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ON THE CONVERGENCE OF DOUBLE VILENKIN SERIES

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Abstract. In this paper, we proved regular convergence and convergence in L^r-norm $(0<r<1)$ of double orthogonal series with respect to any multiplicative (bounded Vilenkin) systems with coefficients of generalized bounded variation.

1. Introduction

We will study the convergence behavior of the double Vilenkin series of the form

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}\chi_j(x)\chi_k(y) \tag{1}$$

where $\{c_{jk}\}_{i,j=0}^{\infty}$ is a sequence of real (or complex) numbers and $\{\chi_i(x)\chi_j(y)\}_{i,j=0}^{\infty}$ is the Vilenkin orthonormal system. For that, first we present introductory material for multiplicative (bounded Vilenkin) systems as follows. Let $\mathbf{P}=\{p_j\}_{j=1}^{\infty}$ be a sequence of natural numbers such that $2\leqslant p_j\leqslant B$ for all $j\in\mathbb{N}$ and $\mathbb{Z}(p_j)=\{0,1,\dots,p_j-1\}$. If $m_0=1$, $m_j=p_1\cdots p_j$ for $j\in\mathbb{N}$, then every $x\in[0,1)$ admits a representation of the form

$$x=\sum_{j=1}^{\infty} x_j m_j^{-1},\ \ x_j\in\mathbb{Z}(p_j). \tag{2}$$

It is uniquely defined if for number of the type $x=\frac{k}{m_n}$, $0<k<m_n$, $k,n\in\mathbb{Z}_+:=\{0,1,2,3,\dots\}$, we choose representations, where only a finite number of $x_j\neq 0$. Let $G(\mathbf{P})$ be the group of sequences $\bar{x}=(x_1,x_2,\dots,x_n,\dots)$, $x_j\in\mathbb{Z}(p_j)$ with the operation $\bar{x}\oplus\bar{y}=\bar{z}$, where $z_j=x_j+y_j \pmod{p_j}$, $j\in\mathbb{N}$. In a similar way, we can define the inverse operation $\bar{x}\ominus\bar{y}=\bar{z}$. That is, $\bar{x}\ominus\bar{y}=\bar{w}$, where

$$w_j=x_j-y_j \pmod{p_j}=\begin{cases} x_j-y_j, & x_j\geqslant y_j,\\ p_j+x_j-y_j, & x_j<y_j.\end{cases}$$

Every $k\in\mathbb{Z}_+$ can be uniquely represented in the form

$$k=\sum_{j=1}^{\infty} k_j m_{j-1},\ \ k_j\in\mathbb{Z}(p_j), \tag{3}$$

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where the sum is finite. If $n,k \in \mathbb{Z}_+$ are represented in the form (3), then we define $n \oplus k := l = \sum_{j=1}^{\infty} l_j m_{j-1}$, where $l_j = n_j + k_j \pmod{p_j}$. We also define $n \ominus k$ analogously. That is, $n \ominus k = s$, where

$$s_j = n_j - k_j \pmod{p_j} = \begin{cases} n_j - k_j, & n_j \geqslant k_j, \\ p_j + n_j - k_j, & n_j < k_j. \end{cases}$$

For $x \in [0,1)$, written in the form (2), and $k \in \mathbb{Z}_+$ of the form (3) we put, by definition,

$$\chi_k(x) = \exp\left(2\pi i \sum_{j=1}^{\infty} \frac{x_j k_j}{p_j}\right).$$

The system $\{\chi_k(x)\}_{k=0}^{\infty}$ is said to be multiplicative; this fact is connected with the following properties (for the proofs see [4, p. 30]):

(1) $\chi_m(x)\chi_n(x) = \chi_{m\oplus n}(x)$ for all $m,n \in \mathbb{Z}_+$, $x \in [0,1)$;

(2) $\chi_n(x \oplus y) = \chi_n(x)\chi_n(y)$ for all $n \in \mathbb{Z}_+$ and for almost all $y \in [0,1)$ for fixed $x \in [0,1)$.

