscr{M} \subset \mathbb{N}$. The λ-density of $\mathscr{M}$, denoted by $\delta_\lambda(\mathscr{M})$, is defined as:

$$\delta_\lambda(\mathscr{M}) = \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in J_n : k \in \mathscr{M}\}|,$$

provided the limit exists. Notably, the λ-density of a finite set is always zero, and for subsets $\mathscr{A} \subseteq \mathscr{B}$, it holds that $\delta_\lambda(\mathscr{A}) \leqslant \delta_\lambda(\mathscr{B})$.

DEFINITION 1. ($\mathscr{I}_\lambda$-density) [19] Let $\mathscr{M} \subseteq \mathbb{N}$. Then the $\mathscr{I}_\lambda$-density of $\mathscr{M}$, denoted by $\delta_\lambda^{\mathscr{I}}(\mathscr{M})$, is defined as

$$\delta_\lambda^{\mathscr{I}}(\mathscr{M}) = \mathscr{I} - \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in J_n : k \in \mathscr{M}\}|,$$

provided the limit exists, where the vertical bars denote the cardinality of the enclosed set.

Let $\mathscr{M} = \{k_1 < k_2 < \cdots < k_p < \cdots\}$ and $\{w_{k_p}\}$ is a subsequence of $\{w_k\}$. If $\delta_\lambda^{\mathscr{I}}(\mathscr{M}) = 0$, then $\{w_{k_p}\}$ is referred to as thin subsequence of $\{w_k\}$. Meanwhile, if $\delta_\lambda^{\mathscr{I}}(\mathscr{M}) \neq 0$, which means either $\mathscr{I}_\lambda$-density of $\mathscr{M}$ is positive number or it fails to have $\mathscr{I}_\lambda$-density, then $\{w_{k_p}\}$ is referred to as nonthin subsequence of $\{w_k\}$.

REMARK 1. If $\lambda_n = n$, then Definition 1 coincides with the notion of $\mathscr{I}$-natural density [44]. And, if $\lambda_n = n$ and $\mathscr{I} = \mathscr{I}_f$, then Definition 1 coincides with the notion of natural density [13].

Continuous t-norm and t-conorm

[50] A binary operation $\circledast : \mathscr{J} \times \mathscr{J} \to \mathscr{J}$, defined on $\mathscr{J} = [0,1]$, is termed a continuous t-norm if it meets the following criteria, for each $v_1, v_2, v_3, v_4 \in \mathscr{J}$:

1. $\boxast$ exhibits both associativity and commutativity;
2. $\boxast$ exhibits continuous behavior;
3. $v_1 \boxast 1 = v_1$ for all $v_1 \in \mathscr{J}$;
4. $v_1 \boxast v_2 \leqslant v_3 \boxast v_4$ whenever $v_1 \leqslant v_3$ and $v_2 \leqslant v_4$.

A binary operation $\circledast : \mathscr{J} \times \mathscr{J} \to \mathscr{J}$, where $\mathscr{J} = [0,1]$ is named to be a continuous t-conorm if for each $v_1, v_2, v_3, v_4 \in \mathscr{J}$, the below conditions hold:

1. $\circledast$ exhibits both associativity and commutativity;
2. $\circledast$ exhibits continuous behavior;
3. $v_1 \circledast 0 = v_1$ for all $v_1 \in \mathscr{J}$;
4. $v_1 \circledast v_2 \leqslant v_3 \circledast v_4$ whenever $v_1 \leqslant v_3$ and $v_2 \leqslant v_4$.

The continuous t-norms are $v_1 \boxast v_2 = \min\{v_1, v_2\}$ and $v_1 \boxast v_2 = v_1.v_2$. On the other hand, continuous t-conorms are $v_1 \circledast v_2 = \max\{v_1, v_2\}$ and $v_1 \circledast v_2 = v_1 + v_2 - v_1.v_2$ [28].

n-normed linear space

[18] Let $n \in \mathbb{N}$ and $\mathscr{W}$ be a real vector space having dimension $d \geqslant n$ (d is finite or infinite). A real valued function $\|\cdot, \ldots, \cdot\|$ on $\underbrace{\mathscr{W} \times \mathscr{W} \times \ldots \times \mathscr{W}}_{n \text{ times}} = \mathscr{W}^n$, gratifying the below four axioms:

1. $\|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\| = 0$ if and only if $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ are linearly dependent;
2. $\|\vartheta_1, \vartheta_2, \ldots, \vartheta_n\|$ remains invariant under any permutation of $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$;
3. $\|\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \alpha\vartheta_n\| = |\alpha| \, \|\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \vartheta_n\|$ for $\alpha \in \mathbb{R}$ (set of real numbers);
4. $\|\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \tau + \omega\| \leqslant \|\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \tau\| + \|\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \omega\|$;

is called an n-norm on $\mathscr{W}$ and the pair $(\mathscr{W},\|\cdot,\ldots,\cdot\|)$ is named to be an n-normed linear space. As an illustration of n-normed linear space we take $\mathscr{W}=\mathbb{R}^n$ equipped with the Euclidean norm

$$\|w_1,w_2,\ldots,w_n\| = abs\left(\begin{vmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{vmatrix}\right)$$

where $w_i=(w_{i1},w_{i2},\ldots,w_{in})\in\mathbb{R}^n$. For instance, we get $\|w_1,w_2,\ldots,w_n\|\geqslant 0$ in an n-normed linear space.

Neutrosophic n-normed linear space

[40] Let $\mathscr{W}$ be a vector space over $\mathscr{F}$ and $\circledast$ and $\circledast$ be continuous t-norm and t-conorm respectively. Let $\mathscr{G},\mathfrak{R},\mathscr{Y}$ be the functions from $\mathscr{W}^n\times(0,\infty)$ to $[0,1]$. Then a six tuple $(\mathscr{W},\mathscr{G},\mathfrak{R},\mathscr{Y},\boxast,\circledast)$ is named to be a neutrosophic n-normed linear space (in short Nn-NLS), $(w_1,w_2,\ldots,w_{n-1},w_n;\zeta)\in\mathscr{W}^n\times(0,\infty)\to[0,1]$, if the below conditions hold:

1. $\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)+\mathfrak{R}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)$ $+\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)\leqslant 3$;
2. $\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)>0$;
3. $\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)=1$ if and only if w_j are linearly dependent, $1\leqslant j\leqslant n$;
4. $\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)$ is invariant under any permutation of $w_1,w_2,\ldots,w_n$;
5. $\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\kappa w_n;\zeta)=\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\frac{\zeta}{|\kappa|})$, $\kappa\neq 0$ and $\kappa\in\mathscr{F}$;
6. $\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n+w_n';\zeta+\tau)$ $\geqslant\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)\boxast\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n';\tau)$;
7. $\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)$ is non-decreasing continuous in ζ;
8. $\lim_{\zeta\to\infty}\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)=1$ and $\lim_{\zeta\to 0}\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)=0$;
9. $\mathfrak{R}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)>0$;
10. $\mathfrak{R}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)=0$ if and only if w_j are linearly dependent, $1\leqslant j\leqslant n$;
11. $\mathfrak{R}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\zeta)$ is invariant under any permutation of $w_1,w_2,\ldots,w_n$;
12. $\mathfrak{R}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\kappa w_n;\zeta)=\mathfrak{R}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_n;\frac{\zeta}{|\kappa|})$, $\kappa\neq 0$ and $\kappa\in\mathscr{F}$;

