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$\mathscr{A}_f$–STATISTICAL CONVERGENCE AND $\mathscr{A}_f$–STATISTICAL BOUNDEDNESS BY MODULI IN $\mathscr{A}$–METRIC SPACES

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Abstract. In recent years, researchers have been intensely studying the important concepts of summability theory in various generalizations of metric spaces. This has motivated us to investigate the concepts of statistical convergence, statistical boundedness, and Cesàro summability, which respect the modulus function, in $\mathscr{A}$-metric spaces, a generalization of usual metric spaces. In this study, we introduce $\mathscr{A}_f$-statistical convergence, $\mathscr{A}_f$-statistical boundedness, and $\mathscr{A}_f$-strong Cesàro summability with respect to a modulus for sequences in $\mathscr{A}$-metric spaces. We explore the relationships between $\mathscr{A}_f$-statistically convergent sequences and $\mathscr{A}_f$-statistically bounded sequences with respect to a modulus. Additionally, we investigate the connections between the set of $\mathscr{A}_f$-statistically convergent sequences and the set of $\mathscr{A}_f$-strongly Cesàro summable sequences, defined using the modulus function.

1. Introduction

The concept of metric spaces, based on the notion of a distance function, has a wide range of applications in various fields such as Mathematics and Engineering. It was first introduced by Fréchet [13] in 1906. In the following years, particularly with the aim of generalizing metric spaces, various studies were conducted. The concepts of 2-metric spaces by Gähler [15] and D-metric spaces by Dhage [10] were proposed as generalizations of metric spaces; however, some issues arose with both of these generalized concepts. Based on the experiences gained from these developments, Mustafa and Sims [31, 32] introduced G-metric spaces as a generalization of metric spaces, and later, Sedghi et al. [37] introduced S-metric spaces, which provide a stronger structure than G-metric spaces, thus offering another generalization of metric spaces. In 2015, Abbas et al. [2] generalized S-metric spaces by introducing $\mathscr{A}$-metric spaces.

The concept of statistical convergence was first introduced independently by Fast [11] and Steinhaus [38] in 1951, and later studied by Schoenberg [36] in 1959. In the following years, this concept has been applied under various names to several theories, including summability theory, number theory, Fourier analysis theory, measure theory, optimization theory, trigonometric series, and Banach spaces, leading to significant results. For details, see the references [6, 7, 12, 14, 42].

Over the past 15 years, studies have been conducted on statistical boundedness and statistical convergence in metric spaces. We briefly discuss some of these studies. In 2012, Küçükaslan and Değer [27] investigated the concept of statistical boundedness

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in metric spaces. In 2014, Küçükaslan et al. [28] studied the statistical convergence of metric-valued sequences and their subsequences, examining the relationship between statistical and ordinary convergence in metric spaces. In 2015, Bilalov and Nazarova [5] explored the concepts of statistical convergence and p-strong convergence in metric spaces, examining the relationships between these concepts and providing Tauberian theorems related to statistical convergence in metric spaces. Recently, these concepts have been extensively studied within generalized metric spaces, yielding significant results. For more detailed information, please refer to references [1, 4, 8, 16, 17, 18, 20, 22, 23, 24, 25, 26, 34, 39, 40, 41].

The concept of a modulus function was introduced by Nakano [33] in 1953. In 1973, Ruckle [35] and in 1993, Maddox [29] defined certain sequence spaces using the modulus function. Connor [9] investigated the relationships between statistical convergence and strongly Cesàro summability with respect to the modulus function. In the following years, the modulus function has been explored by numerous researchers, among which an important study was conducted by Kayan and Çolak [21] in 2019.

In [21], the concepts of df-statistical convergence, df-statistical boundedness, and df-strongly Cesàro summability, which respect the modulus function, were introduced and the relationships between these concepts were examined in metric spaces. This motivated us to adopt a similar approach and investigate more comprehensive definitions and relationships in $\mathscr{A}$-metric spaces, which are one of the most important generalizations of usual metric spaces.

The aim of this study is to investigate the concepts of $\mathscr{A}_f$-statistical convergence, $\mathscr{A}_f$-statistical Cauchy, $\mathscr{A}_f$-statistical boundedness, and $\mathscr{A}_f$-strongly Cesàro summability, derived using the modulus function, as well as the relationships between these concepts. This study will not only contribute to advances in theoretical mathematics but will also make a significant contribution to the development of innovative and adaptable solutions to practical problems in various fields of science and engineering.

The results and techniques presented in this paper are generally analogous to the results and techniques in [21] and constitute an extension of the findings in [21].

Now let us give the basic definitions and notations that we will use in our study.

For a set $\mathscr{K} \subseteq \mathbb{N}$, the asymptotic (or natural) density is defined as follows,

$$\delta(\mathscr{K}) = \lim_{t\to\infty} \frac{1}{t} |\{k \leqslant t : k \in \mathscr{K}\}|$$

where $|\{k \leqslant t : k \in \mathscr{K}\}|$ denotes the number of elements of $\mathscr{K}$ not exceeding t, (see [14]).

A sequence (a_k) is said to be statistically convergent to ξ, if for every $\varepsilon > 0$,

$$\lim_t \frac{1}{t} |\{k \leqslant t : |a_k - \xi| \geqslant \varepsilon\}| = 0, \quad ([14]).$$

Now, let's recall the modulus function (see [30, 33, 35]).