Analogous properties take place for $\chi_{n\ominus m}(x)$ and $\chi_n(x \ominus y)$. That is, $\chi_m(x)\overline{\chi_n(x)} = \chi_{m\ominus n}(x)$ and $\chi_n(x \ominus y) = \chi_n(x)\overline{\chi_n(y)}$ for $m,n \in \mathbb{Z}_+$ and $x,y \in [0,1)$. The system $\{\chi_k(x)\}_{k=0}^{\infty}$ is orthonormal and complete in $L^1[0,1)$ (see [4, p. 25]). Therefore, for $f \in L^1[0,1)$ we can define its Fourier series and Fourier coefficients according to the formulas

$$\sum_{j=0}^{\infty} \hat{f}(j)\chi_j(x), \quad \hat{f}(j) = \int_0^1 f(x)\overline{\chi_j(x)}dx, \ \ j \in \mathbb{Z}_+. \tag{4}$$

The k-th Fourier partial sum of the series (4) is defined as

$$S_k(f)(x) = \sum_{j=0}^{k} \hat{f}(j)\chi_j(x), \ \ k \in \mathbb{N}. \tag{5}$$

For more details about the system $\{\chi_k(x)\}_{k=0}^{\infty}$ see [4, Section 1.5]. As usual, the space $L^r[0,1)$, $1 \leqslant r < \infty$, is equipped with the norm $\|f\|_r = \left(\int_0^1 |f(t)|^r dt\right)^{\frac{1}{r}}$.

The sum $\sum_{k=0}^{n-1} \chi_k(x) := D_n(x)$ is called the nth Dirichlet kernel, and the sum $\frac{1}{n}\sum_{k=1}^{n} D_k(x) := F_n(x)$ is called the nth Fejér kernel with respect to the multiplicative system $\{\chi_k(x)\}$. For all $k \in \mathbb{N}$ and $x \in (0,1)$, it is well-known (see [1, Chap. 4, Sec. 3] and [10, Lemma 4] or [6]) that

$$|D_k(x)| \leqslant \frac{B}{x}, \tag{6}$$

$$|kF_k(x)| \leqslant \frac{C}{x^2}, \tag{7}$$

where N is such that $p_i \leqslant B$ for all $i \in \mathbb{N}$, and C.

The system $\{\chi_i(x)\chi_j(y)\}_{i,j=0}^{\infty}$ is orthonormal and complete in $L[0,1)^2$, which allows to define for $f \in L[0,1)^2$ the Fourier coefficients

$$\hat{f}(i,j) = \int_0^1 \int_0^1 f(x,y)\overline{\chi_i(x)\chi_j(y)}dxdy, \quad i,j \in \mathbb{Z}_+, \tag{8}$$

and the partial Fourier sums

$$s_{mn}(f;x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \hat{f}(i,j)\chi_i(x)\chi_j(x), \quad m,n \in \mathbb{N}. \tag{9}$$

The space $L^r[0,1)^2$ is equipped with the norm

$$\|f\|_r = \left(\int_0^1 \int_0^1 |f(x,y)|^r dxdy\right)^{1/r}.$$

For $m,n \geqslant 0$, the rectangular partial sums of series (1) is defined as

$$S_{mn}(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} c_{jk}\chi_j(x)\chi_k(y).$$

Following Hardy [5], series (1) is said to be regular convergent if it converges in Pringsheim's sense (see, e.g. [11, Vol. 2, Ch. 17]), and, in addition, each "row series" of (1) (i.e., when we delete $\sum_{k=0}^{\infty}$ in (1) and the summation is done only with respect to j for each k) as well as each "column series" converges in the ordinary sense of convergence of single series. F. Móricz [9] rediscovered the notion of regular convergence by the following equivalent condition: the sum

$$S(R;x,y) = \sum_{j=m}^{M}\sum_{k=n}^{N} c_{jk}\chi_j(x)\chi_k(y) \tag{10}$$

tends to zero as $\max(m,n) \to \infty$, independently of the choices of $M(\geqslant m)$ and $N(\geqslant n)$, where $R = \{(j,k) : m \leqslant j \leqslant M \text{ and } n \leqslant k \leqslant N\}$.

