Journal of Classical Analysis
Volume 26, Number 2 (2025), 239–253 doi:10.7153/jca-2025-26-15

STATISTICAL CONVERGENCE OF BIVARIATE FUNCTIONS WITH RESPECT TO A FØLNER SEQUENCE

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Abstract. This study extends the notion of statistical convergence and its related concepts to bivariate functions defined on discrete countable amenable semigroups. We demonstrate that the space of bounded bivariate functions, $m(H\times H)$, forms a Banach space under the supremum norm, establishing a fundamental framework for our analysis. The paper rigorously investigates two-dimensional Følner sequences, statistical convergence, strong p-summability, and statistical Cauchy functions. Additionally, we characterize statistical limit and cluster points, proving the equivalence between statistical convergence and the statistical Cauchy property. Through illustrative examples, we emphasize the significance of nonthin subsets in the study of statistical limit and cluster points, thereby enriching the understanding of summability and convergence within amenable semigroups.

1. Introduction and background

In mathematical analysis, the study of convergence in various mathematical structures has attracted considerable attention, particularly within the framework of amenable semigroups. Let H be a discrete countable amenable semigroup possessing an identity element and satisfying both right and left cancellation laws. The Cartesian product $H\times H$ forms a discrete countable amenable semigroup under similar conditions.

We define $w(H\times H)$ as the space of all bivariate real-valued functions on $H\times H$, while $m(H\times H)$ denotes the space of all bounded bivariate real-valued functions on $H\times H$.

THEOREM 1. *The space $m(H\times H)$, consisting of all bounded functions $\phi : H\times H\to\mathbb{R}$, is a Banach space when endowed with the supremum norm*

$$\|\phi\|_\infty = \sup\{|\phi(\nu,\omega)| : (\nu,\omega)\in H\times H\}.$$

Proof. To show that $m(H\times H)$ is a Banach space, we verify the following:

For any $\phi\in m(H\times H)$, the supremum norm is well-defined since ϕ is bounded. The properties of a norm (non-negativity, definiteness, homogeneity, and the triangle inequality) are straightforward to check for $\|\cdot\|_\infty$.

Mathematics subject classification (2020): 40A05, 40C05, 40D05.
Keywords and phrases: Følner sequence, amenable group, statistical convergence, bivariate function.
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Paper JCA-26-15

Let (ϕ_n) be a Cauchy sequence in $m(H\times H)$. For every $\varepsilon>0$, there exists N such that for all $m,n\geqslant N$,

$$\|\phi_n-\phi_m\|_\infty<\varepsilon.$$

This ensures that $\{\phi_n(v,\omega)\}$ forms a Cauchy sequence in $\mathbb{R}$ for each $(v,\omega)\in H\times H$. Since $\mathbb{R}$ is a complete space, it follows that $\phi_n(v,\omega)\to\phi(v,\omega)$ pointwise. Define the limit function as

$$\phi(v,\omega)=\lim_{n\to\infty}\phi_n(v,\omega).$$

Since ϕ_n are uniformly bounded, $\phi\in m(H\times H)$, and

$$\|\phi_n-\phi\|_\infty\to 0\quad\text{as } n\to\infty.$$

Thus, $m(H\times H)$ is complete. □

The concept of a Følner sequence is a cornerstone in the study of amenable groups. For $H\times H$, a countable amenable group, a two-dimensional Følner sequence is a sequence of finite subsets $\{F_n\times F_n\}$ satisfying the following properties:

(1) $H\times H=\bigcup_{n=1}^{\infty}(F_n\times F_n)$, ensuring that the entire semigroup is covered by the sequence.

(2) $F_n\times F_n\subseteq F_{n+1}\times F_{n+1}$ for all $n\in\mathbb{N}$, which implies that the sequence is nested.

(3) For all $(v,\omega)\in H\times H$,

$$\lim_{n\to\infty}\frac{|(F_n\times F_n)(v,\omega)\cap(F_n\times F_n)|}{|F_n\times F_n|}=1$$

and

$$\lim_{n\to\infty}\frac{|(v,\omega)(F_n\times F_n)\cap(F_n\times F_n)|}{|F_n\times F_n|}=1.$$

Here, $|A|$ denotes the cardinality of a finite set A. Any sequence satisfying these conditions is termed a two-dimensional Følner sequence for $H\times H$.

A canonical example of a two-dimensional Følner sequence in $\mathbb{N}\times\mathbb{N}$ is

$$F_n\times F_n=\{(i,j):i,j\in\{0,1,\dots,n-1\}\}.$$

Similarly, in the group $\mathbb{Z}\times\mathbb{Z}$, a widely used example is the sequence of double intervals

$$F_n\times F_n=\{a_n,a_n+1,\dots,a_n+n-1\}\times\{a_n,a_n+1,\dots,a_n+n-1\},$$

where (a_n) is a sequence in $\mathbb{Z}$.