13. $\mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n + w_n^{'}; \zeta + \tau)$
$\leqslant \mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta) \circledast \mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n^{'}; \tau)$;

14. $\mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta)$ is non-increasing continuous in ζ;

15. $\lim_{\zeta \to \infty} \mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta) = 0$ and
$\lim_{\zeta \to 0} \mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta) = 1$;

16. $\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta) > 0$;

17. $\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta) = 0$ if and only if w_j are linearly dependent, $1 \leqslant j \leqslant n$;

18. $\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta)$ is invariant under any permutation of $w_1, w_2, \ldots, w_n$;

19. $\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \kappa w_n; \zeta) = \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \frac{\zeta}{|\kappa|})$, $\kappa \neq 0$ and $\kappa \in \mathscr{F}$;

20. $\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n + w_n^{'}; \zeta + \tau)$
$\leqslant \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta) \circledast \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n^{'}; \tau)$;

21. $\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta)$ is non-increasing continuous in ζ;

22. $\lim_{\zeta \to \infty} \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta) = 0$ and
$\lim_{\zeta \to 0} \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta) = 1$.

In the sequal, we shall use the notation $\mathscr{H}$ for neutrosophic n-normed linear space instead of $(\mathscr{W}, \mathscr{G}, \mathfrak{R}, \mathscr{Y}, \boxast, \circledast)$ and we denote $\mathscr{N}_n$ to mean neutrosophic n-norm on $\mathscr{H}$.

EXAMPLE 1. [40] Let $(\mathscr{W}, \|\cdot, \ldots, \cdot\|)$ be an n-normed linear space. Also, let $v_1 \boxast v_2 = \min(v_1, v_2)$ and $v_1 \circledast v_2 = \max(v_1, v_2)$ for every $v_1, v_2 \in [0,1]$. If we define $\mathscr{G}, \mathfrak{R}$ and $\mathscr{Y}$ as

$$\mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta) = \frac{\zeta}{\zeta + \|\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n\|}$$
$$\mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta) = \frac{\|\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n\|}{\zeta + \|\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n\|}$$

and

$$\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n; \zeta) = \frac{\|\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_n\|}{\zeta}.$$

Then $(\mathscr{W}, \mathscr{G}, \mathfrak{R}, \mathscr{Y}, \boxast, \circledast)$ is a neutrosophic n-normed linear space.

Convergence in $\mathscr{H}$

[40] A sequence $\{w_k\}$ belonging to $\mathscr{H}$ is named to be convergent to $\upsilon \in \mathscr{W}$ in relation to $\mathscr{N}_n$ (in short $\mathscr{N}_n$-convergence) if for every $\sigma > 0$, $\zeta > 0$ and $\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1} \in \mathscr{W}$ there can be found $k_0 \in \mathbb{N}$ in a way that

$$\begin{cases} \mathscr{G}(\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, w_k - \upsilon; \zeta) > 1 - \sigma \quad \text{and} \\ \Re(\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, w_k - \upsilon; \zeta) < \sigma, \\ \mathscr{Y}(\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, w_k - \upsilon; \zeta) < \sigma \end{cases}$$

for all $k \geqslant k_0$. In this scenario, it is denoted as $\mathscr{N}_n - \lim w_k = \upsilon$ or $w_k \xrightarrow{\mathscr{N}_n} \upsilon$.

A sequence $\{w_k\}$ belonging to $\mathscr{H}$ is referred to as Cauchy sequence in relation to $\mathscr{N}_n$ if for every $\sigma > 0, \zeta > 0$ there can be found $k_0 \in \mathbb{N}$ in a way that

$$\begin{cases} \mathscr{G}(\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, w_k - w_m; \zeta) > 1 - \sigma \quad \text{and} \\ \Re(\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, w_k - w_m; \zeta) < \sigma, \\ \mathscr{Y}(\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, w_k - w_m; \zeta) < \sigma \end{cases}$$

for all $k, m \geqslant k_0$.

THEOREM 1. [40] *Let $\{w_k\}$ be a sequence in $\mathscr{W}$. Then $\{w_k\}$ is convergent in $(\mathscr{W}, \|\cdot, \dots, \cdot\|)$ iff $\{w_k\}$ is convergent in $\mathscr{H}$ with respect to neutrosophic n-norm as defined in Example 1.*

Recall from [25] that an Orlicz function is a mapping $\Phi : [0,\infty) \to [0,\infty)$ that is continuous, convex, and non-decreasing, satisfying $\Phi(0) = 0$, $\Phi(\vartheta) > 0$ and $\Phi(\vartheta) \to \infty$ as $\vartheta \to \infty$. If the convexity condition is replaced by the inequality $\Phi(\vartheta + \alpha) \leqslant \Phi(\vartheta) + \Phi(\alpha)$, the function is known as a modulus function. This concept, introduced by Nakano [46], was further explored with applications to sequence spaces by Musielak [43], and Tripathy [53]. It is well known that if Φ is a convex function with $\Phi(0) = 0$ then it satisfies the inequality $\Phi(\lambda\vartheta) \leqslant \lambda\Phi(\vartheta)$ for all λ with $0 < \lambda < 1$. Lindenstrauss and Tzafriri [42] utilized the notion of Orlicz functions to define the sequence space

$$\ell_\Phi = \left\{ w = \{w_k\} \in \omega : \sum_{k=1}^{\infty} \Phi\left(\frac{|w_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}$$

which is known as the Orlicz sequence space. This space forms a Banach space with the norm

$$\|w\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} \Phi\left(\frac{|w_k|}{\rho}\right) \leqslant 1 \right\}.$$

For $\Phi(\vartheta) = \vartheta^p$ with $1 \leqslant p < \infty$, the space ℓ_Φ coincides with the classical sequence space l_p. An Orlicz function is said to satisfy the condition Δ_2-condition if there exists a constant $K > 0$ such that $\Phi(\gamma\vartheta) \leqslant K\gamma\Phi(\vartheta)$ for all $\gamma > 1$. Building on the concept of $\mathscr{I}$-convergence, Tripathy and Hazarika [53] introduced a new class of sequence spaces defined using the Orlicz function, further enriching the study of sequence spaces and their structural properties. In recent years, this notion has garnered significant attention and development by numerous researchers including Bilgin [8], Debnath et al. [10], Esi [11], and Hazarika [22] across diverse mathematical frameworks such as 2-normed spaces [20], n-normed spaces [12], and intuitionistic fuzzy n-normed spaces [32].

2. Main results

In this section, we explore the concept of λ-convergence and the associated ideal $\mathscr{I}_\lambda$ in neutrosophic n-normed linear spaces. We introduce novel neutrosophic n-normed $\mathscr{I}_\lambda$-convergent sequence spaces, incorporating the Orlicz function Φ, and investigate their fundamental topological and algebraic properties. Additionally, we establish various inclusion relations related to these newly developed spaces, providing deeper insights into their structural framework.

DEFINITION 2. Consider $\{w_k\}$ to be a sequence in a Nn-NLS $\mathscr{H}$. Then $\{w_k\}$ is referred to as λ-convergent to $\upsilon \in \mathscr{W}$ in relation to $\mathscr{N}_n$ if for every $\zeta > 0$, $\sigma \in (0,1)$ and nonzero $\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1} \in \mathscr{W}$ there can be found $n_0 \in \mathbb{N}$ in a way that

$$\begin{cases} \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) > 1 - \sigma \quad \text{and} \\ \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) < \sigma, \\ \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) < \sigma \end{cases}$$

for all $n \geqslant n_0$. In this scenario, we express $\mathscr{N}_n(\lambda) - \lim w_k = \upsilon$.