A function $f : [0,\infty) \to [0,\infty)$ is said to be a modulus ([33]) if it satisfies the following conditions:

i) $f(u) = 0$ if and only if $u = 0$,

ii) $f(u+v) \leqslant f(u)+f(v)$ for all $u,v \geqslant 0$,

iii) f is increasing

iv) f is continuous from the right at 0.

These properties clearly indicate that a modulus function must be continuous on $[0,\infty)$. A modulus may be bounded or unbounded. Examples of bounded and unbounded modulus functions can be given as $f(u)=\frac{u}{1+u}$ and $f(u)=u^m$ for $0<m\leqslant 1$, respectively. For any modulus function f, we have the property: $f(nu)\leqslant nf(u)$ and $f(n)\leqslant nf(1)$ for all $n\in\mathbb{N}$.

The *f-density* $\delta_f(\mathscr{K})$ of a set $\mathscr{K}\subseteq\mathbb{N}$ with respect to a modulus function f is defined as:

$$\delta_f(\mathscr{K}) = \lim_{t\to\infty}\frac{f(|\{k\leqslant t : k\in\mathscr{K}\}|)}{f(t)},$$

if the limit exists (see [3]). This definition generalizes the natural density when $f(x)=x$, so that $\delta_f(\mathscr{K})$ coincides with the natural density $\delta(\mathscr{K})$ in this case. The equation $\delta_f(\mathscr{K})+\delta_f(\mathbb{N}\setminus\mathscr{K})=1$ does not hold in general. If $\delta_f(\mathscr{K})=0$ then $\delta_f(\mathbb{N}\setminus\mathscr{K})=1$. For any unbounded modulus function f and a set $\mathscr{K}\subseteq\mathbb{N}$, if $\delta_f(\mathscr{K})=0$, then it must be that $\delta(\mathscr{K})=0$. However, the converse may not necessarily hold in general. For more details, refer to [3, 21].

DEFINITION 1. [2] Let X be a nonempty set. A function $\mathscr{A}: X^n\to[0,\infty)$ is called an $\mathscr{A}$-metric on X if for any $x_i, a\in X, i=1,2,\dots,n$ the following conditions are satisfied:

(A1) $\mathscr{A}(x_1,x_2,\dots,x_{n-1},x_n)\geqslant 0$,

(A2) $\mathscr{A}(x_1,x_2,\dots,x_{n-1},x_n)=0 \Leftrightarrow x_1=x_2=\dots=x_n$,

(A3) $\mathscr{A}(x_1,x_2,\dots,x_{n-1},x_n)\leqslant \sum_{k=1}^{n}\mathscr{A}(\underbrace{x_k,x_k,\cdots,x_k}_{n-1},a)$.

The pair $(X,\mathscr{A})$ is referred to as an $\mathscr{A}$-metric space.

EXAMPLE 1. [2] Let $X=\mathbb{R}$. A function $\mathscr{A}: X^n\to[0,\infty)$ is defined by

$$\mathscr{A}(x_1,x_2,\dots,x_{n-1},x_n)=\sum_{k=1}^{n}\sum_{k<j}|x_k-x_j|.$$

Then $(X,\mathscr{A})$ is an $\mathscr{A}$-metric space.

LEMMA 1. [2] *Let $(X,\mathscr{A})$ be an $\mathscr{A}$-metric space. Then, for all $x,y\in X$, the following equality holds:*

$$\mathscr{A}(x,x,\dots,x,y)=\mathscr{A}(y,y,\dots,y,x).$$

LEMMA 2. [2] *Let* $(X,\mathscr{A})$ *be an* $\mathscr{A}$*-metric space. For all* $x,y\in X$*, the following inequalities hold:*

$$\mathscr{A}(x,x,\dots,x,z)\leqslant(n-1)\mathscr{A}(x,x,\dots,x,y)+\mathscr{A}(y,y,\dots,y,z) \text{ and}$$
$$\mathscr{A}(x,x,\dots,x,z)\leqslant(n-1)\mathscr{A}(x,x,\dots,x,y)+\mathscr{A}(z,z,\dots,z,y).$$

DEFINITION 2. [2] The $\mathscr{A}$ - metric space $(X,\mathscr{A})$ is called bounded if there exists an $r>0$ such that $\mathscr{A}(y,y,\dots,y,x)\leqslant r$ for all $x,y\in X$. Otherwise, X is unbounded.

DEFINITION 3. [2] Let $(X,\mathscr{A})$ be an $\mathscr{A}$ - metric space and let (a_k) be a sequence in this space. The following definitions describe the convergence and Cauchy properties of the sequence:

(i) The sequence (a_k) is said to be convergent to ξ, if for every $\varepsilon>0$, there exists a positive integer k_0 such that for all $k\geqslant k_0$, we have $\mathscr{A}(a_k,a_k,\dots,a_k,\xi)<\varepsilon$.

(ii) The sequence (a_k) is said to be a Cauchy sequence if for every $\varepsilon>0$, there exists a positive integer k_0 such that for all $k,m\geqslant k_0$, we have $\mathscr{A}(a_k,a_k,\dots,a_k,a_m)<\varepsilon$.

2. Main results

In this section, we introduce and examine the concepts of $\mathscr{A}_f$-statistical convergence and $\mathscr{A}_f$-statistical boundedness of a sequence in an $\mathscr{A}$-metric space using an unbounded modulus function f.

Initially, we define the notion of $\mathscr{A}_f$-statistical convergence in $\mathscr{A}$-metric spaces.

