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THE ANGER DIFFERENCE EQUATION

MEGAN BENKENDORF, TOM CUCHTA*, TIM LUND AND GAURAVPREET SINGH

Abstract. A nonhomogeneous Bessel difference equation, dubbed the Anger difference equation, is solved and properties of its solution are investigated. Numerical evidence that the solution is oscillatory is presented and difficulties in finding a proof for it are discussed.

1. Introduction

We investigate the second-order nonhomogeneous difference equation

$$t(t-1)\Delta^2 y(t-2)+t\Delta y(t-1)+t(t-1)y(t-2)-\nu^2 y(t)=\frac{(t-\nu)\sin(\pi\nu)}{\pi}, \qquad (1)$$

where $\nu\in\mathbb{R}\setminus\mathbb{Z}$, which we dub the Anger difference equation. We argue in this article that (1) is the discrete analogue of the classical Anger differential equation [15, 11.10.5], [12, 8.584]

$$t^2y''(t)+ty'(t)+\left(t^2-\nu^2\right)y(t)=\frac{(t-\nu)\sin(\pi\nu)}{\pi}, \qquad (2)$$

which was introduced in 1855 [2] and has a long history. Recently, the solutions $\mathfrak{J}_\nu$ of (2), called Anger functions, have been applied to topics such as techniques for the generation of X-rays [11], the reflection of cylindrical waves at planar interfaces [19], and truncated circular electromagnetic structures [9].

The $\mathfrak{J}_\nu$ have series representation [12, 8.581]

$$\mathfrak{J}_\nu(z)=\cos\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{2^{2k}\Gamma\left(k+\frac{\nu}{2}+1\right)\Gamma\left(k-\frac{\nu}{2}+1\right)} +\sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{2^{2k+1}\Gamma\left(k+\frac{\nu}{2}+\frac{3}{2}\right)\Gamma\left(k-\frac{\nu}{2}+\frac{3}{2}\right)}. \qquad (3)$$

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* Corresponding author.

When $\nu \in \mathbb{Z}$, the right-hand side of (1) vanishes and the resulting homogeneous difference equation is the Bessel difference equation, which was the subject of [6]. There, the discrete Bessel functions of the first kind, denoted by J_ν, were first defined and their properties explored. Such J_ν have since been used to solve discrete wave equations [4] [17, Theorem 3.1], extended to the "modified" discrete Bessel functions I_ν for solving semidiscrete diffusion problems [18, Lemma 2.2], used in the discrete time heat kernel in field theories of physics [13], and generalized to a broader class of domains to include the quantum calculus via the theory of time scales calculus [20]. For this reason, we shall assume that ν is non-integer throughout this work.

The new contributions appear in both Sections 3 and 4. In Section 3, we solve the Anger difference equation, find its hypergeometric series representation, and find two recurrences for the discrete Anger functions. In Section 4, we discuss the difficulties in proving that the discrete Anger functions are oscillatory.

2. Preliminaries and definitions

The difference operator Δ is defined by $\Delta f(t) = f(t+1) - f(t)$. We refer readers to the monograph [14] for a thorough introduction to the subject. We will consider the homogeneous second-order Sturm–Liouville difference equation of the form

$$\Delta\big(p(t)\Delta x(t)\big) + q(t)x(t+1) = 0 \tag{4}$$

and its nonohomogeneous counterpart

$$\Delta\big(p(t)\Delta x(t)\big) + q(t)x(t+1) = f(t). \tag{5}$$

The discrete Laplace transform (a shifted and scaled $\mathscr{Z}$-transform [5, Section 3.1]) is defined by

$$\mathscr{L}_{\mathbb{N}_0}\{f\}(z) = \sum_{k=t_0}^{\infty} \frac{f(k)}{(1-z)^k}. \tag{6}$$

We recall that the well-known Γ function [3] obeys the recurrence

$$x\Gamma(x) = \Gamma(x+1), \tag{7}$$

which may be rearranged to

$$\frac{1}{\Gamma(x)} = \frac{x}{\Gamma(x+1)}. \tag{8}$$

For $z \in \mathbb{C} \setminus \mathbb{Z}$, the Γ function also satisfies the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{9}$$