DEFINITION 3. Consider $\{w_k\}$ to be a sequence in a Nn-NLS $\mathscr{H}$. Then $\{w_k\}$ is referred to as $\mathscr{I}_\lambda$-convergent to $\upsilon \in \mathscr{W}$ in relation to $\mathscr{N}_n$ if for every $\zeta > 0$, $\sigma \in (0,1)$ and nonzero $\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1} \in \mathscr{W}$ the set

$$\left\{ n \in \mathbb{N} : \begin{array}{l} \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) \leqslant 1 - \sigma \quad \text{or} \\ \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) \geqslant \sigma \quad \text{and} \\ \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) \geqslant \sigma \end{array} \right\} \in \mathscr{I}.$$

In this scenario, we express $\mathscr{I}_\lambda(\mathscr{N}_n) - \lim w_k = \upsilon$ or $w_k \xrightarrow{\mathscr{I}_\lambda(\mathscr{N}_n)} \upsilon$. And, υ is called $\mathscr{I}_\lambda(\mathscr{N}_n)$-limit of $\{w_k\}$.

LEMMA 1. *Consider $\{w_k\}$ to be a sequence in a Nn-NLS $\mathscr{H}$. Then, for every $\zeta > 0$, $\sigma \in (0,1)$ and nonzero $\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1} \in \mathscr{W}$, the statements enlisted below are met:*

1. $w_k \xrightarrow{\mathscr{I}_\lambda(\mathscr{N}_n)} \upsilon$;

2. $\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in J_n} \mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) \leqslant 1-\sigma\right\} \in \mathscr{I}$;
$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in J_n} \mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) \geqslant \sigma\right\} \in \mathscr{I}$;
$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in J_n} \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) \geqslant \sigma\right\} \in \mathscr{I}$;

3.
$$\left\{ n \in \mathbb{N} : \begin{array}{l} \frac{1}{\lambda_n}\sum_{k\in J_n} \mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) > 1-\sigma \quad and \\ \frac{1}{\lambda_n}\sum_{k\in J_n} \mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) < \sigma, \\ \frac{1}{\lambda_n}\sum_{k\in J_n} \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) < \sigma \end{array} \right\} \in \mathscr{F}(\mathscr{I});$$

4. $\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in J_n} \mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) > 1-\sigma\right\} \in \mathscr{F}(\mathscr{I})$;
$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in J_n} \mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) < \sigma\right\} \in \mathscr{F}(\mathscr{I})$;
$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in J_n} \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) \geqslant \sigma\right\} \in \mathscr{F}(\mathscr{I})$;

5. $\mathscr{I}_\lambda(\mathscr{N}_n) - \lim \mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) = 1$,
$\mathscr{I}_\lambda(\mathscr{N}_n) - \lim \mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) = 0$ *and*
$\mathscr{I}_\lambda(\mathscr{N}_n) - \lim \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta) = 0$.

THEOREM 2. *Consider $\{w_k\}$ to be a sequence in a Nn-NLS $\mathscr{H}$. Then, $\mathscr{I}_\lambda(\mathscr{N}_n)$-limit of $\{w_k\}$ is uniquely determined.*

Proof. If possible, let $\mathscr{I}_\lambda(\mathscr{N}_n) - \lim w_k = \upsilon_1$ and $\mathscr{I}_\lambda(\mathscr{N}_n) - \lim w_k = \upsilon_2$ where $\upsilon_1 \neq \upsilon_2$. For a specified $\sigma \in (0,1)$, choose $\varpi \in (0,1)$ such that $(1-\varpi) \boxast (1-\varpi) > 1-\sigma$ and $\varpi \circledast \varpi < \sigma$. Then, for any $\zeta > 0$ and nonzero $\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1} \in \mathscr{W}$, we define

$$\mathscr{C}_{\mathscr{G},1}(\varpi,\zeta) = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in J_n} \mathscr{G}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}\right) \leqslant 1-\varpi\right\},$$

$$\mathscr{C}_{\mathfrak{R},1}(\varpi,\zeta) = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in J_n} \mathfrak{R}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}\right) \geqslant \varpi\right\},$$

$$\mathscr{C}_{\mathscr{Y},1}(\varpi,\zeta) = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in J_n} \mathscr{Y}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}\right) \geqslant \varpi\right\},$$

$$\mathscr{D}_{\mathscr{G},2}(\varpi,\zeta) = \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in J_n} \mathscr{G}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon_2; \frac{\zeta}{2}\right) \leqslant 1-\varpi\right\},$$

$$\mathscr{D}_{\mathfrak{R},2}(\varpi,\zeta)=\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in J_n}\mathfrak{R}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_2;\frac{\zeta}{2}\right)\geqslant\varpi\right\},$$

$$\mathscr{D}_{\mathscr{Y},2}(\varpi,\zeta)=\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in J_n}\mathscr{Y}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_2;\frac{\zeta}{2}\right)\geqslant\varpi\right\}.$$

Since $\mathscr{I}_\lambda(\mathscr{N}_n)-\lim w_k=\upsilon_1$, all of $\mathscr{C}_{\mathscr{G},1}(\varpi,\zeta)$, $\mathscr{C}_{\mathfrak{R},1}(\varpi,\zeta)$ and $\mathscr{C}_{\mathscr{Y},1}(\varpi,\zeta)$ belong to $\mathscr{I}$. Again, since $\mathscr{I}_\lambda(\mathscr{N}_n)-\lim w_k=\upsilon_2$, $\mathscr{D}_{\mathscr{G},2}(\varpi,\zeta)$, $\mathscr{D}_{\mathfrak{R},2}(\varpi,\zeta)$ and $\mathscr{D}_{\mathscr{Y},2}(\varpi,\zeta)$ $\in\mathscr{I}$. Let $\mathscr{A}(\sigma,\zeta)=\{\mathscr{C}_{\mathscr{G},1}(\varpi,\zeta)\cup\mathscr{D}_{\mathscr{G},2}(\varpi,\zeta)\}\cap\{\mathscr{C}_{\mathfrak{R},1}(\varpi,\zeta)\cup\mathscr{D}_{\mathfrak{R},2}(\varpi,\zeta)\}\cap\{\mathscr{C}_{\mathscr{Y},1}(\varpi,\zeta)\cup\mathscr{D}_{\mathscr{Y},2}(\varpi,\zeta)\}$. So, $\mathscr{A}(\sigma,\zeta)\in\mathscr{I}$ and consequently, $\mathscr{A}^c(\sigma,\zeta)\in\mathscr{F}(\mathscr{I})$. So, there is $n\in\mathscr{A}^c(\sigma,\zeta)$. So, whenever $n\in\{\mathscr{C}_{\mathscr{G},1}(\varpi,\zeta)\cup\mathscr{D}_{\mathscr{G},2}(\varpi,\zeta)\}^c$, we get

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_1;\frac{\zeta}{2}\right)>1-\varpi$$
$$\frac{1}{\lambda_n}\sum_{k\in J_n}\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_2;\frac{\zeta}{2}\right)>1-\varpi.$$