DEFINITION 4. Let (a_k) be a sequence in an $\mathscr{A}$-metric space $(X,\mathscr{A})$, and let f be an unbounded modulus function. The sequence (a_k) is said to be $\mathscr{A}_f$-statistically converge to ξ, if there exists an element $\xi\in X$ such that for every $\varepsilon>0$, the following condition holds:

$$\lim_{t\to\infty}\frac{1}{f(t)}f\left(|\{k\leqslant t:\mathscr{A}(a_k,a_k,\dots,a_k,\xi)\geqslant\varepsilon\}|\right)=0.$$

In this case, we write $\mathscr{S}\mathscr{A}_f-\lim a_k=\xi$ or $a_k\xrightarrow{\mathscr{S}\mathscr{A}_f}\xi$ if (a_k) is $\mathscr{A}_f$-statistically convergent to ξ. Throughout this paper, $\mathscr{S}\mathscr{A}_f$ will denote the set of sequences in the an $\mathscr{A}$-metric space (X,A) that are $\mathscr{A}_f$-statistically convergent.

In $\mathscr{A}$-metric spaces, $\mathscr{A}_f$-statistical convergence becomes equivalent to statistical convergence when $f(u)=u$, and it is denoted by $\mathscr{S}\mathscr{A}$, see [39].

In light of the above knowledge, we can state the relationship between statistical convergence and $\mathscr{A}_f$-statistical convergence in $\mathscr{A}$-metric spaces as follows.

COROLLARY 1. *In* $\mathscr{A}$*-metric spaces, every* $\mathscr{A}_f$*-statistically convergent sequence is also statistically convergent; however, the converse does not generally hold for arbitrary unbounded modulus functions* f*.*

LEMMA 3. *Let (a_k) be a sequence in an $\mathscr{A}$-metric space $(X,\mathscr{A})$. If the sequence (a_k) is $\mathscr{A}_f$-statistically convergent, then its limit is unique.*

Proof. Let $a_k \stackrel{\mathscr{S}\mathscr{A}_f}{\longrightarrow} \xi_0$ and $a_k \stackrel{\mathscr{S}\mathscr{A}_f}{\longrightarrow} \xi_1$ such that $\xi_0 \neq \xi_1$. For $\varepsilon > 0$, we define the sets

$$\mathscr{K}_1(\varepsilon) = \{k \in \mathbb{N} : \mathscr{A}(a_k, a_k, \dots, a_k, \xi_0) \geqslant \frac{\varepsilon}{n}\} \quad \text{and}$$
$$\mathscr{K}_2(\varepsilon) = \{k \in \mathbb{N} : \mathscr{A}(a_k, a_k, \dots, a_k, \xi_1) \geqslant \frac{\varepsilon}{n}\}.$$

Since $a_k \stackrel{\mathscr{S}\mathscr{A}_f}{\longrightarrow} \xi_0$, we have $\delta_f(\mathscr{K}_1(\varepsilon)) = 0$. Similarly, since $a_k \stackrel{\mathscr{S}\mathscr{A}_f}{\longrightarrow} \xi_1$, we have $\delta_f(K_2(\varepsilon)) = 0$. Let $\mathscr{K}(\varepsilon) := \mathscr{K}_1(\varepsilon) \cup \mathscr{K}_2(\varepsilon)$. Then $\delta_f(\mathscr{K}(\varepsilon)) = 0$, and therefore, we have $\delta_f(\mathbb{N} \setminus \mathscr{K}(\varepsilon)) = 1$. This implies that $\mathbb{N} \setminus \mathscr{K}(\varepsilon) \neq \emptyset$. Hence, for a $k \in \mathbb{N} \setminus \mathscr{K}(\varepsilon)$, we have

$$\begin{aligned}
\mathscr{A}(\xi_0, \xi_0, \dots, \xi_0, \xi_1) &\leqslant (n-1)\mathscr{A}(\xi_0, \xi_0, \dots, \xi_0, a_k) + \mathscr{A}(a_k, a_k, \dots, a_k, \xi_1) \\
&< (n-1)\frac{\varepsilon}{n} + \frac{\varepsilon}{n} \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have $\mathscr{A}(\xi_0, \xi_0, \dots, \xi_0, \xi_1) = 0$, i.e., $\xi_0 = \xi_1$. □

LEMMA 4. *Let (a_k) be a sequence in an $\mathscr{A}$-metric space $(X,\mathscr{A})$, and let f and g be two unbounded modulus functions. If $a_k \stackrel{\mathscr{S}\mathscr{A}_f}{\longrightarrow} \xi_0$ and $a_k \stackrel{\mathscr{S}\mathscr{A}_g}{\longrightarrow} \xi_1$, then $\xi_0 = \xi_1$.*

Proof. Let $a_k \stackrel{\mathscr{S}\mathscr{A}_f}{\longrightarrow} \xi_0$ and $a_k \stackrel{\mathscr{S}\mathscr{A}_g}{\longrightarrow} \xi_1$. By Corollary 1, an $\mathscr{A}_f$-statistically convergent sequence is also statistically convergent for any unbounded modulus f. So, the sequence (a_k) is statistically convergent to both ξ_0 and ξ_1. Since the limit of a statistically convergent sequence is unique (Theorem 2.5 in [39]), it can be concluded that $\xi_0 = \xi_1$. □

It is straightforward to verify that in $\mathscr{A}$- metric spaces, every convergent sequence is $\mathscr{A}_f$-statistically convergent for any unbounded modulus f. However, the reverse is not necessarily true. For example, let $r, s \in X$ be two distinct fixed points and consider the sequence (a_k) defined by

$$a_k := \begin{cases} r, & \text{if } k = j^2,\ j \in \mathbb{N}, \\ s, & \text{if } k \neq j^2. \end{cases}$$

We consider the unbounded modulus $f(u) = u^m$, where $0 < m \leqslant 1$. It is evident that, (a_k) is not convergent, but since

$$\lim_{t\to\infty} \frac{f(|\{k \leqslant t : \mathscr{A}(a_k, a_k, \dots, a_k, s) \geqslant \varepsilon\}|)}{f(t)} \leqslant \lim_{t\to\infty} \frac{f(\sqrt{t})}{f(t)} = \lim_{t\to\infty} \frac{(\sqrt{t})^m}{t^m} = 0,$$

the sequence (a_k) is $\mathscr{A}_f$-statistically convergent.