DEFINITION 1. A double sequence $\{c_{jk}\}_{i,j=0}^{\infty}$ of complex numbers is called a null sequence if it satisfies

$$c_{jk} = o(1) \quad \text{as} \quad \max\{j,k\} \to \infty. \tag{11}$$

For positive integers p and q, the finite order differences $\Delta_{pq}c_{jk}$ are defined by

$$\Delta_{00}c_{jk} = c_{jk};$$
$$\Delta_{pq}c_{jk} = \Delta_{p-1,q}c_{jk} - \Delta_{p-1,q}c_{j+1,k} \quad (p \geqslant 1);$$
$$\Delta_{pq}c_{jk} = \Delta_{p,q-1}c_{jk} - \Delta_{p,q-1}c_{j,k+1} \quad (q \geqslant 1).$$

DEFINITION 2. A double null sequence $\{c_{jk}\}_{i,j=0}^{\infty}$ is said to be of bounded variation if

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{11}c_{jk}|<\infty. \tag{12}$$

DEFINITION 3. A double sequence $\{c_{jk}\}_{i,j=0}^{\infty}$ is said to be of bounded variation of order $p \geqslant 2$, if the following three conditions are satisfied:

$$\lim_{k\to\infty}\sum_{j=0}^{\infty}|\Delta_{p0}c_{jk}|=0, \tag{13}$$

$$\lim_{j\to\infty}\sum_{k=0}^{\infty}|\Delta_{0p}c_{jk}|=0, \tag{14}$$

and

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{pp}c_{jk}|<\infty. \tag{15}$$

Some authors (see, e.g., [2], [7]) call conditions (13)–(15) as conditions of bounded variation of order $(p,0)$, $(0,p)$, and (p,p), respectively.

If a double sequence $\{c_{jk}\}_{i,j=0}^{\infty}$ is of bounded variation of order p, then it is of bounded variation of order q $(p \leqslant q)$. But the converse is not true in general, that is, there is a double sequence which is of bounded variation of order 2, but not of bounded variation (see, e.g., [3]).

2. Results

Our first main result is an analogue of a result of Móricz [8, Theorem], for any multiplicative orthogonal series.

THEOREM 1. *Let a double sequence $\{c_{jk}\}_{i,j=0}^{\infty}$ satisfies the conditions* (11) *and* (12). *Then the series* (1)

(i) converges regularly to some function $f(x,y)$ for all $x,y \in (0,1)$;

(ii) converges in the $L^r(0,1)^2$-metric to f for all $0<r<1$.

If we take $p_j = 2$ for each $j \in \mathbb{N}$, the Vilenkin system reduces to the Walsh system, so Móricz's result [9, Theorem 1] follows from our Theorem 1. Our second result is an analogue of a result of Móricz [9, Theorem 2], for any multiplicative orthogonal series.

THEOREM 2. *Let a double sequence $\{c_{jk}\}_{i,j=0}^{\infty}$ satisfies the condition* (11) *and for $p=2$,* (15),

$$\sum_{j=0}^{\infty}|\Delta_{20}c_{jk}| \quad \textit{is finite for each } k \textit{ and tends to } 0 \textit{ as } k\to\infty, \tag{16}$$

and

$$\sum_{k=0}^{\infty} |\Delta_{02} c_{jk}| \quad \text{is finite for each } j \text{ and tends to } 0 \text{ as } j \to \infty. \tag{17}$$

Then the series (1)

(i) converges regularly to some function $f(x,y)$ *for all* $x, y \in (0,1)$;

(ii) converges in the $L^r(0,1)^2$*-metric to* f *for all* $0 < r < 1/2$.

3. Proofs

For proving our results, we need the following lemmas.

LEMMA 1. [9, Lemma 1] *If* $\{c_{jk}\}_{i,j=0}^{\infty}$ *satisfies the condition* (11) *and for* $p, q \geqslant 1$,

$$C_{pq} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{pq} c_{jk}| < \infty \tag{18}$$

then

$$\sum_{j=0}^{\infty} |\Delta_{p,q-1} c_{jk}| \leqslant C_{pq} \quad (k = 0, 1, 2, \dots),$$

$$\sum_{j=0}^{\infty} |\Delta_{p,q-1} c_{jk}| \to 0 \quad as \quad k \to \infty,$$

$$\sup_k \sum_{j=m}^{\infty} |\Delta_{p,q-1} c_{jk}| \to 0 \quad as \quad m \to \infty.$$

Analogous statements hold true for $\Delta_{p-1,q} c_{jk}$ *under the same conditions* (11) *and* (18) *if the roles of* j *and* k *are interchanged.*