Let $i=\max\left\{\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_1;\frac{\zeta}{2}\right),\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_2;\frac{\zeta}{2}\right):k\in J_n\right\}$. Then,

$$\begin{aligned}&\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_i-\upsilon_1;\frac{\zeta}{2}\right)\\&>\frac{1}{\lambda_n}\sum_{k\in J_n}\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_1;\frac{\zeta}{2}\right)\\&>1-\varpi\end{aligned}$$

and

$$\begin{aligned}&\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_i-\upsilon_2;\frac{\zeta}{2}\right)\\&>\frac{1}{\lambda_n}\sum_{k\in J_n}\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_2;\frac{\zeta}{2}\right)\\&>1-\varpi.\end{aligned}$$

Therefore, we arrive at

$$\begin{aligned}&\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\upsilon_1-\upsilon_2;\zeta)\\&\geqslant\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_i-\upsilon_1;\frac{\zeta}{2}\right)\circledast\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_i-\upsilon_2;\frac{\zeta}{2}\right)\\&>(1-\varpi)\circledast(1-\varpi)\\&>1-\sigma.\end{aligned}$$

Since, $\sigma > 0$ is arbitrary, $\mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \upsilon_1 - \upsilon_2; \zeta) = 1$. This, yields $\upsilon_1 = \upsilon_2$. For other two cases in relation to $\mathfrak{R}$ and $\mathscr{Y}$, we will arrive at $\upsilon_1 = \upsilon_2$. Hence, $\mathscr{I}_\lambda(\mathscr{N}_n)$-limit of $\{w_k\}$ is unique. Thus, the proof stands established. □

We now introduce the concept of λ-Cauchy sequences within the framework of ideals and explore some results concerning $\mathscr{I}_\lambda$-convergence and $\mathscr{I}_\lambda$-Cauchy sequences in the setting of neutrosophic n-normed linear spaces.

DEFINITION 4. Consider $\{w_k\}$ to be a sequence in a Nn-NLS $\mathscr{H}$. Then $\{w_k\}$ is referred to as λ-Cauchy sequence in relation to $\mathscr{N}_n$ if for every $\zeta > 0$, $\sigma \in (0,1)$ and nonzero $\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1} \in \mathscr{W}$ there can be found $p_0, p \in \mathbb{N}$ in a way that the following inequalities are satisfied:

$$\begin{cases} \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - w_p; \zeta) > 1 - \sigma \quad \text{and} \\ \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - w_p; \zeta) < \sigma, \\ \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - w_p; \zeta) < \sigma \end{cases}$$

for all $n \geqslant p_0$.

DEFINITION 5. Consider $\{w_k\}$ to be a sequence in a Nn-NLS $\mathscr{H}$. Then $\{w_k\}$ is referred to as $\mathscr{I}_\lambda$-Cauchy in relation to $\mathscr{N}_n$ if for every $\zeta > 0$, $\sigma \in (0,1)$ and nonzero $\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1} \in \mathscr{W}$ there is a $p \in \mathbb{N}$ such that the set

$$\left\{ n \in \mathbb{N} : \begin{array}{l} \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - w_p; \zeta) > 1 - \sigma \quad \text{and} \\ \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - w_p; \zeta) < \sigma, \\ \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - w_p; \zeta) < \sigma \end{array} \right\} \in \mathscr{F}(\mathscr{I}).$$

THEOREM 3. *Let $\{w_k\}$ be a sequence in a neutrosophic n-normed linear space (Nn-NLS) $\mathscr{H}$. If $\{w_k\}$ is a λ-Cauchy sequence, then it is also an $\mathscr{I}_\lambda$-Cauchy sequence with respect to the neutrosophic n-norm $\mathscr{N}_n$.*

Proof. The proof follows directly from the definitions and is therefore omitted for brevity. □

THEOREM 4. *Let $\{w_k\}$ be a sequence in a neutrosophic n-normed linear space (Nn-NLS) $\mathscr{H}$. If $\{w_k\}$ is a $\mathscr{I}_\lambda$-convergent sequence, then it is also an $\mathscr{I}_\lambda$-Cauchy sequence with respect to the neutrosophic n-norm $\mathscr{N}_n$.*

Proof. Let, $\{w_k\}$ is a $\mathscr{I}_\lambda$-convergent to $\upsilon \in \mathscr{W}$. For a specified $\sigma \in (0,1)$, choose $\varpi \in (0,1)$ such that $(1-\varpi) \circledast (1-\varpi) > 1-\sigma$ and $\varpi \circledast \varpi < \sigma$. Then, for any $\zeta > 0$ and nonzero $\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1} \in \mathscr{W}$, we define

$$\mathscr{C}_1(\varpi,\zeta) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mathscr{G}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) > 1-\varpi \right\},$$
$$\mathscr{C}_2(\varpi,\zeta) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \Re\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \varpi \right\},$$
$$\mathscr{C}_3(\varpi,\zeta) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mathscr{Y}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \varpi \right\}.$$

Since, $\{w_k\}$ is a $\mathscr{I}_\lambda$-convergent to υ, all of $\mathscr{C}_1(\varpi,\zeta), \mathscr{C}_2(\varpi,\zeta)$ and $\mathscr{C}_3(\varpi,\zeta)$ belong to $\mathscr{F}(\mathscr{I})$. Let

$$\mathscr{C}(\varpi,\zeta) = \mathscr{C}_1(\varpi,\zeta) \cap \mathscr{C}_2(\varpi,\zeta) \cap \mathscr{C}_3(\varpi,\zeta).$$

Hence $\mathscr{C}(\varpi,\zeta) \in \mathscr{F}(\mathscr{I})$ and thus, $\mathscr{C}(\varpi,\zeta)$ is non-empty. Let $n \in \mathscr{C}(\varpi,\zeta)$. So, we can choose a fixed $p \in \mathscr{C}(\varpi,\zeta)$. Then, we have

$$\begin{aligned}
&\mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - w_p; \zeta) \\
&\geqslant \mathscr{G}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) \circledast \mathscr{G}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_p - \upsilon; \frac{\zeta}{2}\right) \\
&> (1-\varpi) \circledast (1-\varpi) \\
&> 1-\sigma.
\end{aligned}$$

This results in

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - w_p; \zeta) > 1-\sigma. \tag{1}$$

And,

$$\begin{aligned}
&\Re(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - w_p; \zeta) \\
&\leqslant \Re\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) \circledast \Re\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_p - \upsilon; \frac{\zeta}{2}\right) \\
&< \varpi \circledast \varpi \\
&< \sigma.
\end{aligned}$$

This yields that

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \Re(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - w_p; \zeta) < \sigma. \tag{2}$$

Also,

$$\begin{aligned}
&\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-w_p;\zeta)\\
&\leqslant \mathscr{Y}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon;\frac{\zeta}{2}\right)\circledast\mathscr{Y}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_p-\upsilon;\frac{\zeta}{2}\right)\\
&<\varpi\circledast\varpi\\
&<\sigma.
\end{aligned}$$

This gives

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-w_p;\zeta)<\sigma. \tag{3}$$

Therefore, from the equations (1), (2) and (3), we conclude that

$$\mathscr{C}(\varpi,\zeta)\subseteq\left\{n\in\mathbb{N}:\begin{array}{c}\frac{1}{\lambda_n}\sum_{k\in J_n}\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-w_p;\zeta)>1-\sigma \quad\text{and}\\ \frac{1}{\lambda_n}\sum_{k\in J_n}\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-w_p;\zeta)<\sigma,\\ \frac{1}{\lambda_n}\sum_{k\in J_n}\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-w_p;\zeta)<\sigma\end{array}\right\}.$$