THEOREM 1. *Let (a_k) be a sequence in an $\mathscr{A}$ - metric space $(X,\mathscr{A})$, and let f be an unbounded modulus function satisfying $\lim_{r\to\infty}\frac{f(r)}{r}>0$. Then, if the sequence (a_k) is statistically convergent, it is also $\mathscr{A}_f$-statistically convergent.*

Proof. We know that $|\{k\leqslant t:\mathscr{A}(a_k,a_k,\ldots,a_k,\xi)\geqslant\varepsilon\}|$ is a positive integer. Hence, we can write

$$f(|\{k\leqslant t:\mathscr{A}(a_k,a_k,\ldots,a_k,\xi)\geqslant\varepsilon\}|)\leqslant|\{k\leqslant t:\mathscr{A}(a_k,a_k,\ldots,a_k,\xi)\geqslant\varepsilon\}|\cdot f(1)$$

and hence

$$\begin{aligned}&\frac{f(|\{k\leqslant t:\mathscr{A}(a_k,a_k,\ldots,a_k,\xi)\geqslant\varepsilon\}|)}{f(t)}\\&\leqslant\frac{t}{f(t)}\cdot\frac{|\{k\leqslant t:\mathscr{A}(a_k,a_k,\ldots,a_k,\xi)\geqslant\varepsilon\}|\cdot f(1)}{t}.\end{aligned}$$

Since $\lim_{r\to\infty}\frac{f(r)}{r}>0$ and $(a_k)\in\mathscr{S}\mathscr{A}$, we get $(a_k)\in\mathscr{S}\mathscr{A}_f$. □

DEFINITION 5. Let (a_k) be a sequence in an $\mathscr{A}$ - metric space $(X,\mathscr{A})$, and let f be an unbounded modulus function. The sequence (a_k) is said to be an $\mathscr{A}_f$-statistically Cauchy sequence if for every $\varepsilon>0$, there exists a positive integer T such that

$$\lim_{t\to\infty}\frac{1}{f(t)}f(|\{k\leqslant t:\mathscr{A}(a_k,a_k,\ldots,a_k,x_T)\geqslant\varepsilon\}|)=0.$$

THEOREM 2. *Let f be an unbounded modulus function. Every $\mathscr{A}_f$-statistically convergent sequence is also an $\mathscr{A}_f$-statistically Cauchy sequence in an $\mathscr{A}$-metric space $(X,\mathscr{A})$.*

Proof. Let (a_k) be a sequence $\mathscr{A}_f$-statistically convergent. For $\varepsilon>0$, there exists an element $\xi\in X$ such that

$$\lim_{t\to\infty}\frac{1}{f(t)}f(|\{k\leqslant t:\mathscr{A}(a_k,a_k,\ldots,a_k,\xi)\geqslant\varepsilon\}|)=0.$$

We define the set $\mathscr{K}(\varepsilon)$ as

$$\mathscr{K}(\varepsilon)=\left\{k\in\mathbb{N}:\mathscr{A}(a_k,a_k,\ldots,a_k,\xi)\geqslant\frac{\varepsilon}{n}\right\}.$$

Since (a_k) is $\mathscr{A}_f$-statistically convergent to ξ, we get $\delta_f(\mathscr{K}(\varepsilon))=0$ and $\delta_f(\mathbb{N}\setminus\mathscr{K}(\varepsilon))=1$. Now, $\varepsilon>0$ be arbitrarily preassigned. Then there exists a fixed $T\in\mathbb{N}\setminus\mathscr{K}(\varepsilon)$ such that

$$\begin{aligned}\mathscr{A}(a_k,a_k,\ldots,a_k,a_T)&\leqslant(n-1)\mathscr{A}(a_k,a_k,\ldots,a_k,\xi)+\mathscr{A}(\xi,\xi,\ldots,\xi,a_T)\\&<(n-1)\frac{\varepsilon}{n}+\frac{\varepsilon}{n}\\&=\varepsilon,\end{aligned}$$

for every $k \in \mathbb{N} \setminus \mathscr{K}(\varepsilon)$. Then, we get

$$\mathbb{N} \setminus \mathscr{K}(\varepsilon) \subseteq \{k \in \mathbb{N} : \mathscr{A}(a_k, a_k, \ldots, a_k, a_T) < \varepsilon\},$$

and so

$$\{k \in \mathbb{N} : \mathscr{A}(a_k, a_k, \ldots, a_k, a_T) \geqslant \varepsilon\} \subseteq \mathscr{K}(\varepsilon).$$

Since $\delta_f(\mathscr{K}(\varepsilon)) = 0$, we get

$$\delta_f(\{k \in \mathbb{N} : \mathscr{A}(a_k, a_k, \ldots, a_k, a_T) \geqslant \varepsilon\}) = 0.$$

Consequently, (a_k) is an $\mathscr{A}_f$-statistically Cauchy sequence in an $\mathscr{A}$-metric space $(X, \mathscr{A})$. $\square$

The concept of statistical boundedness in metric spaces was defined in [27]. Subsequently, Kayan and Çolak [19] extended this concept in metric spaces using the modulus function. Now, we define the concept of statistical boundedness in $\mathscr{A}$-metric spaces using the modulus function.