The notion of statistical convergence, introduced by Fast [7], extends classical convergence by incorporating natural density. For a subset $S\subseteq\mathbb{N}$, the natural density is given by

$$\delta(S)=\lim_{n\to\infty}\frac{1}{n}|\{i\leqslant n:i\in S\}|,$$

where $|\{i \leqslant n : i \in S\}|$ represents the count of elements in K not exceeding n. A sequence (ξ_i) is said to be *statistically convergent* to ξ if, for every $\varepsilon > 0$,

$$\delta(\{i : |\xi_i - \xi| \geqslant \varepsilon\}) = 0.$$

This definition extends classical convergence by allowing exceptions on a subset of indices with zero natural density. An illustrative example is the sequence $\xi_i = i$ when i is a perfect square and $\xi_i = 0$ otherwise, which is statistically convergent to 0.

The concept of statistical convergence has been generalized in various contexts. Mursaleen and Edely [15] extended it to double sequences, while Moricz [13] introduced its application to multiple sequences. The study of summability in amenable semigroups was explored in [11, 12], where Douglas extended the arithmetic mean to amenable semigroups, providing insights into almost convergence [4]. Further developments include rough convergence [6], ideal limit points [19], and recurrent points with positive densities [1]. The work in [17] extended statistical convergence to functions on discrete countable amenable semigroups.

This paper builds upon these foundations by developing the theory of statistical convergence for bivariate functions on $H \times H$. We introduce statistical convergence and strong p-summability in this setting, along with statistical limit and statistical cluster points, offering a deeper understanding of their behavior in the multidimensional framework.

2. Statistical convergence

A double sequence of real numbers is a function

$$x : \mathbb{N} \times \mathbb{N} \to \mathbb{R}.$$

We denote a double sequence by $(x(n,m))$ or simply $(x_{n,m})$. Unlike ordinary sequences, which are indexed by a single variable, double sequences exhibit unique structural properties due to their two-dimensional indexing.

Similarly, a double sequence on $H \times H$ is a bivariate function $\phi : H \times H \to \mathbb{R}$, where H is a discrete countable amenable semigroup. We denote such functions by $\phi(v, \omega)$, emphasizing their dependence on two independent variables $v, \omega \in H$. From this point forward, $H \times H$ will represent a countable amenable semigroup with an identity element, satisfying both right and left cancellation laws.

DEFINITION 1. A function $\phi \in w(H \times H)$ is said to *converge to* s with respect to a given two-dimensional Følner sequence $\{F_n \times F_n\}$ in $H \times H$ if, for every $\varepsilon > 0$, there exists an index $k_0 \in \mathbb{N}$ such that

$$|\phi(v, \omega) - s| < \varepsilon$$

for all $n > k_0$ and for any $(v, \omega) \in (H \times H) \setminus (F_n \times F_n)$.

This notion of convergence utilizes the Følner sequence to examine the behavior of ϕ outside the finite subsets $F_n \times F_n$, ensuring that ϕ asymptotically approaches s in a uniform manner over the complement of these sets.

EXAMPLE 1. Let $H \times H = \mathbb{Z} \times \mathbb{Z}$ and consider the two-dimensional Følner sequence $\{F_n \times F_n\}$ defined as

$$F_n \times F_n = \{(\nu, \omega) : -n \leqslant \nu, \omega \leqslant n\}.$$

Define the function $\phi(\nu, \omega)$ as

$$\phi(\nu, \omega) = \frac{|\nu| + |\omega|}{\nu^2 + \omega^2 + 1}.$$

We show that $\phi(\nu, \omega)$ converges to 0 with respect to the given two-dimensional Følner sequence $\{F_n \times F_n\}$.

For any $\varepsilon > 0$, select $k_0 \in \mathbb{N}$ such that for all $n > k_0$ and $(\nu, \omega) \in (H \times H) \setminus (F_n \times F_n)$, the inequality

$$\frac{|\nu| + |\omega|}{\nu^2 + \omega^2 + 1} < \varepsilon$$

holds.

Since outside $F_n \times F_n$, either $|\nu| > n$ or $|\omega| > n$, it follows that

$$\nu^2 + \omega^2 \geqslant n^2 \quad \text{and} \quad |\nu| + |\omega| \leqslant 2\sqrt{\nu^2 + \omega^2}.$$

Thus, we obtain

$$\frac{|\nu| + |\omega|}{\nu^2 + \omega^2 + 1} \leqslant \frac{2\sqrt{\nu^2 + \omega^2}}{\nu^2 + \omega^2 + 1}.$$

Since the denominator $\nu^2 + \omega^2 + 1$ grows faster than the numerator $2\sqrt{\nu^2 + \omega^2}$ as $n \to \infty$, we conclude

$$\frac{|\nu| + |\omega|}{\nu^2 + \omega^2 + 1} \to 0.$$

For sufficiently large $n > k_0$, we ensure that

$$\frac{|\nu| + |\omega|}{\nu^2 + \omega^2 + 1} < \varepsilon.$$

Hence, the function $\phi(\nu, \omega) = \frac{|\nu| + |\omega|}{\nu^2 + \omega^2 + 1}$ converges to $s = 0$ with respect to the two-dimensional Følner sequence $\{F_n \times F_n\}$ on $H \times H = \mathbb{Z} \times \mathbb{Z}$.