As $\mathscr{C}(\varpi,\zeta)\in\mathscr{F}(\mathscr{I})$, $\{w_k\}$ is an $\mathscr{I}_\lambda$ -Cauchy sequence. Thus, the proof stands established. □

Now, we introduce novel generalized ideal convergent sequence space defined by Orlicz function Φ in neutrosophic n-normed linear spaces. We further explore their topological and algebraic properties and establish key inclusion relations for these newly defined spaces. For some $\upsilon\in\mathscr{W}$ and $\rho>0$, we define new sequence spaces in relation to $\mathscr{N}_n$ as below:

$$\begin{aligned}
&\mathscr{C}^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)\\
&=\left\{\{w_k\}\in\mathscr{W}:\left\{n\in\mathbb{N}:\begin{array}{c}\frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon;\zeta)}{\rho}\right)\leqslant 1-\sigma\text{ or}\\ \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon;\zeta)}{\rho}\right)\geqslant\sigma\text{ and}\\ \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon;\zeta)}{\rho}\right)\geqslant\sigma\end{array}\right\}\in\mathscr{I}\right\},
\end{aligned}$$

$$\begin{aligned}
&\mathscr{C}_0^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)\\
&=\left\{\{w_k\}\in\mathscr{W}:\left\{n\in\mathbb{N}:\begin{array}{c}\frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k;\zeta)}{\rho}\right)\leqslant 1-\sigma\text{ or}\\ \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k;\zeta)}{\rho}\right)\geqslant\sigma\text{ and}\\ \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k;\zeta)}{\rho}\right)\geqslant\sigma\end{array}\right\}\in\mathscr{I}\right\}.
\end{aligned}$$

We define an open ball centered at $w = \{w_k\} \in \mathscr{W}$ with radius σ, relative to $\zeta > 0$ as follows:

$$\mathscr{B}(w,\sigma,\zeta)(\Phi) = \left\{ \{\tau_k\} \in \mathscr{W} : \left\{ n \in \mathbb{N} : \begin{array}{c} \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\tau_k;\zeta)}{\rho}\right) \leqslant 1-\sigma \text{ or} \\ \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\tau_k;\zeta)}{\rho}\right) \geqslant \sigma \text{ and} \\ \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\tau_k;\zeta)}{\rho}\right) \geqslant \sigma \end{array} \right\} \in \mathscr{I} \right\}.$$

THEOREM 5. *The* $\mathscr{C}^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$ *and* $\mathscr{C}_0^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$ *form linear spaces over* $\mathbb{R}$.

Proof. Let $w = \{w_k\}$ and $\tau = \{\tau_k\}$ are in $\mathscr{C}^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$ and γ_1, γ_2 are nonzero scalars. For a specified $\sigma \in (0,1)$, choose $\varpi \in (0,1)$ such that $(1-\varpi) \circledast (1-\varpi) > 1-\sigma$ and $\varpi \circledast \varpi < \sigma$. Then, there are $\upsilon_1, \upsilon_2 \in \mathscr{W}$ for which the sets

$$\mathscr{M}_1 = \left\{ n \in \mathbb{N} : \begin{array}{l} \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_1;\frac{\zeta}{2|\gamma_1|}\right)}{\rho_1}\right) > 1-\varpi \text{ and} \\ \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\Re\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_1;\frac{\zeta}{2|\gamma_1|}\right)}{\rho_1}\right) < \varpi, \\ \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\mathscr{Y}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_1;\frac{\zeta}{2|\gamma_1|}\right)}{\rho_1}\right) < \varpi \end{array} \right\} \in \mathscr{F}(\mathscr{I})$$

and

$$\mathscr{M}_2 = \left\{ n \in \mathbb{N} : \begin{array}{l} \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\tau_k-\upsilon_2;\frac{\zeta}{2|\gamma_2|}\right)}{\rho_2}\right) > 1-\varpi \text{ and} \\ \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\Re\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\tau_k-\upsilon_2;\frac{\zeta}{2|\gamma_2|}\right)}{\rho_2}\right) < \varpi, \\ \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\mathscr{Y}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\tau_k-\upsilon_2;\frac{\zeta}{2|\gamma_2|}\right)}{\rho_2}\right) < \varpi \end{array} \right\} \in \mathscr{F}(\mathscr{I}),$$

for some $\rho_1, \rho_2 > 0$. Now, $\sigma \in (0,1)$ and ρ_3 are chosen in a way that for any Orlicz function Φ and the sequence λ we can have $\frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{1-\sigma}{\rho_3}\right) > 1-\sigma$ and

$\frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\sigma}{\rho_3}\right) < \sigma$. Therefore, by the properties of Φ we will get

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\gamma_1 w_k+\gamma_2\tau_k-(\gamma_1\upsilon_1+\gamma_2\upsilon_2);\zeta)}{\rho_3}\right)$$
$$\geqslant \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_1;\frac{\zeta}{2|\gamma_1|}\right)\circledast\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\tau_k-\upsilon_2;\frac{\zeta}{2|\gamma_2|}\right)}{\rho_3}\right)$$
$$> \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{(1-\varpi)\circledast(1-\varpi)}{\rho_3}\right) > \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{1-\sigma}{\rho_3}\right)$$
$$> 1-\sigma,$$
$$\frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\gamma_1 w_k+\gamma_2\tau_k-(\gamma_1\upsilon_1+\gamma_2\upsilon_2);\zeta)}{\rho_3}\right)$$
$$\leqslant \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\Re\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_1;\frac{\zeta}{2|\gamma_1|}\right)\circledast\Re\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\tau_k-\upsilon_2;\frac{\zeta}{2|\gamma_2|}\right)}{\rho_3}\right)$$
$$< \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\varpi\circledast\varpi}{\rho_3}\right) < \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\sigma}{\rho_3}\right)$$
$$< \sigma,$$

and

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\gamma_1 w_k+\gamma_2\tau_k-(\gamma_1\upsilon_1+\gamma_2\upsilon_2);\zeta)}{\rho_3}\right)$$
$$\leqslant \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{Y}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon_1;\frac{\zeta}{2|\gamma_1|}\right)\circledast\mathscr{Y}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\tau_k-\upsilon_2;\frac{\zeta}{2|\gamma_2|}\right)}{\rho_3}\right)$$
$$< \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\varpi\circledast\varpi}{\rho_3}\right) < \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\sigma}{\rho_3}\right)$$
$$< \sigma.$$

So,

$$\mathscr{M}_1\cap\mathscr{M}_2$$
$$\subseteq \left\{ n\in\mathbb{N}: \begin{array}{l} \frac{1}{\lambda_n}\sum\limits_{k\in J_n}\Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\gamma_1 w_k+\gamma_2\tau_k-(\gamma_1\upsilon_1+\gamma_2\upsilon_2);\zeta)}{\rho_3}\right) > 1-\sigma \text{ and} \\ \frac{1}{\lambda_n}\sum\limits_{k\in J_n}\Phi\left(\frac{\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\gamma_1 w_k+\gamma_2\tau_k-(\gamma_1\upsilon_1+\gamma_2\upsilon_2);\zeta)}{\rho_3}\right) < \sigma, \\ \frac{1}{\lambda_n}\sum\limits_{k\in J_n}\Phi\left(\frac{\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\gamma_1 w_k+\gamma_2\tau_k-(\gamma_1\upsilon_1+\gamma_2\upsilon_2);\zeta)}{\rho_3}\right) < \sigma \end{array} \right\} \in \mathscr{F}(\mathscr{I}).$$