For $n \in \{0,1,2,\ldots\}$, the Pochhammer symbol $(a)_n$ is defined by

$$(a)_n = a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{10}$$

and the related falling powers are defined as the polynomial

$$t^{\underline{n}} = t(t-1)(t-2)\dots(t-n+1),$$

which are the discrete analogue of power functions since $\Delta t^{\underline{n}} = nt^{\underline{n-1}}$. The falling powers satisfy the formula

$$t^{\underline{n}}(t-n)^{\underline{m}} = t^{\underline{n+m}}. \tag{11}$$

The classical generalized hypergeometric series ${}_p\mathscr{F}_q$ is defined by the series

$${}_p\mathscr{F}_q(a_1,\dots,a_p;b_1,\dots,b_q;t) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k t^k}{(b_1)_k \dots (b_q)_k k!}. \tag{12}$$

The discrete generalized hypergeometric series were introduced in [7] and are defined by

$${}_pF_q\left(a_1,\dots,a_p;b_1,\dots,b_q;t,n,\xi\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\xi^k t^{\underline{nk}}}{k!}. \tag{13}$$

It is known that (12) and (13) are related by the formula [10, (5)]

$$\begin{aligned} &{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;t,n,\xi) \\ &\quad = {}_{p+n}\mathscr{F}_q\left(a_1,\dots,a_p,-\frac{t}{n},\dots,\frac{-t+n-1}{n};b_1,\dots,b_q;\xi(-n)^n\right). \end{aligned} \tag{14}$$

The Mathematica command `FunctionExpand[AngerJ[ν, z]]` reveals that (3) has classical hypergeometric representation of the form

$$\begin{aligned} \mathfrak{J}_\nu(z) = &\frac{\cos\left(\frac{\pi\nu}{2}\right)}{\Gamma\left(1-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\nu}{2}\right)} {}_1\mathscr{F}_2\left(1;1-\frac{\nu}{2},1+\frac{\nu}{2};-\frac{z^2}{4}\right) \\ &+ \frac{z\sin\left(\frac{\pi\nu}{2}\right)}{2\Gamma\left(\frac{3-\nu}{2}\right)\Gamma\left(\frac{3+\nu}{2}\right)} {}_1\mathscr{F}_2\left(1;\frac{3-\nu}{2},\frac{3+\nu}{2};-\frac{z^2}{4}\right), \end{aligned} \tag{15}$$

which can be found online but seems absent from the literature. Below, we establish the discrete analogue of (15), and its simple proof can be directly adapted to prove (15) itself. Similarly, the Mathematica command `LaplaceTransform[AngerJ[ν, t], t, z]` reveals

$$\begin{aligned} \mathscr{L}\{\mathfrak{J}_\nu\}(z) = &\frac{\cos\left(\frac{\pi\nu}{2}\right)}{z\Gamma\left(1-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\nu}{2}\right)} {}_3\mathscr{F}_2\left(\frac{1}{2},1,1;1-\frac{\nu}{2};1+\frac{\nu}{2};-\frac{1}{z^2}\right) \\ &+ \frac{\sin\left(\frac{\pi\nu}{2}\right)}{2z^2\Gamma\left(\frac{3}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{3}{2}+\frac{\nu}{2}\right)} {}_3\mathscr{F}_2\left(1,1,\frac{3}{2};\frac{3}{2}-\frac{\nu}{2},\frac{3}{2}+\frac{\nu}{2};-\frac{1}{z^2}\right), \end{aligned} \tag{16}$$

which is similar to a more difficult formula appearing in [16, 3.18] involving the Meijer G function. Our proof below of the discrete analogue of (15) can be adapted to prove it directly for computing the Laplace transform of the $\mathfrak{J}_\nu$.