DEFINITION 2. A bivariate function $\phi(\nu, \omega) \in w(H \times H)$ is said to be *regularly convergent* if it is convergent and, in addition, the following limits exist

$$\lim_{\omega \to \infty} \phi(\nu, \omega) \quad \text{and} \quad \lim_{\nu \to \infty} \phi(\nu, \omega).$$

Regular convergence of $\phi(\nu, \omega)$ implies convergence, but the converse does not necessarily hold. One key limitation of standard convergence is that a convergent bivariate function $\phi(\nu, \omega)$ may not be bounded in general. However, the notion of regular convergence overcomes this drawback, ensuring that the function remains well-behaved in terms of boundedness.

DEFINITION 3. A function $\phi \in w(H \times H)$ is called a *Cauchy sequence* with respect to a given two-dimensional Følner sequence $\{F_n \times F_n\}$ in $H \times H$ if, for every $\varepsilon > 0$, there exists an index $k_0 \in \mathbb{N}$ such that

$$|\phi(v,\omega) - \phi(p,q)| < \varepsilon$$

for all $n > k_0$ and for any pairs $(v,\omega),(p,q) \in (H \times H) \setminus (F_n \times F_n)$.

This definition ensures that the function $\phi(v,\omega)$ exhibits Cauchy behavior outside the finite sets determined by the Følner sequence, contributing to a well-structured notion of convergence in the semigroup setting.

EXAMPLE 2. Consider the discrete countable amenable semigroup $H \times H = \mathbb{N} \times \mathbb{N}$, and define the two-dimensional Følner sequence $\{F_n \times F_n\}$ as

$$F_n \times F_n = \{(v,\omega) \mid 0 \leqslant v,\omega < n\}.$$

Let the function $\phi : H \times H \to \mathbb{R}$ be given by

$$\phi(v,\omega) = \begin{cases} \frac{1}{v+\omega+1}, & \text{if } v,\omega \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

We now verify whether $\phi(v,\omega)$ satisfies the Cauchy criterion with respect to the given Følner sequence. For any $\varepsilon > 0$, select $k_0 \in \mathbb{N}$ such that for all $n > k_0$, the following holds for any pairs (v,ω) and (p,q) from the complement of $F_n \times F_n$

$$(v,\omega),(p,q) \in (H \times H) \setminus (F_n \times F_n).$$

The absolute difference between function values at these points satisfies

$$\left|\phi(v,\omega) - \phi(p,q)\right| = \left|\frac{1}{v+\omega+1} - \frac{1}{p+q+1}\right|.$$

Since both denominators increase unboundedly as $v,\omega,p,q \to \infty$, it follows that

$$\frac{1}{v+\omega+1} \to 0 \quad \text{and} \quad \frac{1}{p+q+1} \to 0.$$

For sufficiently large $n > k_0$, we conclude that

$$|\phi(v,\omega) - \phi(p,q)| < \varepsilon.$$

Thus, the function $\phi(v,\omega)$ is a Cauchy sequence under the given two-dimensional Følner sequence.

DEFINITION 4. A function $\phi \in w(H \times H)$ is said to be *Cesàro summable* to s with respect to a given two-dimensional Følner sequence $\{F_n \times F_n\}$, if the following limit holds

$$\lim_{n\to\infty} \frac{1}{|F_n \times F_n|} \sum_{(\nu,\omega)\in F_n\times F_n} \phi(\nu,\omega) = s.$$

This definition ensures that the average value of $\phi(\nu,\omega)$ over the Følner sequence $\{F_n \times F_n\}$ asymptotically approaches s as $n \to \infty$.

EXAMPLE 3. Let $H \times H = \mathbb{N} \times \mathbb{N}$ and the two-dimensional Følner sequence $\{F_n \times F_n\}$ be defined as

$$F_n \times F_n = \{(v,w) : 0 \leqslant v,w < n\}.$$

Define the function $\phi(v,w)$ as

$$\phi(v,w) = 1 + \frac{1}{(v+1)(w+1)}.$$

We investigate whether $\phi(v,w)$ is Cesàro summable. The Cesàro mean is given by

$$\frac{1}{|F_n \times F_n|} \sum_{(v,w)\in F_n\times F_n} \phi(v,w),$$

where $|F_n \times F_n| = n^2$, since $F_n \times F_n$ is a grid of size $n \times n$. The function $\phi(v,w)$ consists of two terms:

- A constant term 1,
- A variable term $\frac{1}{(v+1)(w+1)}$.