LEMMA 2. [9, Lemma 2] *Let* $\{a_{jk}\}_{i,j=0}^{\infty}$ *and* $\{b_{jk}\}_{i,j=0}^{\infty}$ *be two double sequences of numbers and* $B_{mn} = \sum_{j=0}^{m} \sum_{k=0}^{n} b_{jk}$ $(m, n = 0, 1, \dots)$ *be the rectangular partial sums of* $\{b_{jk}\}$. *Then, for all* $0 \leqslant m \leqslant M$ *and* $0 \leqslant n \leqslant N$,

$$\begin{aligned}\sum_{j=m}^{M} \sum_{k=n}^{N} b_{jk} a_{jk} = &\sum_{j=m}^{M} \sum_{k=n}^{N} B_{jk} \Delta_{11} a_{jk} + \sum_{j=m}^{M} B_{jN} \Delta_{10} a_{j,N+1} - \sum_{j=m}^{M} B_{j,n-1} \Delta_{10} a_{jn} \\ &+ \sum_{k=n}^{N} B_{Mk} \Delta_{01} a_{M+1,k} - \sum_{k=n}^{N} B_{m-1,k} \Delta_{01} a_{mk} + B_{MN} a_{M+1,N+1} \\ &- B_{M,n-1} a_{M+1,n} - B_{m-1,N} a_{m,N+1} + B_{m-1,n-1} a_{mn}.\end{aligned}$$

LEMMA 3. [9, Lemma 3] *If* $\{c_{jk}\}$ *satisfies the condition* (11)*, then for all* $m,n \geqslant 0$*,*

$$\sum_{R_{mn}} b_{jk}a_{jk} = \sum_{R_{mn}} B_{jk}\Delta_{11}a_{jk} - \sum_{j=0}^{m} B_{jn}\Delta_{10}a_{j,n+1} - \sum_{k=0}^{n} B_{mk}\Delta_{01}a_{m+1,k} - B_{mn}a_{m+1,n+1},$$

where $R_{mn} = \{(j,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : \text{ either } j \geqslant m+1 \text{ or } k \geqslant n+1\}$ *and* $\sum_{R_{mn}}$ *stands for* $\sum_{(j,k)} \sum_{\in R_{mn}}$.

Proof of Theorem 1. Proof of (i). Let $0 \leqslant m \leqslant M$ and $0 \leqslant n \leqslant N$. By using the notation (10) and Lemma 2, we have

$$\begin{aligned} &S(R;x,y) \\ &= \sum_{j=m}^{M}\sum_{k=n}^{N} D_j(x)D_k(y)\Delta_{11}c_{jk} + \sum_{j=m}^{M} D_j(x)D_N(y)\Delta_{10}c_{j,N+1} - \sum_{j=m}^{M} D_j(x)D_{n-1}(y)\Delta_{10}c_{jn} \\ &\quad + \sum_{k=n}^{N} D_M(x)D_k(y)\Delta_{01}c_{M+1,k} - \sum_{k=n}^{N} D_{m-1}(x)D_k(y)\Delta_{01}c_{mk} + c_{M+1,N+1}D_M(x)D_N(y) \\ &\quad - c_{M+1,n}D_M(x)D_{n-1}(y) - c_{m,N+1}D_{m-1}(x)D_N(y) + c_{mn}D_{m-1}(x)D_{n-1}(y). \end{aligned} \tag{19}$$

By using (6), for $x,y \in (0,1)$, we get

$$\begin{aligned} B^{-2}xy\,|S(R;x,y)| \leqslant{} & \sum_{j=m}^{M}\sum_{k=n}^{N} |\Delta_{11}c_{jk}| + \sum_{j=m}^{M} |\Delta_{10}c_{j,+1N}| + \sum_{j=m}^{M} |\Delta_{10}c_{jn}| \\ & + \sum_{k=n}^{N} |\Delta_{01}c_{M+1,k}| + \sum_{k=n}^{N} |\Delta_{01}c_{mk}| + |c_{M+1,N+1}| \\ & + |c_{M+1,n}| + |c_{m,N+1}| + |c_{mn}|. \end{aligned}$$

Since $\{c_{jk}\}_{i,j=0}^{\infty}$ satisfies conditions (11) and (15), making use of Lemma 1 (with $p = q = 1$) we can see that each term on the right-hand side tends to zero as $\max(m,n) \to \infty$. Thus, the series (1) converges to the some function $f(x,y)$ for all $x,y \in (0,1)$.