Hence, $(\gamma_1 w+\gamma_2\tau)\in\mathscr{C}^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$, i.e., $\mathscr{C}^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$ is a linear space over $\mathbb{R}$. By similar argument we can easily verify that $(\gamma_1 w+\gamma_2\tau)\in\mathscr{C}_0^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$. Thus, the proof stands established. □

THEOREM 6. *Every open ball $\mathscr{B}(w,\sigma,\zeta)(\Phi)$ is an open set in $\mathscr{C}^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$.*

Proof. Let $\tau=\{\tau_k\}\in\mathscr{B}^c(w,\sigma,\zeta)(\Phi)$. Then,

$$\begin{cases}\dfrac{1}{\lambda_n}\displaystyle\sum_{k\in J_n}\Phi\left(\dfrac{\mathscr{G}(\vartheta_1,\vartheta_2,\dots,\vartheta_{n-1},w_k-\tau_k;\zeta)}{\rho}\right)>1-\sigma;\\ \dfrac{1}{\lambda_n}\displaystyle\sum_{k\in J_n}\Phi\left(\dfrac{\Re(\vartheta_1,\vartheta_2,\dots,\vartheta_{n-1},w_k-\tau_k;\zeta)}{\rho}\right)<\sigma \quad\text{and}\\ \dfrac{1}{\lambda_n}\displaystyle\sum_{k\in J_n}\Phi\left(\dfrac{\mathscr{Y}(\vartheta_1,\vartheta_2,\dots,\vartheta_{n-1},w_k-\tau_k;\zeta)}{\rho}\right)<\sigma\end{cases}\tag{4}$$

holds good. Since $\frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\dots,\vartheta_{n-1},w_k-\tau_k;\zeta)}{\rho}\right)>1-\sigma$, $\exists\ \zeta_0\in(0,\zeta_0)$ such that

$$\begin{cases}\dfrac{1}{\lambda_n}\displaystyle\sum_{k\in J_n}\Phi\left(\dfrac{\mathscr{G}(\vartheta_1,\vartheta_2,\dots,\vartheta_{n-1},w_k-\tau_k;\zeta_0)}{\rho}\right)>1-\sigma;\\ \dfrac{1}{\lambda_n}\displaystyle\sum_{k\in J_n}\Phi\left(\dfrac{\Re(\vartheta_1,\vartheta_2,\dots,\vartheta_{n-1},w_k-\tau_k;\zeta_0)}{\rho}\right)<\sigma \quad\text{and}\\ \dfrac{1}{\lambda_n}\displaystyle\sum_{k\in J_n}\Phi\left(\dfrac{\mathscr{Y}(\vartheta_1,\vartheta_2,\dots,\vartheta_{n-1},w_k-\tau_k;\zeta_0)}{\rho}\right)<\sigma.\end{cases}\tag{5}$$

Suppose $\sigma_0=\frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\dots,\vartheta_{n-1},w_k-\tau_k;\zeta_0)}{\rho}\right)>1-\sigma$. From (5) $\sigma_0>1-\sigma$. So, $\exists\ \varpi\in(0,1)$ such that $\sigma_0>1-\varpi>1-\sigma$. Consequently, $\sigma_1,\sigma_2\in(0,1)$ such that $\sigma_0\circledast\sigma_1>1-\varpi$ and $(1-\sigma_0)\circledast(1-\sigma_2)<\varpi$. Let $\sigma_3=\max\{\sigma_1,\sigma_2\}$. Then,

$$1-\varpi<\sigma_0\circledast\sigma_3,\ \text{and}\ (1-\sigma_0)\circledast(1-\sigma_3)<\varpi.\tag{6}$$

Now, we show that $\mathscr{B}^c(w,\sigma,\zeta)(\Phi)\supseteq\mathscr{B}^c(\tau,1-\sigma_3,\zeta-\zeta_0)(\Phi)$. Let $\{\alpha_k\}\in\mathscr{B}^c(\tau,1-\sigma_3,\zeta-\zeta_0)(\Phi)$. Then,

$$\begin{cases}\dfrac{1}{\lambda_n}\displaystyle\sum_{k\in J_n}\Phi\left(\dfrac{\mathscr{G}(\vartheta_1,\vartheta_2,\dots,\vartheta_{n-1},\tau_k-\alpha_k;\zeta-\zeta_0)}{\rho}\right)>1-(1-\sigma_3)=\sigma_3;\\ \dfrac{1}{\lambda_n}\displaystyle\sum_{k\in J_n}\Phi\left(\dfrac{\Re(\vartheta_1,\vartheta_2,\dots,\vartheta_{n-1},\tau_k-\alpha_k;\zeta-\zeta_0)}{\rho}\right)<\sigma_3 \quad\text{and}\\ \dfrac{1}{\lambda_n}\displaystyle\sum_{k\in J_n}\Phi\left(\dfrac{\mathscr{Y}(\vartheta_1,\vartheta_2,\dots,\vartheta_{n-1},\tau_k-\alpha_k;\zeta-\zeta_0)}{\rho}\right)<\sigma_3.\end{cases}\tag{7}$$

So,

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\alpha_k;\zeta)}{\rho}\right)$$
$$\geqslant \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\tau_k;\zeta_0)\circledast\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\tau_k-\alpha_k;\zeta-\zeta_0)}{\rho}\right)$$
$$>\sigma_0\circledast\sigma_3>1-\varpi>1-\sigma,$$

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\alpha_k;\zeta)}{\rho}\right)$$
$$\leqslant \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\tau_k;\zeta_0)\circledast\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\tau_k-\alpha_k;\zeta-\zeta_0)}{\rho}\right)$$
$$<(1-\sigma_0)\circledast(1-\sigma_3)<\varpi<\sigma,$$

and

$$\frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\alpha_k;\zeta)}{\rho}\right)$$
$$\leqslant \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\tau_k;\zeta_0)\circledast\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},\tau_k-\alpha_k;\zeta-\zeta_0)}{\rho}\right)$$
$$<(1-\sigma_0)\circledast(1-\sigma_3)<\varpi<\sigma.$$

So, $\{\alpha_k\}\in\mathscr{B}^c(w,\sigma,\zeta)(\Phi)$, i.e., $\mathscr{B}^c(w,\sigma,\zeta)(\Phi)\supseteq\mathscr{B}^c(\tau,1-\sigma_3,\zeta-\zeta_0)(\Phi)$. Thus, the proof stands established. □

DEFINITION 6. Define $\mathscr{T}^{\mathscr{I}_\lambda}(\mathscr{N}_n)=\{\mathscr{M}\subset\mathscr{C}^{\mathscr{I}_\lambda}(\mathscr{N}_n)$: for each $w\in\mathscr{M}$ one may find out $\zeta>0$ and $\sigma\in(0,1)$ such that $\mathscr{B}(w,\sigma,\zeta)\subset\mathscr{M}\}$.

REMARK 2. $\mathscr{T}^{\mathscr{I}_\lambda}(\mathscr{N}_n)$ is a topology on $\mathscr{C}^{\mathscr{I}_\lambda}(\mathscr{N}_n)$.