Thus, the summation can be written as

$$\frac{1}{n^2} \sum_{(v,w)\in F_n\times F_n} \phi(v,w) = \frac{1}{n^2} \sum_{(v,w)\in F_n\times F_n} 1 + \frac{1}{n^2} \sum_{(v,w)\in F_n\times F_n} \frac{1}{(v+1)(w+1)}.$$

For the first term, we compute

$$\frac{1}{n^2} \sum_{(v,w)\in F_n\times F_n} 1 = 1.$$

For the second term, using a double integral approximation, we obtain

$$S_n = \sum_{(v,w)\in F_n\times F_n} \frac{1}{(v+1)(w+1)} \approx (\ln(n+1))^2.$$

Thus, we have

$$\frac{1}{n^2} S_n = \frac{(\ln(n+1))^2}{n^2}.$$

Taking the limit:

$$\lim_{n\to\infty} \frac{(\ln(n+1))^2}{n^2} = 0.$$

Therefore, the Cesàro mean simplifies to

$$\frac{1}{|F_n \times F_n|} \sum_{(v,w)\in F_n\times F_n} \phi(v,w) \to 1 \quad \text{as } n\to\infty.$$

This confirms that the function $\phi(v,w) = 1 + \frac{1}{(v+1)(w+1)}$ is Cesàro summable to $s=1$ for the given two-dimensional Følner sequence $\{F_n \times F_n\}$.

DEFINITION 5. A function $\phi \in w(H\times H)$ is said to be strongly Cesàro summable to s with respect to a given two-dimensional Følner sequence $\{F_n \times F_n\}$ for $H\times H$, if

$$\lim_{n\to\infty} \frac{1}{|F_n \times F_n|} \sum_{(\nu,\omega)\in F_n\times F_n} |\phi(\nu,\omega) - s| = 0.$$

EXAMPLE 4. Let $H\times H = \mathbb{N}\times\mathbb{N}$ and the two-dimensional Følner sequence $\{F_n \times F_n\}$ be defined as

$$F_n \times F_n = \{(\nu,\omega) : 0 \leqslant \nu,\omega < n\}.$$

Define the function $\phi(\nu,\omega)$ as:

$$\phi(\nu,\omega) = \begin{cases} \omega, & \text{if } \nu = 1, \\ \nu, & \text{if } \omega = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We show that $\phi(\nu,\omega)$ is strongly Cesàro summable. For this, we calculate the Cesàro mean

$$\frac{1}{|F_n \times F_n|} \sum_{(\nu,\omega)\in F_n\times F_n} |\phi(\nu,\omega) - s|,$$

where $|F_n \times F_n| = n^2$ and s is the Cesàro sum.

The function $\phi(\nu,\omega)$ is nonzero only for

- $\nu = 1$: In this case, $\phi(1,\omega) = \omega$ for $\omega = 0,1,\dots,n-1$. There are n elements, and their sum is
$$\sum_{\omega=0}^{n-1} \omega = \frac{(n-1)n}{2}.$$

- $\omega = 1$: In this case, $\phi(\nu,1) = \nu$ for $\nu = 0,1,\dots,n-1$, excluding $\nu = 1$. There are $n-1$ elements, and their sum is
$$\sum_{\nu=0,\nu\neq 1}^{n-1} \nu = \frac{(n-1)n}{2} - 1.$$

The total sum of $\phi(\nu,\omega)$ is

$$\sum_{(\nu,\omega)\in F_n\times F_n}\phi(\nu,\omega)=\frac{(n-1)n}{2}+\left(\frac{(n-1)n}{2}-1\right)=(n-1)n-1.$$

The Cesàro mean is calculated as

$$\frac{1}{|F_n\times F_n|}\sum_{(\nu,\omega)\in F_n\times F_n}\phi(\nu,\omega)=\frac{(n-1)n-1}{n^2}.$$

Taking the limit as $n\to\infty$, we find

$$\lim_{n\to\infty}\frac{(n-1)n-1}{n^2}=\lim_{n\to\infty}\left(1-\frac{1}{n}-\frac{1}{n^2}\right)=1.$$

Thus, $\phi(\nu,\omega)$ is strongly Cesàro summable with Cesàro sum $s=1$.