Proof of (ii). From (i) above the series (1) converges pointwise to the function f, that is,

$$f(x,y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}\chi_j(x)\chi_k(y),$$

for all $x,y \in (0,1)$. Therefore

$$f(x,y) - S_{mn}(x,y) = \sum_{R_{mn}} c_{jk}\chi_j(x)\chi_k(y).$$

By Lemma 3,

$$\sum_{R_{mn}} c_{jk}\chi_j(x)\chi_k(y) = \sum_{R_{mn}} D_j(x)D_k(y)\Delta_{11}c_{jk} - \sum_{j=0}^{m} D_j(x)D_n(y)\Delta_{10}c_{j,n+1}$$
$$- \sum_{k=0}^{n} D_m(x)D_k(y)\Delta_{01}c_{m+1,k} - D_m(x)D_n(y)c_{m+1,n+1}. \tag{20}$$

By using (6), for all $x,y \in (0,1)$, we get

$$B^{-2}xy|f(x,y) - S_{mn}(x,y)|$$
$$\leqslant \sum_{R_{mn}} |\Delta_{11}c_{jk}| + \sum_{j=0}^{m} |\Delta_{10}c_{j,n+1}| + \sum_{k=0}^{n} |\Delta_{01}c_{m+1,k}| + |c_{m+1,n+1}|.$$

The right-hand side is of $o(1)$ as $\max(m,n) \to \infty$ due to (15) (for $p=1$), Lemma 1, and (11). Now, for $x,y \in (0,1)$ and $0 < r < 1$, we have

$$|f(x,y) - S_{mn}(x,y)|^r \leqslant \{o(1)\}^r \left\{\frac{B^2}{xy}\right\}^r.$$

Therefore

$$\int_0^1\int_0^1 |f(x,y) - S_{mn}(x,y)|^r dxdy \leqslant o(1)B^{2r}\int_0^1\int_0^1 \frac{dxdy}{x^r y^r}$$
$$= o(1)B^{2r}\frac{1}{(1-r)^2}.$$

Hence $||f - S_{mn}||_r \to 0$ as $\max(m,n) \to \infty$. This completes the proof. □

Proof of Theorem 2. Proof of (i). Performing one more summation by parts on the right-hand side double sum of (19) and using the definition of Fejér kernel, we get

$$\sum_{j=m}^{M}\sum_{k=n}^{N} D_j(x)D_k(y)\Delta_{11}c_{jk}$$
$$= \sum_{j=m}^{M}\sum_{k=n}^{N} jF_j(x)kF_k(y)\Delta_{22}c_{jk} + \sum_{j=m}^{M} jF_j(x)NF_N(y)\Delta_{21}c_{j,N+1}$$
$$- \sum_{j=m}^{M} jF_j(x)(n-1)F_{n-1}(y)\Delta_{21}c_{jn} + \sum_{k=n}^{N} MF_M(x)kF_k(y)\Delta_{12}c_{M+1,k}$$
$$- \sum_{k=n}^{N} (m-1)F_{m-1}(x)kF_k(y)\Delta_{12}c_{mk} + \Delta_{11}c_{M+1,N+1}MF_M(x)NF_N(y)$$
$$- \Delta_{11}c_{M+1,n}MF_M(x)(n-1)F_{n-1}(y) - \Delta_{11}c_{m,N+1}(m-1)F_{m-1}NF_N(y)$$
$$+ \Delta_{11}c_{mn}(m-1)F_{m-1}(x)(n-1)F_{n-1}(y).$$

Using (7), for $x,y \in (0,1)$, we have

$$(C_1C_2)^{-1}x^2y^2\left|\sum_{j=m}^{M}\sum_{k=n}^{N}D_j(x)D_k(y)\Delta_{11}c_{jk}\right|$$
$$\leqslant \sum_{j=m}^{M}\sum_{k=n}^{N}\left|\Delta_{22}c_{jk}\right| + \sum_{j=m}^{M}\left|\Delta_{21}c_{j,N+1}\right| + \sum_{j=m}^{M}\left|\Delta_{21}c_{jn}\right| + \sum_{k=n}^{N}\left|\Delta_{12}c_{M+1,k}\right|$$
$$+\sum_{k=n}^{N}\left|\Delta_{12}c_{mk}\right| + \left|\Delta_{11}c_{M+1,N+1}\right| + \left|\Delta_{11}c_{M+1,n}\right| + \left|\Delta_{11}c_{m,N+1}\right| + \left|\Delta_{11}c_{mn}\right|.$$

By virtue of (15), Lemma 1 (with $p=q=2$), and (11), respectively, the right-hand side tends to zero as $\max(m,n)\to\infty$.