PROPOSITION 1. *The spaces $\mathscr{C}^{\mathscr{I}_\lambda}(\mathscr{N}_n)$ and $\mathscr{C}_0^{\mathscr{I}_\lambda}(\mathscr{N}_n)$ are Hausdorff.*

Proof. Suppose that $w=\{w_k\}$ and $\tau=\{\tau_k\}$ are in $\mathscr{C}^{\mathscr{I}_\lambda}(\mathscr{N}_n)$ such that $w\neq\tau$. Then for every $\zeta>0$, nonzero $\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1}\in\mathscr{W}$ and by the properties of $\mathscr{G},\Re$ and $\mathscr{Y}$, we get

$$\begin{cases}0<\dfrac{1}{\lambda_n}\displaystyle\sum_{k\in J_n}\Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\tau_k;\zeta)}{\rho}\right)<1,\\ 0<\dfrac{1}{\lambda_n}\displaystyle\sum_{k\in J_n}\Phi\left(\frac{\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\tau_k;\zeta)}{\rho}\right)<1,\\ 0<\dfrac{1}{\lambda_n}\displaystyle\sum_{k\in J_n}\Phi\left(\frac{\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\tau_k;\zeta)}{\rho}\right)<1.\end{cases}$$

Let

$$\begin{cases} \sigma_1 = \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \tau_k; \zeta)}{\rho}\right), \\ \sigma_2 = \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\Re(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \tau_k; \zeta)}{\rho}\right), \\ \sigma_3 = \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \tau_k; \zeta)}{\rho}\right), \\ \text{and } \sigma = \max\{\sigma_1, 1-\sigma_2, 1-\sigma_3\}. \end{cases}$$

Then, for each $\sigma_0 > \sigma$ there can be found σ_4, σ_5 and σ_6 in a way that $\sigma_4 \circledast \sigma_4 \geqslant \sigma_0$, $(1-\sigma_5) \circledast (1-\sigma_5) \leqslant (1-\sigma_0)$ and $(1-\sigma_6) \circledast (1-\sigma_6) \leqslant (1-\sigma_0)$. Now, put $\sigma_7 = \max\{\sigma_4, \sigma_5, \sigma_6\}$ and consider the open balls $\mathscr{B}\left(w, 1-\sigma_7, \frac{\zeta}{2}\right)$ and $\mathscr{B}\left(\tau, 1-\sigma_7, \frac{\zeta}{2}\right)$ centered at w and τ respectively.

We aim to show that $\mathscr{B}\left(w, 1-\sigma_7, \frac{\zeta}{2}\right) \cap \mathscr{B}\left(\tau, 1-\sigma_7, \frac{\zeta}{2}\right) = \emptyset$. If possible, let $\{\alpha_k\} \in \mathscr{B}\left(w, 1-\sigma_7, \frac{\zeta}{2}\right) \cap \mathscr{B}\left(\tau, 1-\sigma_7, \frac{\zeta}{2}\right)$. Then, it will happen

$$\begin{aligned} \sigma_1 &= \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \tau_k; \zeta)}{\rho}\right) \\ &\geqslant \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{G}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \alpha_k; \frac{\zeta}{2}\right) \circledast \mathscr{G}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \tau_k - \alpha_k; \frac{\zeta}{2}\right)}{\rho}\right) \\ &\geqslant \sigma_7 \circledast \sigma_7 \geqslant \sigma_4 \circledast \sigma_4 > \sigma_0 > \sigma > \sigma_1, \text{ absurd,} \end{aligned}$$

$$\begin{aligned} \sigma_2 &= \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\Re(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \tau_k; \zeta)}{\rho}\right) \\ &\leqslant \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\Re\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \alpha_k; \frac{\zeta}{2}\right) \circledast \Re\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \tau_k - \alpha_k; \frac{\zeta}{2}\right)}{\rho}\right) \\ &\leqslant (1-\sigma_7) \circledast (1-\sigma_7) \leqslant (1-\sigma_5) \circledast (1-\sigma_5) \\ &< (1-\sigma_0) < (1-\sigma) < \sigma_2, \text{ a contradiction,} \end{aligned}$$

and

$$\begin{aligned} \sigma_3 &= \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \tau_k; \zeta)}{\rho}\right) \\ &\leqslant \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{Y}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \alpha_k; \frac{\zeta}{2}\right) \circledast \mathscr{Y}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \tau_k - \alpha_k; \frac{\zeta}{2}\right)}{\rho}\right) \\ &\leqslant (1-\sigma_7) \circledast (1-\sigma_7) \leqslant (1-\sigma_6) \circledast (1-\sigma_6) \\ &< (1-\sigma_0) < (1-\sigma) < \sigma_3, \text{ a contradiction.} \end{aligned}$$

Hence the space $\mathscr{C}^{\mathscr{I}_\lambda}(\mathscr{N}_n)$ is Hausdorff. By similar argument, we can do the same for the space $\mathscr{C}_0^{\mathscr{I}_\lambda}(\mathscr{N}_n)$. Thus, the proof stands established. □

THEOREM 7. *For an element* $\{w_k\} \in \mathscr{C}^{\mathscr{I}_\lambda}(\mathscr{N}_n)$, *converging to* υ *if and only if, for* $\sigma \in (0,1)$, $\zeta > 0$ *and nonzero* $\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1} \in \mathscr{W}$,

$$\begin{cases} \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta)}{\rho}\right) \to 1 \quad \textit{or} \\ \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\Re(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta)}{\rho}\right) \to 0, \quad \textit{and} \\ \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta)}{\rho}\right) \to 0, \end{cases}$$

as $k \to \infty$.

Proof. Suppose that $w_k \xrightarrow{\mathscr{N}_n} \upsilon$. Let $\sigma \in (0,1)$ be given. Then for $\zeta > 0$ there is a number $n_0 \in \mathbb{N}$ such that $w \in \mathscr{B}(\upsilon, \sigma, \zeta)$ for all $k \geqslant n_0$. Building upon this, we arrive at

$$\begin{cases} 1 - \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta)}{\rho}\right) < \sigma \quad \text{or} \\ \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\Re(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta)}{\rho}\right) < \sigma, \quad \text{and} \\ \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta)}{\rho}\right) < \sigma. \end{cases}$$

Hence,

$$\begin{cases} \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta)}{\rho}\right) \to 1 \quad \text{or} \\ \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\Re(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta)}{\rho}\right) \to 0, \quad \text{and} \\ \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta)}{\rho}\right) \to 0, \end{cases}$$

as $k \to \infty$. We can easily prove the converse part of the Theorem. So, we omit details. □

We now present the novel sequence spaces $\ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi)$ and $\ell_\infty^{(\mathscr{N}_n)}(\Phi)$, defined as follows:

$$\ell_\infty^{(\mathscr{N}_n)}(\Phi) = \left\{ \{w_k\} \in \mathscr{W} : \exists\, \zeta_0, \sigma \in (0,1) : \forall k \in \mathbb{N} : \begin{array}{l} \Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k;\zeta_0)}{\rho}\right) > 1-\sigma \text{ and} \\ \Phi\left(\frac{\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k;\zeta_0)}{\rho}\right) < \sigma, \\ \Phi\left(\frac{\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k;\zeta_0)}{\rho}\right) < \sigma \end{array} \right\}.$$

$$\ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi) = \left\{ \{w_k\} \in \ell_\infty^{(\mathscr{N}_n)}(\Phi) : \left\{ n \in \mathbb{N} : \begin{array}{l} \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon;\zeta)}{\rho}\right) \leqslant 1-\sigma \text{ or} \\ \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon;\zeta)}{\rho}\right) \geqslant \sigma \text{ and} \\ \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon;\zeta)}{\rho}\right) \geqslant \sigma \end{array} \right\} \in \mathscr{I} \right\}.$$