For any two-dimensional Følner sequence $\{F_n\times F_n\}$ for $H\times H$, the two-dimensional upper and lower Følner densities of the set $F\times F\subset H\times H$ are defined by

$$\overline{\delta}(F\times F)=\limsup_{n\to\infty}\frac{1}{|F_n\times F_n|}|\{(\nu,\omega)\in F_n\times F_n:(\nu,\omega)\in F\times F\}|$$

and

$$\underline{\delta}(F\times F)=\liminf_{n\to\infty}\frac{1}{|F_n\times F_n|}|\{(\nu,\omega)\in F_n\times F_n:(\nu,\omega)\in F\times F\}|.$$

In case the above limits exist and $\overline{\delta}(F\times F)=\underline{\delta}(F\times F)$, we say that $F\times F$ has Følner density, and we denote it by

$$\delta(F\times F)=\lim_{n\to\infty}\frac{1}{|F_n\times F_n|}|\{(\nu,\omega)\in F_n\times F_n:(\nu,\omega)\in F\times F\}|.$$

We shall be particularly concerned with sets having two-dimensional Følner density zero.

If ϕ is a function such that $\phi(\nu,\omega)$ satisfies property P for all (ν,ω) except on a set of two-dimensional Følner density zero, then we say that $\phi(\nu,\omega)$ satisfies P for "almost all (ν,ω)", and we abbreviate this by "a.a. (ν,ω)".

DEFINITION 6. A function $\phi\in w(H\times H)$ is said to be statistically convergent to s with respect to a given two-dimensional Følner sequence $\{F_n\times F_n\}$ for $H\times H$, if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{|F_n\times F_n|}|\{(\nu,\omega)\in F_n\times F_n:|\phi(\nu,\omega)-s|\geqslant\varepsilon\}|=0;$$

that is, $|\phi(\nu,\omega)-s|<\varepsilon$ for almost all (ν,ω) (denoted $a.a.(\nu,\omega)$). In this case, we write $st-\lim\phi(\nu,\omega)=s$.

The set of all statistically convergent bivariate functions will be denoted by $F(H\times H)$.

If $\phi(\nu,\omega)$ is a convergent bivariate function, then it is also statistically convergent to the same limit. Moreover, if $\phi(\nu,\omega)$ is statistically convergent to s, then s is determined uniquely.

However, if $\phi(\nu,\omega)$ is statistically convergent, it does not necessarily imply that $\phi(\nu,\omega)$ is convergent or bounded.

EXAMPLE 5. Let $H\times H=\mathbb{N}\times\mathbb{N}$ and $\{F_n\times F_n\}$ be defined as

$$F_n\times F_n=\{(\nu,\omega):0\leqslant \nu,\omega<n\}.$$

Define the function $\phi(\nu,\omega)$ as

$$\phi(\nu,\omega)=\begin{cases}\nu\omega, & \text{if } \nu,\omega \text{ are perfect squares,}\\ 1, & \text{otherwise.}\end{cases}$$

We analyze the statistical convergence of $\phi(\nu,\omega)$ over $F_n\times F_n$. The proportion of points $(\nu,\omega)\in F_n\times F_n$ where $|\phi(\nu,\omega)-1|\geqslant\varepsilon$ is determined by the number of pairs (ν,ω) where both ν and ω are perfect squares. The number of such points is approximately $\sqrt{n}\times\sqrt{n}=n$, while the total number of points in $F_n\times F_n$ is n^2. Thus, the proportion is:

$$\frac{1}{|F_n\times F_n|}|\{(\nu,\omega)\in F_n\times F_n:|\phi(\nu,\omega)-1|\geqslant\varepsilon\}|=\frac{n}{n^2}=\frac{1}{n}.$$

As $n\to\infty$, this proportion tends to 0. Therefore, $\phi(\nu,\omega)$ is statistically convergent to 1.

However, $\phi(\nu,\omega)$ is neither pointwise convergent nor bounded, as $\nu\omega$ can grow arbitrarily large for sufficiently large ν and ω.

DEFINITION 7. A function $\phi\in w(H\times H)$ is said to be regularly statistically convergent to s with respect to a given two-dimensional Følner sequence $\{F_n\times F_n\}$ for $H\times H$, if the following conditions are satisfied

(i) The bivariate function $\phi(\nu,\omega)$ is statistically convergent to s,

(ii) The single-variable function $\{\phi(\nu,\omega):\omega\in H\}$ is statistically convergent to s_ν for each fixed $\nu\in H\setminus F_1$,

(iii) The single-variable function $\{\phi(\nu,\omega):\nu\in H\}$ is statistically convergent to s_ω for each fixed $\omega\in H\setminus F_2$,

where F_1 and F_2 are subsets of H whose Følner density is zero.

DEFINITION 8. A function $\phi(\nu,\omega)\in w(H\times H)$ is said to be a statistically Cauchy function with respect to a given two-dimensional Følner sequence $\{F_n\times F_n\}$ for $H\times H$, if for each $\varepsilon>0$ and $k\geqslant 0$, there exists $(p,q)\in(H\times H)\setminus(F_k\times F_k)$ such that

$$\lim_{n\to\infty}\frac{1}{|F_n\times F_n|}|\{(\nu,\omega)\in F_n\times F_n:|\phi(\nu,\omega)-\phi(p,q)|\geqslant\varepsilon\}|=0;$$

that is, $|\phi(\nu,\omega)-\phi(p,q)|<\varepsilon\quad$ for almost all (ν,ω) (denoted $a.a.(\nu,\omega)$).

