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LACUNARY STATISTICAL MEASURABLE CONVERGENCE WITH APPLICATION TO KOROVKIN–TYPE THEOREM

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Abstract. The purpose of the article is to investigate the concept of lacunary statistical mean convergence, lacunary statistical measurable convergence and lacunary statistical Lebesgue measurable convergence. We demonstrate an advanced perception to Korovkin-type theorem with the help of three algebraic test functions by means of lacunary statistical Lebesgue measurable convergence.

1. Introduction and preliminaries

The theory of statistical convergence was initially presented by Fast [6] and Steinhaus [29] independently. Rath and Tripathy [28] studied statistical convergence from sequence spaces point of view. The concept of statistical convergence was further investigated by Connor [3], Fridy [7], Miller and Orhan [17], Jena et al. [10], İnce and Karaçal [9], Kadak and Mohiuddine [13], Raj and Choudhary [26] and many more. For detailed studies involving statistical convergence one may refer [12], [27].

Let $T \subset \mathbb{N}$ such that $T_m = \{n : n \leqslant m \text{ and } n \in T\}$. Then the asymptotic density of T is defined as

$$d(T) = \lim_{m\to\infty} \frac{|T_m|}{m},$$

where $|T_m|$ is cardinality of the enclosed set and $d(T)$ is finite. A sequence $z = (z_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to z_0, if for given $\varepsilon > 0$,

$$T_\varepsilon = \{n : n \in \mathbb{N} \text{ and } |z_n - z_0| \geqslant \varepsilon\},$$

has zero density (see [6], [29]), which implies that

$$d(T_\varepsilon) = \lim_{m\to\infty} \frac{|T_{\varepsilon,m}|}{m} = 0, \quad \text{where} \quad T_{\varepsilon,m} = \{n : n \leqslant m, n \in T_\varepsilon\}.$$

We may write it as

$$St \lim_{m\to\infty} z_m = z.$$

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For more results on statistical convergence one may refer [4], [5], [12], [18], [26].

Consider a measure space (T,Σ,μ). Then a sequence $g_m : T \to \mathbb{R}$ of measurable functions (MF) is said to be measurable convergent (MC) to g for a given $\varepsilon > 0$,

$$\lim_{m\to\infty} \mu(\{l \in T : |g_m(l) - g(l)| > \varepsilon\}) = 0.$$

We write it as

$$\mu \lim_{m\to\infty} g_m = g.$$

EXAMPLE 1. Let $X = [0,1]$ and μ be the Lebesgue measure on X. Then define a sequence of functions as:

$$g_m(l) = \begin{cases} m, & \text{if } l \in [0, \frac{1}{m}]; \\ 0, & \text{otherwise.} \end{cases}$$

Each g_m is a simple function (taking only two values, m and 0), and its pre-images

$$g_m^{-1}(\{m\}) = [0, \tfrac{1}{m}], \quad g_m^{-1}(\{0\}) = (\tfrac{1}{m}, 1]$$

are intervals in $[0,1]$, hence they are Lebesgue measurable sets. Therefore, (g_m) is a sequence of measurable functions.

Now, $\lim\limits_{m\to\infty} \mu(\{l \in T : |g_m(l) - g(l)| > \varepsilon\}) = \lim\limits_{m\to\infty} \dfrac{1}{m} = 0$.
Hence, $\{g_m\}$ is measurable convergence to $g \equiv 0$.

Throughout the paper, we denote sequence of measurable functions by (SMF) and measurable function by (MF). A SMF (g_m) is known as statistically measurable convergent (St_{MC}) to a MF g, if

$$\lim_{m\to\infty} \frac{1}{m} \Big|\{n : n \leqslant m \text{ and } \mu(\{l \in T : |g_n(l) - g(l)| > \varepsilon\}) \geqslant \delta\}\Big| = 0,$$

for given $\varepsilon, \delta > 0$. We may write it as

$$St_{MC} g_m(l) \to g(l).$$

For detailed study on measurable convergence one may refer [8].