The $\mathfrak{J}_\nu$ also obey numerous recurrence relations [12, 8.582] such as

$$z\mathfrak{J}_{\nu-1}(z) + z\mathfrak{J}_{\nu+1}(z) = 2\nu\mathfrak{J}_\nu(z) - \frac{2\sin(\pi\nu)}{\pi}, \tag{17}$$

and

$$\mathfrak{J}_{\nu-1}(z) - \mathfrak{J}_{\nu+1}(z) = 2\mathfrak{J}'_\nu(z). \tag{18}$$

3. The discrete Anger functions

Analogous to (3), we define the discrete Anger function $\mathbf{J}_\nu$ by

$$\mathbf{J}_\nu(t) := \cos\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{\underline{2k}}}{2^{2k}\Gamma\left(k+\frac{\nu}{2}+1\right)\Gamma\left(k-\frac{\nu}{2}+1\right)} + \sin\left(\frac{\pi\nu}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^{\underline{2k+1}}}{2^{2k+1}\Gamma\left(k+\frac{\nu}{2}+\frac{3}{2}\right)\Gamma\left(k-\frac{\nu}{2}+\frac{3}{2}\right)}. \tag{19}$$

We present some plots of the $\mathbf{J}_\nu$ below in Figure 1. We now formalize the idea that (19) is the discrete analogue of (3).

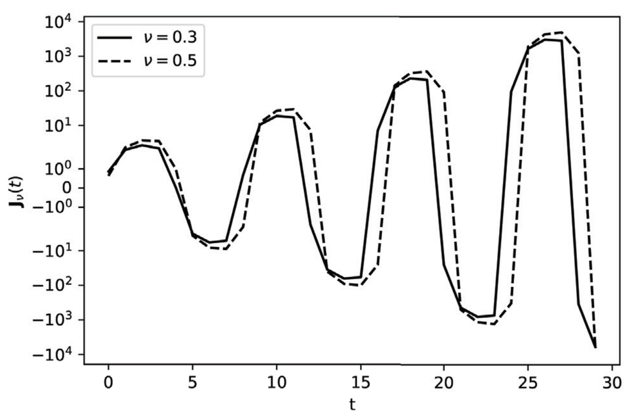

Figure 1: *Plot of the discrete Anger functions of orders $\nu = 0.3$ and $\nu = 0.5$ with a logarithmic y-axis.*

THEOREM 1. *The function $y(t) = \mathbf{J}_\nu(t)$ solves* (1).

Proof. Define

$$\alpha_k := \frac{\cos\left(\frac{\pi\nu}{2}\right)(-1)^k}{2^{2k}\Gamma\left(k+\frac{\nu}{2}+1\right)\Gamma\left(k-\frac{\nu}{2}+1\right)} \quad \text{and} \quad \beta_k := \frac{\sin\left(\frac{\pi\nu}{2}\right)(-1)^k}{2^{2k+1}\Gamma\left(k+\frac{\nu}{2}+\frac{3}{2}\right)\Gamma\left(k-\frac{\nu}{2}+\frac{3}{2}\right)},$$

so that

$$\mathbf{J}_\nu(t) = \sum_{k=0}^{\infty} \alpha_k t^{\underline{2k}} + \sum_{k=0}^{\infty} \beta_k t^{\underline{2k+1}} = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k t^{\underline{2k}} + \beta_0 t + \sum_{k=1}^{\infty} \beta_k t^{\underline{2k+1}}.$$

Computing the first two differences, we have

$$\Delta \mathbf{J}_\nu(t) = \sum_{k=1}^{\infty} 2k\alpha_k t^{\underline{2k-1}} + \beta_0 + \sum_{k=1}^{\infty} (2k+1)\beta_k t^{\underline{2k}},$$

and

$$\Delta^2 \mathbf{J}_\nu(t) = \sum_{k=1}^{\infty} (2k)(2k-1)\alpha_k t^{\underline{2k-2}} + \sum_{k=1}^{\infty} (2k+1)(2k)\beta_k t^{\underline{2k-1}}.$$