THEOREM 2. *A function $\phi(\nu,\omega) \in w(H\times H)$ is statistically convergent with respect to a given two-dimensional Følner sequence $\{F_n \times F_n\}$ if and only if $\phi(\nu,\omega)$ is a statistically Cauchy function.*

Proof. Assume that $\phi(\nu,\omega)$ is statistically convergent to some limit s. Then, for every $\varepsilon > 0$, the following holds

$$\lim_{n\to\infty} \frac{1}{|F_n \times F_n|} |\{(\nu,\omega) \in F_n \times F_n : |\phi(\nu,\omega) - s| \geqslant \varepsilon\}| = 0.$$

Choosing any pair (p,q) where $|\phi(p,q) - s| \geqslant \varepsilon$, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{|F_n \times F_n|} |\{(\nu,\omega) \in F_n \times F_n : |\phi(\nu,\omega) - \phi(p,q)| \geqslant \varepsilon\}| \\ &\leqslant \lim_{n\to\infty} \frac{1}{|F_n \times F_n|} |\{(\nu,\omega) \in F_n \times F_n : |\phi(\nu,\omega) - s| \geqslant \varepsilon\}| \\ &\quad + \lim_{n\to\infty} \frac{1}{|F_n \times F_n|} |\{(p,q) \in F_n \times F_n : |\phi(p,q) - s| \geqslant \varepsilon\}| = 0. \end{aligned}$$

This establishes that $\phi(\nu,\omega)$ satisfies the statistical Cauchy property.

Conversely, suppose that $\phi(\nu,\omega)$ is a statistically Cauchy function but does not statistically converge. Then, there must exist some (p,q) such that

$$\lim_{n\to\infty} \frac{1}{|F_n \times F_n|} |\{(\nu,\omega) \in F_n \times F_n : |\phi(\nu,\omega) - \phi(p,q)| \geqslant \varepsilon\}| = 0.$$

From this, we conclude:

$$\lim_{n\to\infty} \frac{1}{|F_n \times F_n|} |\{(\nu,\omega) \in F_n \times F_n : |\phi(\nu,\omega) - \phi(p,q)| < \varepsilon\}| = 1.$$

Applying the inequality:

$$|\phi(\nu,\omega) - \phi(p,q)| \leqslant 2|\phi(\nu,\omega) - s| < \varepsilon, \tag{1}$$

whenever $|\phi(\nu,\omega) - s| < \frac{\varepsilon}{2}$, we reach a contradiction. Since $\phi(\nu,\omega)$ is assumed not to be statistically convergent, we obtain

$$\lim_{n\to\infty} \frac{1}{|F_n \times F_n|} |\{(\nu,\omega) \in F_n \times F_n : |\phi(\nu,\omega) - s| < \varepsilon\}| = 0.$$

Thus, using (1),

$$\lim_{n\to\infty} \frac{1}{|F_n \times F_n|} |\{(\nu,\omega) \in F_n \times F_n : |\phi(\nu,\omega) - \phi(p,q)| < \varepsilon\}| = 0,$$

which contradicts our earlier assumption that $\phi(\nu,\omega)$ is statistically Cauchy. Therefore, $\phi(\nu,\omega)$ must be statistically convergent. □

THEOREM 3. *A bivariate function $\phi(\nu,\omega)$ is statistically convergent to a limit s with respect to any Følner sequence $\{F_n \times F_n\}$ in $H \times H$ if and only if there exists a subset $U \subseteq H \times H$ such that $\delta(U) = 1$ and*

$$\lim_{(\nu,\omega)\in U} \phi(\nu,\omega) = s.$$

Proof. Assume that $\phi(\nu,\omega)$ is statistically convergent to s under a given two-dimensional Følner sequence $\{F_n \times F_n\}$. Define the sets

$$U_r = \left\{(\nu,\omega) \in H \times H : |\phi(\nu,\omega) - s| \geqslant \frac{1}{r}\right\}$$

$$V_r = \left\{(\nu,\omega) \in H \times H : |\phi(\nu,\omega) - s| < \frac{1}{r}\right\}, \quad r = 1,2,3,\ldots$$

Since $\phi(\nu,\omega)$ is statistically convergent to s, we have $\delta(U_r) = 0$ and the sequence of sets satisfies:

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq \ldots \supseteq V_r \supseteq V_{r+1} \supseteq \ldots$$

with $\delta(V_r) = 1$ for all $r \geqslant 1$. To establish pointwise convergence on V_r, assume for contradiction that $\phi(\nu,\omega)$ does not converge to s. Then, there exists $\varepsilon > 0$ such that

$$|\phi(\nu,\omega) - s| \geqslant \varepsilon$$

for infinitely many terms. Define

$$V_\varepsilon = \{(\nu,\omega) \in H \times H : |\phi(\nu,\omega) - s| < \varepsilon\}.$$

Since $\varepsilon > \frac{1}{r}$ for all r, it follows that $\delta(V_\varepsilon) = 0$. Since $V_r \subseteq V_\varepsilon$, we obtain $\delta(V_r) = 0$, contradicting $\delta(V_r) = 1$. Thus, $\phi(\nu,\omega)$ must converge to s on U.