DEFINITION 6. Let (a_k) be a sequence in an $\mathscr{A}$- metric space $(X, \mathscr{A})$, and let f be an unbounded modulus function. The sequence (a_k) is said to be $\mathscr{A}_f$-statistically bounded, if there exists an element $\xi \in X$ and a positive real number M such that

$$\lim_{t \to \infty} \frac{1}{f(t)} f(|\{k \leqslant t : \mathscr{A}(a_k, a_k, \ldots, a_k, \xi) > M\}|) = 0.$$

The following theorem demonstrates the relationship between boundedness and $\mathscr{A}_f$-statistical boundedness in $\mathscr{A}$-metric spaces.

THEOREM 3. *Let (a_k) be a sequence in an $\mathscr{A}$- metric space $(X, \mathscr{A})$, and let f be an unbounded modulus function. If the sequence (a_k) is bounded, then it is $\mathscr{A}_f$-statistically bounded.*

Proof. Let the sequence (a_k) be bounded. Then, there exist an $M \in \mathbb{R}^+$ and $\xi \in X$ such that

$$\mathscr{A}(a_k, a_k, \ldots, a_k, \xi) \leqslant M$$

for every $k \in \mathbb{N}$. Then, the set $\{k \in \mathbb{N} : \mathscr{A}(a_k, a_k, \ldots, a_k, \xi) > M\}$ is an empty set. So, we get

$$\lim_{t \to \infty} \frac{1}{f(t)} f(|\{k \leqslant t : \mathscr{A}(a_k, a_k, \ldots, a_k, \xi) > M\}|) = \lim_{t \to \infty} \frac{f(0)}{f(t)} = 0.$$

Therefore, the sequence (a_k) is $\mathscr{A}_f$-statistically bounded. $\square$

The converse of Theorem 3 does not hold in general. Let $X = \mathbb{R}$ with the $\mathscr{A}$-metric defined by

$$\mathscr{A}(x_1, x_2, \ldots, x_{n-1}, x_n) = \sum_{k=1}^{n} \sum_{k<j} |a_k - x_j|,$$

and let the modulus $f(u) = u^m$, where $0 < m \leqslant 1$. Then, we define the sequence (a_k) as follows:

$$a_k = \begin{cases} k, & \text{if } k = j^3,\ j \in \mathbb{N}, \\ 0, & \text{if } k \neq j^3. \end{cases}$$

The sequence (a_k) is not bounded. Take $a_0 = 0$ and choose a sufficiently large $\mathscr{M} := \dfrac{M}{n-1} > 0$, where $n \geqslant 2$ is a positive integer. Then,

$$|\{k \leqslant t : (n-1)|a_k| > M\}| = |\{k \leqslant t : |a_k| > \mathscr{M}\}| = \left|\{k \leqslant t : k = j^3, k > \mathscr{M}\}\right| \leqslant \sqrt[3]{t}.$$

So,

$$\lim_{t\to\infty} \frac{1}{f(t)} f\left(|\{k \leqslant t : |a_k| > \mathscr{M}\}|\right) = \lim_{t\to\infty} \frac{f\left(\sqrt[3]{t}\right)}{f(t)} = \lim_{t\to\infty} \frac{\left(\sqrt[3]{t}\right)^m}{t^m} = 0,$$

because f is increasing. Consequently, the sequence (a_k) is $\mathscr{A}_f$-statistically bounded.

The following theorem presents a relationship between $\mathscr{A}_f$-statistical convergence and $\mathscr{A}_f$-statistical boundedness in $\mathscr{A}$-metric spaces.

THEOREM 4. *Let (a_k) be a sequence in an $\mathscr{A}$-metric space (X,A), and let f be an unbounded modulus. If the sequence (a_k) is $\mathscr{A}_f$-statistically convergent, then it is also $\mathscr{A}_f$-statistically bounded.*

Proof. For any $\varepsilon > 0$ and a sufficiently large $M > 0$, we can write

$$\{k \in \mathbb{N} : \mathscr{A}(a_k, a_k, \ldots, a_k, \xi) > M\} \subseteq \{k \in \mathbb{N} : \mathscr{A}(a_k, a_k, \ldots, a_k, \xi) \geqslant \varepsilon\}$$

and so

$$|\{k \in \mathbb{N} : \mathscr{A}(a_k, a_k, \ldots, a_k, \xi) > M\}| \leqslant |\{k \in \mathbb{N} : \mathscr{A}(a_k, a_k, \ldots, a_k, \xi) \geqslant \varepsilon\}|.$$

Since f is increasing and (a_k) is $\mathscr{A}_f$-statistically convergent, we have

$$\lim_{t\to\infty} \frac{1}{f(t)} f\left(|\{k \leqslant t : \mathscr{A}(a_k, a_k, \ldots, a_k, \xi) > M\}|\right) = 0.$$

Therefore, the proof is complete. □

The converse of Theorem 4 does not hold in general. Consider the $\mathscr{A}$-metric space in Example 1. Then, let us consider the sequence (a_k) as follows:

$$a_k = \begin{cases} 1, & \text{if } k \text{ is even} \\ 0, & \text{if } k \text{ is odd} \end{cases}$$

in an $\mathscr{A}$-metric space $(X, \mathscr{A})$, Let the modulus be $f(u) = \log(u+1)$. Then, it is clear that the sequence (a_k) is $\mathscr{A}_f$-statistically bounded, but it is not $\mathscr{A}_f$-statistically convergent.