Thus using (11),

$$t(t-1)\mathbf{J}_\nu(t-2) = \sum_{k=0}^{\infty} \alpha_k t^{\underline{2k+2}} + \sum_{k=0}^{\infty} \beta_k t^{\underline{2k+3}} = \sum_{k=1}^{\infty} \alpha_{k-1} t^{\underline{2k}} + \sum_{k=1}^{\infty} \beta_{k-1} t^{\underline{2k+1}},$$

$$t\Delta \mathbf{J}_\nu(t-1) = \sum_{k=1}^{\infty} 2k\alpha_k t^{\underline{2k}} + \beta_0 t + \sum_{k=1}^{\infty} (2k+1)\beta_k t^{\underline{2k+1}},$$

and

$$t(t-1)\Delta^2 \mathbf{J}_\nu(t-2) = \sum_{k=1}^{\infty} (2k)(2k-1)\alpha_k t^{\underline{2k}} + \sum_{k=1}^{\infty} (2k+1)(2k)\beta_k t^{\underline{2k+1}}.$$

Substituting these quantities into the left-hand side of (1), we obtain

$$\beta_0 t - \nu^2(\alpha_0 + \beta_0 t) + \sum_{k=1}^{\infty} \Big[(2k)(2k-1)\alpha_k + 2k\alpha_k + \alpha_{k-1} - \nu^2\alpha_k\Big] t^{\underline{2k}}$$
$$+ \sum_{k=1}^{\infty} \Big[(2k)(2k+1)\beta_k + (2k+1)\beta_k + \beta_{k-1} - \nu^2\beta_k\Big] t^{\underline{2k+1}}. \quad (20)$$

To simplify this, we use (8) to obtain

$$\alpha_{k-1} = \frac{\cos\left(\frac{\pi\nu}{2}\right)(-1)^{k-1}}{2^{2k-2}\Gamma\left(k+\frac{\nu}{2}\right)\Gamma\left(k-\frac{\nu}{2}\right)}$$
$$= \frac{-4\left(k+\frac{\nu}{2}\right)\left(k-\frac{\nu}{2}\right)\cos\left(\frac{\pi\nu}{2}\right)(-1)^k}{2^{2k}\Gamma\left(k+\frac{\nu}{2}+1\right)\Gamma\left(k-\frac{\nu}{2}+1\right)} = -4\left(k+\frac{\nu}{2}\right)\left(k-\frac{\nu}{2}\right)\alpha_k$$

and similarly, $\beta_{k-1} = -4\left(k+\frac{\nu}{2}+\frac{1}{2}\right)\left(k-\frac{\nu}{2}+\frac{1}{2}\right)\beta_k$. Now from (20), we compute

$$2k(2k-1)\alpha_k + 2k\alpha_k + \alpha_{k-1} - \nu^2\alpha_k$$
$$= \alpha_k\left(2k(2k-1) + 2k - 4\left(k+\frac{\nu}{2}\right)\left(k-\frac{\nu}{2}\right) - \nu^2\right) = 0$$

and

$$\begin{aligned}
&2k(2k+1)\beta_k+(2k+1)\beta_k+\beta_{k-1}-\nu^2\beta_k \\
&\quad=\beta_k\left(2k(2k+1)+(2k+1)-4\left(k+\frac{\nu}{2}+\frac{1}{2}\right)\left(k-\frac{\nu}{2}+\frac{1}{2}\right)-\nu^2\right) \\
&\qquad=\beta_k\left(4k^2+2k+2k+1-4\left(k^2+k-\frac{\nu^2}{4}+\frac{1}{4}\right)-\nu^2\right)=0.
\end{aligned}$$

Using (7), (9), the double angle identity for sine, and the cofunction identity relating sine and cosine yields