EXAMPLE 2. Let $X = [0,1]$ and μ be the Lebesgue measure on X. Then define a sequence of functions as:

$$g_n(l) = \begin{cases} 1, & \text{if } n \text{ is square;} \\ 0, & \text{otherwise.} \end{cases}$$

for all $l \in [0,1]$ and let g be the zero function, i.e., $g(l) = 0$ for all $l \in [0,1]$. Then

$$\lim_{m\to\infty} \frac{|\{n : n \leqslant m \text{ and } \mu(\{l \in T : |g_n(l) - g(l)| > \varepsilon\}) \geqslant \delta\}|}{m} \leqslant \lim_{m\to\infty} \frac{\sqrt{m}}{m} = 0.$$

Hence, (St_{MC})-$\lim g_n(l) = 0$.

Lacunary sequence is an increasing integer sequence $\theta = (t_r)$ such that $h_r = t_r - t_{r-1} \to \infty$ as $r \to \infty$. We denote the interval determined by θ as $I_r = (t_{r-1}, t_r]$. Consider $T \subset \mathbb{N}$ and $r \in \mathbb{N}$. Then the r^{th}-partial density of T is denoted as $\delta_\theta^r(T)$ if

$$\delta_\theta^r(T) = \frac{|T \cap I_r|}{h_r}.$$

A number $\delta_\theta(T)$ is said to be lacunary density of T if

$$\delta_\theta(T) = \lim_{r\to\infty} \frac{1}{h_r} |\{t \in I_r : t \in T\}|, \quad (\delta_\theta(T) = \lim_{r\to\infty} \delta_\theta^r(T))$$

exists.

Jena et al. [11] introduced new approach of Korovkin type approximation via deferred Cesàro statistical measurable convergence. After that, Khan et al. [14] established generalized version of Korovkin type approximation theorem. Recently, Narrania et al. [25] studied statistical measurable convergence with respect to power series for double sequences.

Korovkin [15] established a significant result known as the Korovkin theorem. This theorem demonstrates how a continuous function defined on a compact metric space can be uniformly approximated by a sequence of positive linear operators. Over time, the theorem has gained considerable importance across various fields of mathematics. This research inspired our study, leading us to explore the concepts of statistically measurable convergence and statistically Lebesgue measurable convergence by means of lacunary sequences for real-valued functions. Additionally, we examine the connections between these types of convergence. Utilizing these concepts, we establish Korovkin-type approximation theorems.

2. Statistically lacunary measurable convergence

DEFINITION 1. A sequence (g_r) of functions is lacunary statistically convergent (St_{LC}) to real function g, if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{t : t_{r-1} < t \leqslant t_r \text{ and } |g_r(t) - g(t)| > \varepsilon\}| = 0,$$

for given $\varepsilon > 0$. It is represented as

$$St_{LC} \lim_{r\to\infty} g_r = g.$$

DEFINITION 2. A SMF (g_r) is called as statistically lacunary measurable convergent (St_{LMC}) to a MF g, if for given $\varepsilon, \delta > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} |\{t : t_{r-1} < t \leqslant t_r \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\}| = 0.$$

We may write it as

$$St_{LMC}\, g_r(t) \to g(t).$$

THEOREM 1. *If $St_{LC}\, g_r(t) \to g(t)$, then $St_{LMC}\, g_r(t) \to g(t)$.*

Proof. Consider an arbitrary real number $\varepsilon > 0$. Then from Definition 1, we get

$$\lim_{r\to\infty} \frac{1}{h_r} |\{t : t_{r-1} < t \leqslant t_r \text{ and } |g_r(t) - g(t)| \geqslant \varepsilon\}| = 0. \tag{1}$$

Consider $\varepsilon, \delta > 0$. Therefore,

$$\{t : t_{r-1} < t \leqslant t_r \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\} \subseteq \Psi,$$

where $\Psi = \{t : t_{r-1} < t \leqslant t_r \text{ and } |g_r(t) - g(t)| > \varepsilon\}$.