In this section, using a modulus function f, we define and study the $\mathscr{A}_f$-strong Cesàro summability of a sequence in an $\mathscr{A}$-metric space. We also establish the relations between $\mathscr{A}_f$-strong Cesàro summability and $\mathscr{A}_f$-statistical convergence.

DEFINITION 7. Let $(X,\mathscr{A})$ be an $\mathscr{A}$- metric space, and let f be a modulus function. A sequence (a_k) is said to be $\mathscr{A}_f$-strongly Cesàro summable to $\xi \in X$ if there exists an element $\xi \in X$ such that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} f\left[\mathscr{A}(a_k, a_k, \ldots, a_k, \xi)\right] = 0.$$

The set of all $\mathscr{A}_f$-strongly Cesàro summable sequences in the $\mathscr{A}$- metric space $(X,\mathscr{A})$ is denoted by $\mathscr{W}_f^A$, i.e.,

$$\mathscr{W}_f^A = \left\{ (a_k) : \lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} f\left[\mathscr{A}(a_k, a_k, \ldots, a_k, \xi)\right] = 0, \text{ for some } \xi \in X \right\}.$$

Note that in this definition, the modulus function f does not necessarily need to be unbounded.

In the special case where $f(u) = u$, the sequence (a_k) is said to be strongly Cesàro summable to ξ, and the set of all strongly Cesàro summable sequences in the $\mathscr{A}$-metric space (X,A) is denoted by $\mathscr{W}^A$, i.e.,

$$\mathscr{W}^A = \left\{ (a_k) : \lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} \mathscr{A}(a_k, a_k, \ldots, a_k, \xi) = 0, \text{ for some } \xi \in X \right\}.$$

The following theorem presents a relationship between $\mathscr{A}_f$-strong Cesàro summability and strong Cesàro summability in an $\mathscr{A}$-metric space.

THEOREM 5. *Let $(X,\mathscr{A})$ be an $\mathscr{A}$-metric space, and let f be a modulus function. Then, the following inclusion holds:*

$$\mathscr{W}^{\mathscr{A}} \subset \mathscr{W}_f^{\mathscr{A}}.$$

Proof. Let $(a_k) \in \mathscr{W}^{\mathscr{A}}$. Then, there exists an element $\xi \in X$ such that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} \mathscr{A}(a_k, a_k, \ldots, a_k, \xi) = 0.$$

Let $\varepsilon > 0$. Then we choose δ such that $0 < \delta < 1$ and $f(s) < \varepsilon$ for $s \in (0,\delta]$. Let

$$b_k = \mathscr{A}(a_k, a_k, \ldots, a_k, \xi).$$

Now, consider the sum

$$\sum_{k=1}^{t} f(b_k) = \sum_{\substack{k=1 \\ b_k \leqslant \delta}}^{t} f(b_k) + \sum_{\substack{k=1 \\ b_k > \delta}}^{t} f(b_k),$$

For $b_k \leqslant \delta$, we get $f(b_k) < \varepsilon$, so

$$\sum_{\substack{k=1 \\ b_k \leqslant \delta}}^{t} f(b_k) < \varepsilon t.$$

For $b_k > \delta$, we can write

$$b_k < \frac{b_k}{\delta} < 1 + \left[\frac{b_k}{\delta}\right],$$

where $[i]$ denotes the integer part of the real number i. Since f is increasing and subadditive, we can write

$$f(b_k) \leqslant f\left(1 + \left[\frac{b_k}{\delta}\right]\right) \leqslant f(1)\left(1 + \left[\frac{b_k}{\delta}\right]\right) \leqslant 2f(1)\frac{b_k}{\delta}.$$

So, we have

$$\sum_{\substack{k=1 \\ b_k > \delta}}^{t} f(b_k) \leqslant \frac{2f(1)}{\delta} \sum_{\substack{k=1 \\ b_k > \delta}}^{t} b_k.$$

Thus,

$$\frac{1}{t}\sum_{k=1}^{t} f\left[\mathscr{A}(a_k, a_k, \ldots, a_k, \xi)\right] \leqslant \varepsilon + \frac{2f(1)}{\delta} \cdot \frac{1}{t} \sum_{\substack{k=1 \\ \mathscr{A}(a_k, a_k, \ldots, a_k, \xi) > \delta}}^{t} \mathscr{A}(a_k, a_k, \ldots, a_k, \xi).$$

Since $(a_k) \in \mathscr{W}^{\mathscr{A}}$, we conclude that $(a_k) \in \mathscr{W}_f^{\mathscr{A}}$, and this completes the proof. □

The following example clearly demonstrates that the converse of Theorem 5 is not generally valid. Consider the $\mathscr{A}$-metric space in Example 1, and the sequence (a_k) defined by

$$a_k = \begin{cases} k, & k = j^3, \ j \in \mathbb{N}, \\ 0, & k \neq j^3. \end{cases}$$