$$\begin{aligned}
\frac{(1-\nu^2)\sin\left(\frac{\pi\nu}{2}\right)t}{2\Gamma\left(\frac{\nu}{2}+\frac{3}{2}\right)\Gamma\left(-\frac{\nu}{2}+\frac{3}{2}\right)}&=\frac{(1-\nu^2)\sin\left(\frac{\pi\nu}{2}\right)t}{2\left(\frac{\nu}{2}+\frac{1}{2}\right)\left(\frac{\nu}{2}-\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}-\frac{1}{2}\right)\Gamma\left(1-\left(\frac{\nu}{2}-\frac{1}{2}\right)\right)} \\
&=\frac{(1-\nu^2)\sin\left(\frac{\pi\nu}{2}\right)t\sin\left(\frac{\pi(\nu-1)}{2}\right)}{\frac{1}{2}(\nu^2-1)\pi} \\
&=\frac{2\sin\left(\frac{\pi\nu}{2}\right)\cos\left(\frac{\pi\nu}{2}\right)t}{\pi}=\frac{\sin(\pi\nu)t}{\pi}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
-\frac{\nu^2\cos\left(\frac{\pi\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}+1\right)\Gamma\left(-\frac{\nu}{2}+1\right)}&=-\frac{\nu^2\cos\left(\frac{\pi\nu}{2}\right)}{\frac{\nu}{2}\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(1-\frac{\nu}{2}\right)} \\
&=-\frac{2\nu\cos\left(\frac{\pi\nu}{2}\right)\sin\left(\frac{\pi\nu}{2}\right)}{\pi}=-\frac{\nu\sin(\pi\nu)}{\pi}.
\end{aligned}$$

Therefore, we finally obtain

$$\begin{aligned}
\beta_0t-\nu^2(\alpha_0+\beta_0t)&=(1-\nu^2)\beta_0t-\nu^2\alpha_0 \\
&=\frac{(1-\nu^2)\sin\left(\frac{\pi\nu}{2}\right)t}{2\Gamma\left(\frac{\nu}{2}+\frac{3}{2}\right)\Gamma\left(-\frac{\nu}{2}+\frac{3}{2}\right)}-\frac{\nu^2\cos\left(\frac{\pi\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}+1\right)\Gamma\left(-\frac{\nu}{2}+1\right)} \\
&=\frac{\sin(\pi\nu)t}{\pi}-\frac{\nu\sin(\pi\nu)}{\pi}=\frac{(t-\nu)\sin(\pi\nu)}{\pi},
\end{aligned}$$

completing the proof. □

We now prove the discrete analogue of (15).

THEOREM 2. *The $\mathbf{J}_\nu$ have discrete hypergeometric series representation*

$$\begin{aligned}
\mathbf{J}_\nu(t)&=\frac{\cos\left(\frac{\pi\nu}{2}\right)}{\Gamma\left(1-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\nu}{2}\right)}{}_1F_2\left(1;1-\frac{\nu}{2},1+\frac{\nu}{2};t,2,-\frac{1}{4}\right) \\
&\quad+\frac{\sin\left(\frac{\pi\nu}{2}\right)t}{2\Gamma\left(\frac{3}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{3}{2}+\frac{\nu}{2}\right)}{}_1F_2\left(1;\frac{3}{2}-\frac{\nu}{2},\frac{3}{2}+\frac{\nu}{2};t-1,2,-\frac{1}{4}\right). \qquad (21)
\end{aligned}$$

Proof. Using (10), we rewrite the four Γ functions appearing in the series (19) as

$$\frac{1}{\Gamma\left(k \pm \frac{\nu}{2}+1\right)}=\frac{1}{\left(\pm\frac{\nu}{2}+1\right)_k \Gamma\left(\pm\frac{\nu}{2}+1\right)},$$

and

$$\frac{1}{\Gamma\left(k \pm \frac{\nu}{2}+\frac{3}{2}\right)}=\frac{1}{\left(\pm\frac{\nu}{2}+\frac{3}{2}\right)_k \Gamma\left(\pm\frac{\nu}{2}+\frac{3}{2}\right)}.$$

The proof follows by observing that the series defining the right-hand side of (21) matches the series defining $\mathbf{J}_\nu$ term-by-term. □

The relationship (14) immediately yields the following corollary expressing the discrete Anger function in terms of the classical hypergeometric series.