Conversely, assume that there exists a subset $U \subseteq H \times H$ with $\delta(U) = 1$ and

$$\lim_{(\nu,\omega)\in U} \phi(\nu,\omega) = s.$$

This means that for any $\varepsilon > 0$, there exists an index $k_0 \in \mathbb{N}$ such that

$$|\phi(\nu,\omega) - s| < \varepsilon$$

for all $(\nu,\omega) \in U \setminus (F_n \times F_n)$ and $n > k_0$. Define

$$U_\varepsilon = \{(\nu,\omega) \in H \times H : |\phi(\nu,\omega) - s| \geqslant \varepsilon\}.$$

Since $U_\varepsilon \subseteq (H \times H) \setminus \{(\nu,\omega) \in (F_n \times F_n) : n > k_0\}$, it follows that

$$\delta(U_\varepsilon) \leqslant 1 - 1 = 0.$$

Thus, $\phi(\nu,\omega)$ is statistically convergent to s, completing the proof. □

THEOREM 4. *The following conditions are equivalent for a function $\phi \in w(H \times H)$*

(i) ϕ *is statistically convergent.*

(ii) ϕ *satisfies the statistical Cauchy criterion.*

(iii) There exists a function φ *such that* φ *is convergent and* $\phi(\nu,\nu) = \varphi(\nu,\omega)$ *for almost all* $(\nu,\omega) \in H \times H$.

The proof follows a similar approach to Theorem 7 in [17] and is thus omitted.

THEOREM 5. *Let* $\phi \in w(H \times H)$ *and let* p *be a positive real number. Then,*

(a) If ϕ *is strongly* p*-Cesàro summable to* s *with respect to a two-dimensional Følner sequence* $\{F_n \times F_n\}$*, then* ϕ *is statistically convergent to* s.

(b)

$$w_p(H \times H) = F(H \times H) \cap m(H \times H).$$

Proof. (a) Let $\phi \in w(H \times H)$ and suppose ϕ is strongly p-Cesàro summable to s. For any $\varepsilon > 0$, consider

$$\sum_{(\nu,\omega)\in F_n\times F_n} |\phi(\nu,\omega)-s|^p \geqslant |\{(\nu,\omega)\in F_n\times F_n : |\phi(\nu,\omega)-s| \geqslant \varepsilon\}|\cdot\varepsilon^p.$$

Since ϕ is strongly p-Cesàro summable, the right-hand side tends to zero as $n \to \infty$, which implies that ϕ is statistically convergent.

(b) Now, let $\phi \in m(H \times H)$ be statistically convergent to s for the Følner sequence $\{F_n \times F_n\}$. Since ϕ is bounded, define $M = \|\phi\|_\infty + s$. Given $\varepsilon > 0$, choose N_ε such that

$$\frac{1}{|F_n\times F_n|}\left|\left\{(\nu,\omega)\in F_n\times F_n : |\phi(\nu,\omega)-s| \geqslant \left(\frac{\varepsilon}{2}\right)^{1/p}\right\}\right| < \frac{\varepsilon}{2M^p}$$

for all $n > N_\varepsilon$. Define:

$$L_n = \left\{(\nu,\omega)\in F_n\times F_n : |\phi(\nu,\omega)-s| \geqslant \left(\frac{\varepsilon}{2}\right)^{1/p}\right\}.$$

Then we estimate

$$\frac{1}{|F_n\times F_n|}\sum_{(\nu,\omega)\in F_n\times F_n} |\phi(\nu,\omega)-s|^p$$

$$= \frac{1}{|F_n\times F_n|}\left(\sum_{(\nu,\omega)\in L_n} |\phi(\nu,\omega)-s|^p + \sum_{(\nu,\omega)\in (F_n\times F_n)\setminus L_n} |\phi(\nu,\omega)-s|^p\right).$$

Using the boundedness of ϕ, we obtain

$$< \frac{1}{|F_n\times F_n|}\cdot\frac{|F_n\times F_n|\varepsilon}{2M^p}M^p + \frac{1}{|F_n\times F_n|}\frac{\varepsilon}{2} = \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Thus, ϕ is strongly p-Cesàro summable to s under the given Følner sequence. □

3. Statistical limit and statistical cluster points

In [17], the definitions of statistical limit points and statistical cluster points were originally introduced for single-variable functions defined on discrete countable amenable semigroups. In this work, we generalize these notions to bivariate functions over the domain $H\times H$.