Now, we claim that each of the four single sums on the right-hand side of (19) tends to zero as $\max(m,n)\to\infty$, for all $(x,y)\in(0,1)$. We show that in the case of the first single sum. All three other single sums can be estimated analogously. Performing summation by parts

$$\sum_{j=m}^{M}D_j(x)D_N(y)\Delta_{10}c_{j,N+1}$$
$$=\sum_{j=m}^{M}jF_j(x)D_N(y)\Delta_{20}c_{j,N+1}+MF_M(x)D_N(y)\Delta_{10}c_{M,N+1}$$
$$-(m-1)F_{m-1}(x)D_N(y)\Delta_{10}c_{m,N+1}. \tag{21}$$

Using (6) and (7), we have

$$(CB)^{-1}x^2y\left|\sum_{j=m}^{M}D_j(x)D_N(y)\Delta_{10}c_{j,N+1}\right|$$
$$\leqslant \sum_{j=m}^{M}\left|\Delta_{20}c_{j,N+1}\right| + \left|c_{M,N+1}\right| + \left|\Delta_{10}c_{m,N+1}\right|.$$

By using of (16) and (11), the right-hand side tends to zero as $\max(m,n)\to\infty$. Finally, the last four terms on the right-hand side of (19) also tend to zero as $\max(m,n)\to\infty$, by using (6) and (11). Combining all estimates, we have $S(R;x,y)\to 0$ as $\max(m,n)\to\infty$, for $(x,y)\in(0,1)$. Hence, the series (1) converges regularly to some function $f(x,y)$, for $(x,y)\in(0,1)$.

Proof of (ii). Here, we start with (20). By applying Lemma 3 on the first sum on the right-hand side of (20) and using the definition of Fejér kernel, we get

$$\sum_{R_{mn}}D_j(x)D_k(y)\Delta_{11}c_{jk}=\sum_{R_{mn}}jF_j(x)kF_k(y)\Delta_{22}c_{jk}-\sum_{j=0}^{m}jF_j(x)nF_n(y)\Delta_{21}c_{j,n+1}$$
$$-\sum_{k=0}^{n}mF_m(x)kF_k(y)\Delta_{12}c_{m+1,k}-mF_m(x)nF_n(y)\Delta_{11}c_{m+1,n+1}.$$

By using (7), for all $x,y \in (0,1)$, we have

$$\frac{x^2y^2}{C_1C_2}\left|\sum_{R_{mn}} D_j(x)D_k(y)\Delta_{11}c_{jk}\right| \leqslant \sum_{R_{mn}}\left|\Delta_{22}c_{jk}\right| + \sum_{j=0}^{m}\left|\Delta_{21}c_{j,n+1}\right| + \sum_{k=0}^{n}\left|\Delta_{12}c_{m+1,k}\right| + \left|\Delta_{11}c_{m+1,n+1}\right|.$$

Due to (15) (for $p=2$), Lemma 1, and (11), the right-hand side of the above inequality is of $o(1)$ as $\max(m,n)\to\infty$. Therefore, for $x,y\in(0,1)$ and $0<r<1/2$, we have

$$\int_0^1\int_0^1\left|\sum_{R_{mn}} D_j(x)D_k(y)\Delta_{11}c_{jk}\right|^r dxdy \leqslant (o(1))^r\{C_1C_2\}^r\int_0^1\int_0^1\frac{1}{x^{2r}y^{2r}}dxdy = \frac{(C_1C_2)^r}{(1-2r)^2}o(1). \tag{22}$$

Similar to (21), a single summation by parts on the second single sum on the right-hand side of (20) and using the definition of Fejér kernel, we get

$$\sum_{j=0}^{m} D_j(x)D_n(y)\Delta_{10}c_{j,n+1} = \sum_{j=0}^{m} jF_j(x)D_n(y)\Delta_{20}c_{j,n+1} + mF_m(x)D_n(y)\Delta_{10}c_{m,n+1}.$$

By using (6) and (7), for $x,y\in(0,1)$, we have

$$\frac{x^2y}{C_1B}\left|\sum_{j=0}^{m} D_j(x)D_n(y)\Delta_{10}c_{j,n+1}\right| \leqslant \sum_{j=0}^{m}\left|\Delta_{20}c_{j,n+1}\right| + \left|\Delta_{10}c_{m,n+1}\right|.$$