Then,

$$|\{t : t_{r-1} < t \leqslant t_r \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\}| \leqslant |\{t : t_{r-1} < t \leqslant t_r \text{ and } |g_r(t) - g(t)| > \varepsilon\}|.$$

Thus

$$\frac{1}{h_r} |\{t : t_{r-1} < t \leqslant t_r \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\}| \leqslant \frac{1}{h_r} |\{t : t_{r-1} < t \leqslant t_r \text{ and } |g_r(t) - g(t)| > \varepsilon\}|.$$

Taking limit $r \to \infty$ and from equation (1), we get $St_{LMC}\, g_r(t) \to g(t)$. □

DEFINITION 3. The lacunary sequence $\phi = (t_r^{'})$ is a lacunary refinement of lacunary sequence $\theta = (t_r)$ if $(t_r) \subseteq (t_r^{'})$. In this case $I_r \subseteq I_r^{'}$, where $I_r^{'} = (t_{r-1}^{'}, t_r^{'}]$ and we denote $h_r^{'} = t_r^{'} - t_{r-1}^{'}$.

THEOREM 2. *Let $\phi = (t_r^{'})$ be a lacunary refinement of $\theta = (t_r)$ such that the set $\{t : t \in I_r^{'} \setminus I_r\}$ is finite for each r. Then $St_{LMC}\, g_r(t) \to g(t)$ implies $St_{L'MC}\, g_r(t) \to g(t)$.*

Proof. Consider $\varepsilon > 0, \delta > 0$ and $St_{LMC}\, g_r(t) \to g(t)$. Consider the set

$$\begin{aligned}
&\{t : t_{r-1}^{'} < t \leqslant t_r^{'} \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\} \\
&\quad = \{t : t_{r-1}^{'} < t \leqslant t_{r-1} \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\} \\
&\quad \cup \{t : t_{r-1} < t \leqslant t_r^{'} \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\} \\
&\quad \cup \{t : t_r^{'} < t \leqslant t_r \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\}.
\end{aligned}$$

The statistical version can be written as

$$\begin{aligned}
&\frac{1}{t'_r - t'_{r-1}}|\{t : t'_{r-1} < t \leqslant t'_r \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\}| \\
&\leqslant \frac{1}{t_{r-1} - t'_{r-1}}|\{t : t'_{r-1} < t \leqslant t_{r-1} \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\}| \\
&\quad + \frac{1}{t'_r - t_{r-1}}|\{t : t_{r-1} < t \leqslant t'_r \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\}| \\
&\quad + \frac{1}{t_r - t'_r}|\{t : t'_r < t \leqslant t_r \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\}|.
\end{aligned}$$

Taking $r \to \infty$, and using the hypothesis, we get

$$\lim_{r\to\infty} \frac{1}{t'_r - t'_{r-1}}|\{t : t'_{r-1} < t \leqslant t'_r \text{ and } \mu(\{t \in T : |g_r(t) - g(t)| > \varepsilon\}) \geqslant \delta\}| = 0.$$

Hence,

$$St_{L'MC}\, g_r(t) \to g(t). \quad \square$$

3. Lacunary measurable statistical mean convergence

Let $L^s(T)$, for $s \geqslant 1$, be the space of all measurable functions g on T for which the s-norm

$$\|g\|_s = \left(\int_T |g(l)|^s \, d\mu(l)\right)^{1/s}$$

is finite.

DEFINITION 4. A SMF (g_r), $r \in \mathbb{N}$ is said to be lacunary statistically mean convergent $(LMEC)$ for order s to a MF g, if for all $\delta > 0$

$$\lim_{r\to\infty} \frac{1}{h_r}|\{t : t_{r-1} < t \leqslant t_r \text{ and } \|g_r(t) - g(t)\|_s \geqslant \delta\}| = 0.$$

We write it as

$$St_{LMEC}\, g_r(t) \to g(t).$$

DEFINITION 5. A SMF (g_r), $r \in \mathbb{N}$ is said to be lacunary statistically mean convergent in measure $(LMECM)$ for order s to a MF g, if for all $\varepsilon, \delta > 0$

$$\lim_{r\to\infty} \frac{1}{h_r}|\{t : t_{r-1} < t \leqslant t_r \text{ and } \mu(\{t \in T : \|g_r(t) - g(t)\|_s > \varepsilon\}) \geqslant \delta\}| = 0.$$