If $\phi\in w(H\times H)$ and $G\subset H$,we write $R_{\phi}(H\times H)$ to denote the range of $\phi\in w(H\times H)$. If $R_{\phi}(G\times G)\subset R_{\phi}(H\times H)$ and

$$\lim_{n\to\infty}\frac{|(G\times G)\cap(F_n\times F_n)|}{|F_n\times F_n|}=0,$$

then $R_{\phi}(G\times G)$ is called a subset of Følner density zero for any Følner sequence $\{F_n\times F_n\}$ for H, or a *thin* subset. Conversely, if

$$\lim_{n\to\infty}\frac{|(G\times G)\cap(F_n\times F_n)|}{|F_n\times F_n|}\neq 0,$$

then $R_{\phi}(G\times G)$ is a *nonthin* subset of $R_{\phi}(H\times H)$.

DEFINITION 9. A real number s is termed a *statistical limit point* of a function $\phi\in w(H\times H)$ with respect to a given Følner sequence $\{F_n\times F_n\}$ if there exists a nonthin subset of $R_{\phi}(H\times H)$ on which ϕ converges to s.

DEFINITION 10. A real number c is called a *statistical cluster point* of $\phi\in w(H\times H)$ if, for every $\varepsilon>0$, the set

$$\{(\nu,\omega)\in F_n\times F_n:|\phi(\nu,\omega)-c|<\varepsilon\}$$

is not a thin subset for any Følner sequence $\{F_n\times F_n\}$.

EXAMPLE 6. Let $H\times H=\mathbb{Z}\times\mathbb{Z}$ and $F_n\times F_n=[-n,n]\times[-n,n]$. Define the function ϕ as follows

$$\phi(\nu,\omega)=\begin{cases}0, & \text{if } \nu \text{ or } \omega \text{ is a prime number,}\\ 1, & \text{if } \nu \text{ or } \omega \text{ is even and not prime,}\\ 2, & \text{otherwise.}\end{cases}$$

Then, the statistical cluster points of ϕ are

- $c=1$, since the set $\{(\nu,\omega):\phi(\nu,\omega)=1\}$ does not have Følner density zero.
- $c=2$, since the set $\{(\nu,\omega):\phi(\nu,\omega)=2\}$ does not have Følner density zero.

However, $c=0$ is not a statistical cluster point, as the set $\{(\nu,\omega):\phi(\nu,\omega)=0\}$ has Følner density zero.

For $\phi \in w(H\times H)$, we define $L_\phi(H\times H)$ as the set of all ordinary limit points of ϕ. Similarly, the sets of all statistical limit points and statistical cluster points of ϕ are denoted by $\Lambda_\phi(H\times H)$ and $\Gamma_\phi(H\times H)$, respectively.

$$\Lambda_\phi(H\times H)\subseteq \Gamma_\phi(H\times H)\subseteq L_\phi(H\times H).$$

EXAMPLE 7. Take $H\times H=\mathbb{Z}\times\mathbb{Z}$, $R\times R=\{(0,0),(\pm1,\pm1),(\pm4,\pm4),(\pm9,\pm9),\ldots\}$, $F_n\times F_n=[-n,n]\times[-n,n]$, and define ϕ as follows

$$\phi(\nu,\omega)=\begin{cases}0, & \text{if } (\nu,\omega)\in R\times R,\\ 1, & \text{if } (\nu,\omega)\in (H\times H)\setminus (R\times R).\end{cases}$$

Then the set of all limit points of ϕ, denoted $L_\phi(H\times H)$, is $\{0,1\}$. The set of statistical limit points of ϕ, denoted $\Lambda_\phi(H\times H)$, is $\{1\}$, since $R\times R$ is a sparse subset of $H\times H$ and contributes only a statistically negligible portion of the values.

EXAMPLE 8. Take $H\times H=\mathbb{Z}\times\mathbb{Z}$, $R\times R=\{(k,k): k \text{ is a prime number}\}$, and $F_n\times F_n=[-n,n]\times[-n,n]$. Define the function ϕ as follows

$$\phi(\nu,\omega)=\begin{cases}0, & \text{if } (\nu,\omega)\in R\times R,\\ 2, & \text{if } \nu=0 \text{ or } \omega=0,\\ 1, & \text{otherwise.}\end{cases}$$

Then the set of all limit points of ϕ, denoted $L_\phi(H\times H)$, is $\{0,1,2\}$. The set of statistical limit points of ϕ, denoted $\Lambda_\phi(H\times H)$, is $\{1\}$, since $H\times H$ and the coordinate axes represent sparse subsets of $R\times R$ and contribute only statistically negligible values.

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(Received March 12, 2025)

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