The right-hand side is of $o(1)$ as $\max(m,n)\to\infty$ due to (16) (for $p=2$) and (11). Therefore, for $x,y\in(0,1)$ and $0<r<1/2$, we have

$$\int_0^1\int_0^1\left|\sum_{j=0}^{m} D_j(x)D_n(y)\Delta_{10}c_{j,n+1}\right|^r dxdy \leqslant (o(1))^r\{C_1B\}^r\int_0^1\int_0^1\frac{1}{x^{2r}y^{r}}dxdy = \frac{(C_1B)^r}{(1-r)(1-2r)}o(1). \tag{23}$$

Similarly, we can estimate the third single sum of the right-hand side of (20) by

$$\int_0^1\int_0^1\left|\sum_{j=0}^{m} D_m(x)D_k(y)\Delta_{01}c_{m+1,k}\right|^r dxdy \leqslant (o(1))^r\{BC_2\}^r\int_0^1\int_0^1\frac{1}{x^{r}y^{2r}}dxdy = \frac{(BC_2)^r}{(1-2r)(1-r)}o(1). \tag{24}$$

For the last term on the right-hand side of (20), using (11), we have

$$\int_0^1\int_0^1\left|D_m(x)D_n(y)c_{m+1,n+1}\right|^r dxdy \leqslant (o(1))^r B^{2r}\int_0^1\int_0^1\frac{1}{xy}dxdy = \frac{B^2r}{(1-r)^2}o(1). \tag{25}$$

Note that for $0<r<1/2$, $(a+b)^r \leqslant a^r+b^r$ for all $a,b \geqslant 0$. By using all the estimates (22)–(25) in (20), as $\max(m,n) \to \infty$ we have

$$\int_0^1 \int_0^1 \left| \sum_{R_{mn}} c_{jk}\chi_j(x)\chi_k(y) \right|^r dxdy = o(1).$$

Therefore, $||f - S_{mn}||_r \to 0$ as $\max(m,n) \to \infty$ for $0<r<1/2$. That is, the series (1) converges in the $L^r(0,1)^2$-metric to f for all $0<r<1/2$. This completes the proof. □

REFERENCES

[1] G. N. AGAEV, N. YA. VILENKIN, G. M. DZHAFARLI, *Multiplicative Systems of Functions and Harmonic Analysis on Zero-Dimensional Groups*, Élm, Baku, 1981, (in Russian).

[2] C. CHANG-PAO AND W. CHING-TANG, *Double Walsh series with coefficients of bounded variation of higher order*, Trans. Amer. Math. Soc. **350** (1998), no. 1, 395–417.

[3] B. L. GHODADRA AND V. FÜLÖP, *A note on the definition of bounded variation of higher order for double sequences*, Kragujevac J. Math. **44** (2020), no. 4, 563–570.

[4] B. GOLUBOV, A. EFIMOV, V. SKVORTSOV, *Walsh Series and Transforms: Theory and Applications*, Nauka, Moscow, 1987; Kluwer Academic Publishers Dordrecht, Boston, London, 1991.

[5] G. H. HARDY, *On the convergence of certain multiple series*, Proc. Cambridge Phil. Soc. **19** (1916–1919), 86–95.

[6] J. B. SHINGOD AND B. L. GHODADRA, *On the L^r convergence of Vilenkin series for $0<r\leqslant 1$*, Riv. Math. Uni. Pharma (to appear).

[7] K. KULWINDER, S. S. BHATIA AND B. RAM, *Double trigonometric series with coefficients of bounded variation of higher order*, Tamkang J. Math. **35** (2004), no. 3, 267–280.

[8] F. MÓRICZ, *Convergence and integrability of double trigonometric series with coefficients of bounded variation*, Proc. Amer. Math. Soc. **102** (1988), no. 3, 633–640.

[9] F. MÓRICZ, *Double Walsh series with coefficients of bounded variation*, Z. Anal. Anwendungen **10** (1991), no. 1, 3–10.

[10] S. S. VOLOSIVETS, *On convergence of Fourier-Vilenkin series in $L^p[0,1)$, $0<p\leqslant 1$*, (Russian), Izvestiya Sarat. university Mat. Mechanics. Informatics. **3** (2008), 3–9.

[11] A. ZYGMUND, *Trigonometric series*, Cambridge University Press, 1959.

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