We write it as

$$St_{LMECM}\, g_r(t) \to g(t).$$

THEOREM 3. *If $St_{LMEC}\, g_r(t) \to g(t)$, then $St_{LMC}\, g_r(t) \to g(t)$.*

Proof. Consider SMF (g_r) which is not $LMEC$ of order s to a MF g. Then, there exist $\varepsilon, \delta > 0$ such that $St_{LMC}\, g_r(t) \to g(t)$ for infinite values of r. But we have

$$\lim_{r\to\infty} \frac{1}{h_r} |\{t : t_{r-1} < t \leqslant t_r \text{ and } \|g_r(t) - g(t)\|_s \geqslant \delta\}| = 0,$$

which is a contradiction. Thus, $St_{LMEC}\, g_r(t) \to g(t)$ implies $St_{LMC}\, g_r(t) \to g(t)$. □

THEOREM 4. *Suppose that $(t_{r-1}), (t_r), (t'_{r-1}), (t'_r)$ be the sequences of non-negative integers fulfilling $(t_{r-1}) \leqslant (t'_{r-1}) < (t_r) \leqslant (t'_r)$ such that*

$$\lim_{r\to\infty} \frac{t_r - t_{r-1}}{t'_r - t'_{r-1}} = s > 0,$$

then $St_{LMEC}\, g_r(t) \to g(t)$ implies $St_{L'MEC}\, g_r(t) \to g(t)$.

Proof. Suppose that $\delta > 0$ such that

$$|\{t : t'_{r-1} + 1 \leqslant t \leqslant t'_r \text{ and } \|g_r(t) - g(t)\|_s \geqslant \delta\}| \\ \subset |\{t : t_{r-1} + 1 \leqslant t \leqslant t_r \text{ and } \|g_r(t) - g(t)\|_s \geqslant \delta\}|.$$

Clearly,

$$|\{t : t'_{r-1} + 1 \leqslant t \leqslant t'_r \text{ and } \|g_r(t) - g(t)\|_s \geqslant \delta\}| \\ \leqslant |\{t : t_{r-1} + 1 \leqslant t \leqslant t_r \text{ and } \|g_r(t) - g(t)\|_s \geqslant \delta\}|.$$

Thus, we get

$$\frac{1}{t'_r - t'_{r-1}} |\{t : t'_r + 1 \leqslant t \leqslant t'_r \text{ and } \|g_r(t) - g(t)\|_s \geqslant \delta\}| \\ \leqslant \frac{t_r - t_{r-1}}{t'_r - t'_{r-1}} \frac{1}{t_r - t_{r-1}} |\{t : t_{r-1} + 1 \leqslant t \leqslant t_r \text{ and } \|g_r(t) - g(t)\|_s \geqslant \delta\}|.$$

Hence, taking limit $r \to \infty$ on both sides, we get

$$\lim_{r\to\infty} \frac{1}{t'_r - t'_{r-1}} |\{t : t_{r-1} + 1 \leqslant t \leqslant t_r \text{ and } \|g_r(t) - g(t)\|_s \geqslant \delta\}| = 0. \quad \square$$

4. Lacunary statistical measurable convergence under integral

Consider a MF g with finite positive values each on the measurable set $\{v_1, v_2, \cdots, v_r\}$. Throughout this section, let $\{\varphi_r\}$ be a sequence of finite Borel measures on the interval $[0,1]$. For a simple measurable function

$$g = \sum_{i=1}^{k} v_i \chi_{G_i},$$

where $v_i > 0$ and G_i are measurable sets, the integral of g with respect to φ_r is defined in the standard way as

$$\int_0^1 g\, d\varphi_r = \sum_{i=1}^{k} v_i\, \varphi_r(G_i).$$

This definition extends to non-negative measurable functions g via the usual approximation by simple functions, and to real-valued functions by considering positive and negative parts.