Let the modulus be $f(u) = \frac{u}{1+u}$. Given that $f(0) = 0$, we can write

$$\begin{aligned} \frac{1}{t}\sum_{k=1}^{t} f\left[\mathscr{A}(a_k, a_k, \ldots, a_k, 0)\right] &= \frac{1}{t}\sum_{k=1}^{t} f((n-1)|a_k|) \\ &= \frac{1}{t}\sum_{\substack{k=1 \\ k=j^3}}^{t} f((n-1)k) + \frac{1}{t}\sum_{\substack{k=1 \\ k \neq j^3}}^{t} f(0), \end{aligned}$$

where $n \geqslant 2$ is a fixed positive integer. Consequently, this implies that

$$\frac{1}{t}\sum_{\substack{k=1 \\ k=j^3}}^{t} \frac{(n-1)k}{1+(n-1)k} < \frac{1}{t}\sum_{\substack{k=1 \\ k=j^3}}^{t} 1 \leqslant \frac{\sqrt[3]{t}}{t}.$$

Thus, the limit as $t \to \infty$ gives us a value approaching 0, i.e., $(a_k) \in \mathscr{W}_f^{\mathscr{A}}$. However, if we compute

$$\begin{aligned}\frac{1}{t}\sum_{k=1}^{t}\mathscr{A}(a_k,a_k,\ldots,a_k,0) &= \frac{1}{t}\sum_{\substack{k=1\\k=j^3}}^{t}(n-1)\cdot k+\frac{1}{t}\sum_{\substack{k=1\\k\neq j^3}}^{t}0\\ &= \frac{n-1}{t}\cdot\left(1^3+2^3+3^3+\cdots+\left(\sqrt[3]{t}\right)^3\right)\\ &= \frac{(n-1)}{t}\left[\frac{[\sqrt[3]{t}]([\sqrt[3]{t}]+1)}{2}\right]^2\\ &\geqslant \frac{(n-1)}{t}\left[\frac{(\sqrt[3]{t}-1)(\sqrt[3]{t})}{2}\right]^2\\ &= \frac{(n-1)}{t}\frac{(t^{\frac{4}{3}}-2t+t^{\frac{2}{3}})}{4},\end{aligned}$$

where $n \geqslant 2$ is a fixed positive integer. Taking the limit as $t \to \infty$, we observe that $(a_k) \notin \mathscr{W}^{\mathscr{A}}$.

The following theorem demonstrates that the converse of the aforementioned theorem holds under some additional conditions.

THEOREM 6. *Let $(X,\mathscr{A})$ be an $\mathscr{A}$-metric space and let f be an unbounded modulus function. Then if $\lim_{r\to\infty}\frac{f(r)}{r} > 0$ is satisfied then $\mathscr{W}_f^{\mathscr{A}} \subset \mathscr{W}^{\mathscr{A}}$.*

Proof. Let $\lim_{r\to\infty}\frac{f(r)}{r} > 0$. Then we get $\theta = \lim_{r\to\infty}\frac{f(r)}{r} = \inf\{\frac{f(r)}{r} : r > 0\}$ by proposition 1 in [30] and from this we get $f(r) \geqslant \theta \cdot r$ and since $\theta > 0$, we get

$$\frac{1}{t}\sum_{t=1}^{t}\mathscr{A}(a_k,a_k,\ldots,a_k,\xi) \leqslant \frac{1}{\theta}\frac{1}{t}\sum_{k=1}^{t}f\left[\mathscr{A}(a_k,a_k,\ldots,a_k,\xi)\right]$$

It implies that if $(a_k) \in \mathscr{W}_f^{\mathscr{A}}$, then $(a_k) \in \mathscr{W}^{\mathscr{A}}$. □

The following theorem investigates the connection between $\mathscr{A}_f$-strong Cesàro summability and $\mathscr{A}_f$-statistical convergence within the framework of $\mathscr{A}$-metric spaces.

THEOREM 7. *Let (a_k) be a sequence in an $\mathscr{A}$-metric space $(X,\mathscr{A})$, and let f be an unbounded modulus function. Then if $\lim_{r\to\infty}\frac{f(r)}{r} > 0$ is satisfied and (a_k) is $\mathscr{A}_f$-strongly Cesàro summable to a point $\xi \in X$, then the sequence (a_k) is $\mathscr{A}_f$-statistically convergent to ξ.*

Proof. Let $(X,\mathscr{A})$ be an $\mathscr{A}$-metric space, and let the sequence $(a_k) \in X$ be $\mathscr{A}_f$-strongly Cesàro summable to an element $\xi \in X$, and let $\varepsilon > 0$. The set $\mathscr{K}(t)$ defined by

$$\mathscr{K}(t) = \{k \leqslant t : \mathscr{A}(a_k,a_k,\ldots,a_k,\xi) \geqslant \varepsilon\}$$

Since f is increasing and $f(|\mathscr{K}(t)|) \leqslant |\mathscr{K}(t)| f(1)$, we write

$$\begin{aligned}\frac{1}{t}\sum_{k=1}^{t} f\left[\mathscr{A}(a_k,a_k,\ldots,a_k,\xi)\right] &\geqslant \frac{1}{t}\sum_{\substack{k=1\\ k\in\mathscr{K}(t)}} f\left[\mathscr{A}(a_k,a_k,\ldots,a_k,\xi)\right]\\ &\geqslant \frac{1}{t} \mid \mathscr{K}(t) \mid f(\varepsilon)\\ &\geqslant \frac{f(\mid \mathscr{K}(t) \mid)}{f(t)} \frac{f(t)}{t} \frac{f(\varepsilon)}{f(1)}\end{aligned}$$

Since $\lim_{r\to\infty} \frac{f(r)}{r} > 0$ and $(a_k) \in \mathscr{W}_f^{\mathscr{A}}$, we get $(a_k) \in \mathscr{S}\mathscr{A}_f$. Hence, the proof is complete. □

For $f(u) = u$, taking $p = 1$ specifically in Theorem 2.15 (i) in [39], the following result is obtained.