THEOREM 8. *$\ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi)$ is a closed subset of $\ell_\infty^{(\mathscr{N}_n)}(\Phi)$.*

Proof. It is evident that the given space is a subspace of $\ell_\infty^{(\mathscr{N}_n)}(\Phi)$. We shall show that $\ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi)$ is closed, i.e., $\overline{\ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi)} = \ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi)$ where $\overline{\ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi)}$ denotes the closure of $\ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi)$. It is obvious that $\ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi) \subseteq \overline{\ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi)}$. For the reverse inclusion, let $w = \{w_k\} \in \overline{\ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi)}$. Then, $\mathscr{B}(w,\sigma,\zeta)(\Phi) \cap \ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi) \neq \emptyset$. So, let $\tau = \{\tau_k\} \in \mathscr{B}(w,\sigma,\zeta)(\Phi) \cap \ell_\infty^{\mathscr{I}\lambda(\mathscr{N}_n)}(\Phi)$. For a specified $\sigma \in (0,1)$, choose $\varpi \in (0,1)$ such that $(1-\varpi) \boxast (1-\varpi) > 1-\sigma$ and $\varpi \circledast \varpi < \sigma$. Since, $\tau \in \mathscr{B}(w,\sigma,\zeta)(\Phi)$, there exist a set $\mathscr{C} \in \mathscr{F}(\mathscr{I})$ such that for all $n \in \mathscr{C}$ the following inequalities are satisfied.

$$\begin{cases} \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\mathscr{G}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\tau_k;\frac{\zeta}{2}\right)}{\rho}\right) > 1-\varpi \quad \text{and} \\ \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\Re\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\tau_k;\frac{\zeta}{2}\right)}{\rho}\right) < \varpi, \\ \frac{1}{\lambda_n}\sum_{k\in J_n} \Phi\left(\frac{\mathscr{Y}\left(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\tau_k;\frac{\zeta}{2}\right)}{\rho}\right) < \varpi. \end{cases}$$

And, as $\tau \in \ell_\infty^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$, there is a set $\mathscr{D} \in \mathscr{F}(\mathscr{I})$ such that for all $n \in \mathbb{N}$ the following inequalities hold.

$$\begin{cases} \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \Phi\left(\dfrac{\mathscr{G}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \tau_k - \upsilon; \frac{\zeta}{2}\right)}{\rho}\right) > 1 - \varpi \quad \text{and} \\ \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \Phi\left(\dfrac{\mathfrak{R}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \tau_k - \upsilon; \frac{\zeta}{2}\right)}{\rho}\right) < \varpi, \\ \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in J_n} \Phi\left(\dfrac{\mathscr{Y}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \tau_k - \upsilon; \frac{\zeta}{2}\right)}{\rho}\right) < \varpi. \end{cases}$$

Thus, for all $n \in \mathscr{C} \cap \mathscr{D}$ we have

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{G}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta)}{\rho}\right) \\ &\geqslant \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{G}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \tau_k; \frac{\zeta}{2}\right) \boxast \mathscr{G}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \tau_k - \upsilon; \frac{\zeta}{2}\right)}{\rho}\right) \\ &> (1 - \varpi) \boxast (1 - \varpi) \\ &> 1 - \sigma, \end{aligned}$$

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathfrak{R}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta)}{\rho}\right) \\ &\leqslant \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathfrak{R}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \tau_k; \frac{\zeta}{2}\right) \circledast \mathfrak{R}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \tau_k - \upsilon; \frac{\zeta}{2}\right)}{\rho}\right) \\ &< \varpi \circledast \varpi \\ &< \sigma \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{Y}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \upsilon; \zeta)}{\rho}\right) \\ &\leqslant \frac{1}{\lambda_n} \sum_{k \in J_n} \Phi\left(\frac{\mathscr{Y}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, w_k - \tau_k; \frac{\zeta}{2}\right) \circledast \mathscr{Y}\left(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \tau_k - \upsilon; \frac{\zeta}{2}\right)}{\rho}\right) \\ &< \varpi \circledast \varpi \\ &< \sigma. \end{aligned}$$

Therefore,

$$\mathscr{C}\cap\mathscr{D} \subseteq \left\{ n\in\mathbb{N} : \begin{array}{l} \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{G}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon;\zeta)}{\rho}\right) > 1-\sigma \text{ and} \\ \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\Re(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon;\zeta)}{\rho}\right) < \sigma, \\ \frac{1}{\lambda_n}\sum_{k\in J_n}\Phi\left(\frac{\mathscr{Y}(\vartheta_1,\vartheta_2,\ldots,\vartheta_{n-1},w_k-\upsilon;\zeta)}{\rho}\right) < \sigma \end{array} \right\} \in \mathscr{F}(\mathscr{I}).$$

Thus, $\{w_k\} \in \ell_\infty^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$. Hence $\ell_\infty^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi) \supseteq \overline{\ell_\infty^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)}$, i.e., $\overline{\ell_\infty^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)} = \ell_\infty^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$. Thus, the proof stands established. □

Conclusion

In this paper, we have introduced and explored the innovative concepts of the newly introduced spaces $\mathscr{C}^{\mathscr{I}_\lambda}(\mathscr{N}_n)$, $\mathscr{C}_0^{\mathscr{I}_\lambda}(\mathscr{N}_n)$ using $\mathscr{I}_\lambda$-convergence and Orlicz function within summability theory. Through a detailed examination, we have uncovered topological properties of them including Hausdorffness. Furthermore, we introduced the novel sequence spaces $\ell_\infty^{(\mathscr{N}_n)}(\Phi)$ and $\ell_\infty^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$, and established that $\ell_\infty^{\mathscr{I}_\lambda(\mathscr{N}_n)}(\Phi)$ forms a closed subset of $\ell_\infty^{(\mathscr{N}_n)}(\Phi)$.

Applications

Orlicz spaces ℓ_Φ, where Φ is an Orlicz function, serve as a significant generalization of classical l_p-spaces and have been extensively studied in mathematical literature (see, e.g., [26, 47]). Equipped with appropriate norms, these spaces form Banach spaces and provide a natural framework for extending the analysis of various function spaces. Notably, Orlicz spaces are closely connected to Hardy-Littlewood maximal functions and appear as subspaces in certain Sobolev spaces. Furthermore, their structural and functional relationships with l_p-spaces have been thoroughly explored in [5]. Orlicz spaces have found wide ranging applications across several branches of mathematics, mathematical physics, and statistics. Foundational concepts and results can be found in [48]. In recent years, a growing number of research papers have highlighted the versatility and applicability of Orlicz spaces in both pure and applied mathematical sciences. For detailed discussions and applications, see [2, 7, 9, 23, 24]. This concept has also been effectively utilized in summability theory to construct several novel and interesting sequence spaces, particularly through the use of ideals, natural density, and multiple sequences (see, e.g., [3, 6, 35, 49]. Research on sequence convergence in neutrosophic n-normed linear spaces is still in its early stages. In the future, it holds great potential for the development of novel sequence spaces by incorporating Orlicz functions in combination with tools such as lacunary sequences, multiple sequences, ideals, and natural density with respect to neutrosophic n-norm.

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