DEFINITION 6. Suppose that $\{\phi_r^z(t)\}_{r\in\mathbb{N}}$ is the sequence of finite Borel measurable functions on $[0,1]$. Then $\{\phi_r^z(t)\}_{r\in\mathbb{N}}$ is called statistically Lebesgue measurable convergent to a MF ϕ on $[0,1]$, if

$$\Big\{t : t < r \text{ and } \mu\Big(t \in T : \Big|\int_0^1 gd\phi_r - \int_0^1 gd\phi\Big| > \varepsilon\Big) \geqslant \delta\Big\},$$

for every $g \in C[0,1]$ and $\varepsilon, \delta > 0$ has zero natural density, i.e.,

$$\lim_{r\to\infty} \frac{1}{r}\Big|\Big\{t : t < r \text{ and } \mu\Big(t \in T : \Big|\int_0^1 gd\phi_r - \int_0^1 gd\phi\Big| > \varepsilon\Big) \geqslant \delta\Big\}\Big| = 0.$$

We write it as

$$St_{BMC} \lim_{r\to\infty} \int_0^1 gd\phi_r \to \int_0^1 gd\phi.$$

DEFINITION 7. Suppose that $\{\phi_r^z(t)\}_{r\in\mathbb{N}}$ is the sequence of finite Borel measurable function on $[0,1]$. Then $\{\phi_r^z(t)\}_{r\in\mathbb{N}}$ is called lacunary statistically Lebesgue convergent to a MF ϕ on $[0,1]$, if for every $g \in C[0,1]$ and $\varepsilon > 0$,

$$\Big\{t : t_{r-1} < t \leqslant t_r \text{ and } \Big|\int_0^1 gd\phi_r - \int_0^1 gd\phi\Big| \geqslant \varepsilon\Big\},$$

has zero natural density, i.e.,

$$\lim_{r\to\infty} \frac{1}{h_r}\Big|\Big\{t : t_{r-1} < t \leqslant t_r \text{ and } \Big|\int_0^1 gd\phi_r - \int_0^1 gd\phi\Big| \geqslant \varepsilon\Big\}\Big| = 0.$$

We write it as

$$St_{BLC} \lim_{r\to\infty} \int_0^1 gd\phi_r \to \int_0^1 gd\phi.$$

DEFINITION 8. Suppose that $\{\phi_r^z(t)\}_{r\in\mathbb{N}}$ be the sequence of finite Borel measurable functions on $[0,1]$. Then $\{\phi_r^z(t)\}_{r\in\mathbb{N}}$ is called lacunary statistically Lebesgue measurable $(BLMC)$ convergent to a MF ϕ on $[0,1]$, if for every $g \in C[0,1]$, $\exists\, \varepsilon, \delta > 0$, s.t.

$$\Big\{t : t_{r-1} < t \leqslant t_r \text{ and } \mu\Big(t \in T : \Big|\int_0^1 gd\phi_r - \int_0^1 gd\phi\Big| > \varepsilon\Big) \geqslant \delta\Big\},$$

has zero natural density, i.e.,

$$\lim_{r\to\infty}\frac{1}{h_r}\Big|\Big\{t:t_{r-1}<t\leqslant t_r \text{ and } \mu\Big(t\in T:\Big|\int_0^1 gd\phi_r-\int_0^1 gd\phi\Big|>\varepsilon\Big)\geqslant\delta\Big\}=0.$$

We write it as

$$St_{BLMC}\lim_{r\to\infty}\int_0^1 gd\phi_r\to\int_0^1 gd\phi.$$

THEOREM 5. *Suppose that* $\{\phi_r^z(t)\}$ *be the sequence of finite Borel measurable functions such that* $St_{BLC}\int_0^1 g\,d\phi_r\to\int_0^1 g\,d\phi$. *Then* $St_{BLMC}\int_0^1 g\,d\phi_r\to\int_0^1 g\,d\phi$.