COROLLARY 2. *Let (a_k) be a sequence in an $\mathscr{A}$ - metric space $(X,\mathscr{A})$. If the sequence (a_k) is strongly Cesàro summable to an element $\xi \in X$, then it is statistically convergent to ξ.*

COROLLARY 3. *Let (a_k) be a sequence in an $\mathscr{A}$ - metric space $(X,\mathscr{A})$. If the sequence (a_k) is bounded and $\mathscr{A}_f$-statistically convergent to an element $\xi \in X$, then it is $\mathscr{A}_f$-strongly Cesàro summable to ξ.*

Proof. Let $(X,\mathscr{A})$ is an $\mathscr{A}$ - metric space and let a sequence $(a_k) \in X$ be bounded and $\mathscr{A}_f$-statistically convergent. By Corollary 1, (a_k) is also statistically convergent. From Theorem 2.15 (ii) in [39], it is clear that a bounded and statistically convergent sequence is strongly Cesàro summable. From Theorem 5, we conclude that $\mathscr{W}^{\mathscr{A}} \subset \mathscr{W}_f^{A}$, which implies that (a_k) is $\mathscr{A}_f$-strongly Cesàro summable. □

REMARK 1. The condition $\lim_{r\to\infty} \frac{f(r)}{r} > 0$ in Theorem 7 is essential and cannot be omitted. To illustrate, consider an unbounded modulus f such that $\lim_{r\to\infty} \frac{f(r)}{r} = 0$. In this case, $\mathscr{A}_f$-strongly Cesàro summable sequence need not be $\mathscr{A}_f$-statistically convergent. For example, consider the $\mathscr{A}$-metric space in Example 1, and the sequence (a_k) defined by

$$a_k = \begin{cases} 1, & k = j^2,\ j \in \mathbb{N}, \\ 0, & k \neq j^2. \end{cases}$$

We take the modulus $f(u) = \log(u+1)$, then we get $\lim\limits_{r\to\infty} \dfrac{\log(r+1)}{r} = 0$. Then

$$\begin{aligned}\frac{1}{t}\sum_{k=1}^{t} f(\mathscr{A}(a_k,a_k,\ldots,a_k,0)) &= \frac{1}{t}\sum_{k=1}^{t} f((n-1)\mid a_k \mid)\\ &= \frac{1}{t}\sum_{\substack{k=1\\ k=j^2}}^{t} f(n-1) + \frac{1}{t}\sum_{\substack{k=1\\ k\neq j^2}}^{t} f(0)\end{aligned}$$

$$= \frac{1}{t} \sum_{\substack{k=1 \\ k=j^2}}^{t} \log n$$

$$= \frac{\sqrt{t}}{t} \log n \to 0 \quad \text{as} \quad t \to \infty$$

where $n \geqslant 2$ is a fixed positive integer. So we have $(a_k) \in \mathscr{W}_f^{\mathscr{A}}$. However, since

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{f(t)} \cdot |\{k : \mathscr{A}(a_k, a_k, \ldots, a_k, 0) \geqslant \varepsilon\}| \\
&= \lim_{t\to\infty} \frac{1}{f(t)} \cdot f(|\{k : |a_k| \geqslant \frac{\varepsilon}{n-1}\}|) \\
&= \lim_{t\to\infty} \frac{\log(|\{k : |a_k| \geqslant \frac{\varepsilon}{n-1}\}| + 1)}{\log(t+1)} \\
&= \lim_{t\to\infty} \frac{\log(\sqrt{t}+1)}{\log(t+1)} = \frac{1}{2} \neq 0,
\end{aligned}$$

we conclude that $(a_k) \notin \mathscr{S}\mathscr{A}_f$.

3. Conclusions

The concept of a metric has been studied in various fields such as mathematics, basic sciences, medicine, computer science and artificial intelligence, and engineering. In particular, the notion of a generalized metric has provided a suitable framework for the investigation of various problems in these domains. In the field of mathematics, studies on fixed point theory have been especially extensive in the context of generalized metric spaces. In recent years, significant concepts of summability theory have been extensively studied within generalized metric spaces. In this study, we introduce the concepts of $\mathscr{A}_f$-statistical convergence, $\mathscr{A}_f$-statistical boundedness, and $\mathscr{A}_f$-strong Cesàro convergence in $\mathscr{A}$-metric spaces, which are generalizations of metric spaces, by using the modulus function, and we analyze the relationships among these concepts. The results obtained in this study are more comprehensive than those in [21]. Furthermore, when $f(u) = u$, the concept of $\mathscr{A}_f$-statistical convergence reduces to the concept of statistical convergence, making some of the results here more general than those in [39].

Moreover, the findings show that, in the analysis of complex structures where classical metric spaces fall short, particularly in cases involving high measurement uncertainty, the ability of $\mathscr{A}$-metric spaces to model multi-point relationships, together with the flexibility offered by the modulus function, demonstrates that the concepts of statistical convergence, statistical boundedness, and Cesàro convergence can be used as effective and flexible tools for modeling and analyzing various problems encountered across different disciplines. In this respect, beyond its theoretical contributions, this study also offers potential application areas for future research in fields such as optimization, computer science, system stability and control theory, data science, and signal processing.

Building on these results, future studies may focus on investigating the generalized convergence behaviors of double sequences in generalized metric spaces that respect the modulus function, thereby further expanding the theoretical framework and practical applicability of these concepts.

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