Proof. Let $St_{BLC}\int_0^1 g\,d\phi_r\to\int_0^1 g\,d\phi$, then from Definition 7, let $\varepsilon>0$ be such that

$$\lim_{r\to\infty}\frac{1}{h_r}\Big|\Big\{t:t_{r-1}<t\leqslant t_r \text{ and } \Big|\int_0^1 g\,d\phi_r-\int_0^1 g\,d\phi\Big|\geqslant\varepsilon\Big\}\Big|=0.$$

For given $\varepsilon,\delta>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\Big|\Big\{t:t_{r-1}<t\leqslant t_r \text{ and } \mu\Big(\Big\{t\in T:\Big|\int_0^1 g\,d\phi_r-\int_0^1 g\,d\phi\Big|>\varepsilon\Big\}\Big)\geqslant\delta\Big\}\Big|$$
$$\subseteq\lim_{r\to\infty}\frac{1}{h_r}\Big|\Big\{t:t_{r-1}<t\leqslant t_r \text{ and } \Big|\int_0^1 g\,d\phi_r-\int_0^1 g\,d\phi\Big|\geqslant\varepsilon\Big\}\Big|.$$

Hence, by Definition 8 we have $St_{BLMC}\int_0^1 g\,d\phi_r\to\int_0^1 g\,d\phi$. □

THEOREM 6. *Suppose that* (g_r) *be the sequence of non-negative MF and* g *be a MF such that*

$$St_{BLMC}\lim_{r\to\infty}\int_0^1 g\,d\phi_r\to\int_0^1 g\,d\phi.$$

Then

$$St_{BLMC}\lim_{r\to\infty}\int_0^1 g\,d\phi\ \leqslant\ St_{BLMC}\liminf_{r\to\infty}\int_0^1 g_r\,d\phi.$$

Proof. Let $St_{BLMC}\lim_{r\to\infty}\int_0^1 g\,d\phi<\infty$ and

$$St_{BLMC}\lim_{r\to\infty}\int_0^1 g\,d\phi>St_{BLMC}\liminf_{r\to\infty}\int_0^1 g_r\,d\phi.$$

Then there exists δ and (r_i) such that for each i

$$St_{BLMC}\lim_{r\to\infty}\int_0^1 g_{r_i}\,d\phi<St_{BLMC}\liminf_{r\to\infty}\int_0^1 g\,d\phi-\delta.$$

Since $St_{BLMC} \lim_{r\to\infty} g_{r_i} \to g$, so let $St_{BLMC} \lim_{r\to\infty} g_r \to g$. Then there exists a subsequence (r_i) such that $St_{BLMC} \lim_{r\to\infty} g_{r_i} \to g$. Therefore, we get a subsequence (r_i') of (r_i) such that $St_{BLMC} \lim_{r\to\infty} g_{r_i'} \to g$. Now, from Fatou's lemma

$$\begin{aligned} St_{BLMC} \lim_{r\to\infty} \int g\, d\phi &\leqslant St_{BLMC} \lim_{r\to\infty} \inf \int_0^1 g_{r_i'}\, d\phi \\ &\leqslant \int g\, d\phi - \delta, \end{aligned}$$

which is a contradiction.

Next, we suppose that $St_{BLMC} \lim_{r\to\infty} \int g\, d\phi = \infty$ and $St_{BLMC} \lim_{r\to\infty} \inf \int_0^1 g_r\, d\phi < \infty$. Then there exists $N > 0$, and also suppose that a subsequence (g_{r_i}) such that $St_{BLMC} \lim_{r\to\infty} \int g_{r_i}\, d\phi < N$. Moreover, we get a subsequence (r_i') of (r_i) such that $St_{BLMC} \lim_{r\to\infty} g_{r_i} \to g$. From Fatou's lemma, we get $St_{BLMC} \liminf \int g_{r_i}\, d\phi = \infty$, which is again a contradiction. □

5. Application to Korovkin-type approximation theorem

One of the most powerful results in approximation theory is Korovkin's theorem. Several variants of this theorem have been intensively studied by many authors by applying different summability methods (see [1], [2], [16], [20], [21], [22], [23], [24]). The next result is the Korovkin type theorem for our method.

THEOREM 7. *Suppose that $\mathscr{C}[0,1]$ be the space of all real valued continuous functions and $u_r^z : \mathscr{C}[0,1] \to \mathscr{C}[0,1]$ is the Borel measures (finite) lies between $[0,1]$. Then, for all $g \in \mathscr{C}[0,1]$*

$$St_{BLMC} \lim_{r\to\infty} \int_0^1 g(t) u_r^z(t) \to g(z) \tag{2}$$

if and only if

$$St_{BLMC} \lim_{r\to\infty} \int_0^1 t^i u_r^z(t) \to z^i, \quad i = 0,1,2. \tag{3}$$

Proof. For $t^i = \{1, t, t^2\} \in \mathscr{C}[0,1]$ which is continuous, (2) $\Rightarrow$ (3) is trivial. Therefore, it is sufficient to prove only if part where we first take (3) is true. Consider $g \in \mathscr{C}[0,1]$ be such that $|g(z)| \leqslant M$, where M is a constant. We get

$$d(g(t), g(z)) \leqslant 2M, \;\; t, z \in [0,1]. \tag{4}$$

where the symbol $d(\cdot,\cdot)$ denotes the usual metric on the real numbers, i.e., $d(a,b) = |a-b|$.

For given $\varepsilon > 0$, we have

$$d(g(t), g(z)) < \varepsilon, \tag{5}$$

whenever $d(t-z) < \delta$, for all t and $z \in [0,1]$. Choose $\Theta_1 = \Theta_1(t,z) = (t-z)^2$. If $d(t-z) \geqslant \delta$, then

$$d(g(t),g(z)) < \frac{2\mathscr{H}}{\delta^2}\Theta_1(t,z). \tag{6}$$

From (5) and (6), we obtain

$$d(g(t),g(z)) < \varepsilon + \frac{2\mathscr{H}}{\delta^2}\Theta_1(t,z),$$

which implies that

$$-\varepsilon - \frac{2\mathscr{H}}{\delta^2}\Theta_1(t,z) \leqslant d(g(t)-g(z)) \leqslant \varepsilon + \frac{2\mathscr{H}}{\delta^2}\Theta_1(t,z). \tag{7}$$

However, $\int_0^1 du_r^z(t)$ is linear and monotone, so by using $\int_0^1 du_r^z(t)$ to this inequality, we get

$$\int_0^1 du_r^z(t)\Big(-\varepsilon - \frac{2\mathscr{H}}{\delta^2}\Theta_1(t,z)\Big) \leqslant \int_0^1 du_r^z(t)d((g(t)-g(z)) \tag{8}$$

$$\leqslant \int_0^1 du_r^z(t)\Big(\varepsilon + \frac{2\mathscr{H}}{\delta^2}\Theta_1(t,z)\Big). \tag{9}$$

Also, z is fixed and so is $g(z)$. Therefore,

$$-\varepsilon\int_0^1 du_r^z(t) - \frac{2\mathscr{H}}{\delta^2}\int_0^1 \Theta_1 du_r^z(t) \leqslant \int_0^1 g(t)du_r^z(t) - g(z)\int_0^1 du_r^z(t) \tag{10}$$

$$\leqslant \varepsilon\int_0^1 du_r^z(t) + \frac{2\mathscr{H}}{\delta^2}\int_0^1 \Theta_1 du_r^z(t). \tag{11}$$

But

$$\int_0^1 g(t)du_r^z(t) - g(z) = \Big[\int_0^1 g(t)du_r^z(t) - g(z)\int_0^1 du_r^z(t)\Big] + g(z)\Big[\int_0^1 du_r^z(t) - 1\Big]. \tag{12}$$

Using (10) and (12)

$$\int_0^1 g(t)du_r^z(t) - g(z) < \varepsilon\int_0^1 du_r^z(t) + \frac{2\mathscr{H}}{\delta^2}\int_0^1 \Theta_1 du_r^z(t) + g(z)\Big[\int_0^1 du_r^z(t) - 1\Big]. \tag{13}$$

Now, we estimate $\int_0^1 \Theta_1 du_r^z(t)$ as

$$\begin{aligned}\int_0^1 \Theta_1 du_r^z(t) &= \int_0^1 (t-z)^2 du_r^z(t) = \int_0^1 (t^2+z^2-2zt)du_r^z(t)\\ &= \int_0^1 t^2 du_r^z(t) - 2z\int_0^1 t\,du_r^z(t) + z^2\int_0^1 du_r^z(t)\\ &= \Big(\int_0^1 t^2 du_r^z(t) - z^2\Big) - 2z\Big(\int_0^1 t\,du_r^z(t) - z\Big) + z^2\Big(\int_0^1 du_r^z(t) - 1\Big).\end{aligned}$$

Now, from (13), we obtain

$$\begin{aligned}\int_0^1 g(t)du_r^z(t)-g(z) < \varepsilon\Big(\int_0^1 du_r^z(t)-1\Big)+\varepsilon \\ +\frac{2\mathscr{H}}{\delta^2}\Big\{\Big(\int_0^1 t^2du_r^z(t)-z^2\Big)-2z\Big(\int_0^1 tdu_r^z(t)-z\Big) \\ +z^2\Big(\int_0^1 du_r^z(t)-1\Big)\Big\}+g(z)\Big(\int_0^1 du_r^z(t)-1\Big).\end{aligned}$$

Thus, we can write

$$\begin{aligned}\Big|\int_0^1 g(t)du_r^z(t)-g(z)\Big| &\leqslant \varepsilon+\Big(\varepsilon+\frac{2\mathscr{H}}{\delta^2}+M\Big)\Big|\int_0^1 du_r^z(t)-1\Big| \\ &\quad+\frac{4\mathscr{H}}{\delta^2}\Big|\int_0^1 tdu_r^z(t)-z\Big|+\frac{2\mathscr{H}}{\delta^2}\Big|\int_0^1 t^2du_r^z(t)-z^2\Big| \\ &\leqslant S\Big[\Big|\int_0^1 du_r^z(t)-1\Big|+\Big|\int_0^1 tdu_r^z(t)-z\Big|+\Big|\int_0^1 t^2du_r^z(t)-z^2\Big|\Big],\end{aligned}$$

where

$$S=\max\Big(\varepsilon+\frac{2\mathscr{H}}{\delta^2}+\mathscr{H},\frac{4\mathscr{H}}{\delta^2},\frac{2\mathscr{H}}{\delta^2}\Big).$$

Now, for given $w>0$, there exists $\varepsilon,\delta>0$ for which $\varepsilon<w$, then set

$$\Psi_r(z,w)=\Big\{t:t_{r-1}<t<t_r \text{ and } \mu\Big(\Big|\int_0^1 g(t)du_r^z(t)-g(z)\Big|\geqslant w\Big)\Big\}\geqslant\delta$$

and for $i=0,1,2$

$$\Psi_{i,r}(z,w)=\Big\{t:t_{r-1}<t<t_r \text{ and } \mu\Big(\Big|\int_0^1 t^idu_r^z(t)-z^i\Big|>\frac{w-\varepsilon}{3S}\Big)\Big\}\geqslant\delta.$$

Thus, we get

$$\Psi_r(z,w)\leqslant\sum_{i=0}^{2}\Psi_{i,r}(z,w). \tag{14}$$

Therefore,

$$\frac{\|\Psi_r(z,w)\|_{\mathscr{C}[0,1]}}{t_r-t_{r-1}}\leqslant\sum_{i=0}^{2}\frac{\|\Psi_{i,r}(z,w)\|_{\mathscr{C}[0,1]}}{t_r-t_{r-1}}.$$

Hence, by using (3) and Definition 7, the R.H.S of (14) tends to zero as $r\to\infty$.

$$\lim_{r\to\infty}\frac{\|\Psi_r(z,w)\|_{\mathscr{C}[0,1]}}{t_r-t_{r-1}}=0,\quad \delta,w>0.$$

Hence, (2) holds true. □

6. Conclusion

Upon prior analysis, we modified the study of Jena et al. [11] and investigated various aspects of statistically measurable convergence and statistically Lebesgue measurable convergence by means of lacunary sequences. We demonstrated some theorems by introducing certain relationship among lacunary statistical mean convergence, lacunary statistical measurable convergence and lacunary Lebesgue measurable convergence. As an application we studied a new version of Korovkin-type approximation theorem. As a future work, we can study Egorov's theorem by means of lacunary statistically Lebesgue measurable convergence.

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