COROLLARY 1. *The* $\mathbf{J}_\nu$ *have classical hypergeometric representation*

$$\begin{aligned}\mathbf{J}_\nu(t)=&\frac{\cos\left(\frac{\pi\nu}{2}\right)}{\Gamma\left(1-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\nu}{2}\right)}{}_3\mathscr{F}_2\left(1;-\frac{t}{2},\frac{-t+1}{2};1-\frac{\nu}{2},1+\frac{\nu}{2};-1\right)\\&+\frac{\sin\left(\frac{\pi\nu}{2}\right)t}{2\Gamma\left(\frac{3}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{3}{2}+\frac{\nu}{2}\right)}{}_3\mathscr{F}_2\left(1,\frac{-t+1}{2},\frac{-t}{2};\frac{3}{2}-\frac{\nu}{2},\frac{3}{2}+\frac{\nu}{2};-1\right).\end{aligned}$$

We can also now easily derive the discrete Laplace transform of the discrete Anger function in terms of the discrete hypergeometric function (13).

COROLLARY 2. *The discrete Laplace transform of the* $\mathbf{J}_\nu$ *is*

$$\begin{aligned}\mathscr{L}_{\mathbb{N}_0}\{\boldsymbol{J}_\nu(t)\}(z)=&\frac{\cos\left(\frac{\pi\nu}{2}\right)}{z\Gamma(1-\frac{\nu}{2})\Gamma(1+\frac{\nu}{2})}{}_3F_2\left(1,1,\frac{1}{2};1-\frac{\nu}{2},1+\frac{\nu}{2};-\frac{1}{z^2}\right)\\&+\frac{\sin\left(\frac{\pi\nu}{2}\right)}{2z^2\Gamma(\frac{3-\nu}{2})\Gamma(\frac{3+\nu}{2})}{}_3F_2\left(1,1,\frac{3}{2};\frac{3-\nu}{2},\frac{3+\nu}{2};-\frac{1}{z^2}\right).\end{aligned}$$

Proof. Using (6), we have

$$\mathscr{L}_{\mathbb{N}_0}\left\{t^{\underline{2k}}\right\}(z)=\frac{(2k)!}{z^{2k+1}}=\frac{1}{z}\frac{2^{2k}(1)_k\left(\frac{1}{2}\right)_k}{z^{2k}}$$

and

$$\mathscr{L}_{\mathbb{N}_0}\left\{t^{\underline{2k+1}}\right\}(z)=\frac{(2k+1)!}{z^{2k+2}}=\frac{1}{z^2}\frac{2^{2k}(2)_k\left(\frac{3}{2}\right)_k}{z^{2k}}.$$

Substitution of these formulas into the discrete hypergeometric form (21) and comparing with (13) completes the proof. □

The following result is the discrete analogue of (17).

THEOREM 3. *The* $\mathbf{J}_\nu$ *satisfy the recurrence*

$$t\mathbf{J}_{\nu-1}(t-1)+t\mathbf{J}_{\nu+1}(t-1)=2\nu\mathbf{J}_\nu(t)-\frac{2\sin(\pi\nu)}{\pi}.$$

Proof. By definition, (11), and taking $\alpha_k(\nu)$ and $\beta_k(\nu)$ to be the coefficients of $\mathbf{J}_\nu$ as was done in the proof of Theorem 1, we have

$$t\mathbf{J}_{\nu-1}(t-1)=\sum_{k=0}^{\infty}\alpha_k(\nu-1)t^{\underline{2k+1}}+\sum_{k=0}^{\infty}\beta_k(\nu-1)t^{\underline{2k+2}},$$

and

$$t\mathbf{J}_{\nu+1}(t-1)=\sum_{k=0}^{\infty}\alpha_k(\nu+1)t^{\underline{2k+1}}+\sum_{k=0}^{\infty}\beta_k(\nu+1)t^{\underline{2k+2}}.$$

Using (7) and the cofunction identities for sin and cos, we compute

$$\begin{aligned}\alpha_k(\nu-1)+\alpha_k(\nu+1)&=\frac{\sin\left(\frac{\pi\nu}{2}\right)(-1)^k\left[\left(k+\frac{\nu}{2}+\frac{1}{2}\right)-\left(k-\frac{\nu}{2}+\frac{1}{2}\right)\right]}{2^{2k}\Gamma\left(k+\frac{\nu}{2}+\frac{3}{2}\right)\Gamma\left(k-\frac{\nu}{2}+\frac{3}{2}\right)}\\&=\frac{\nu\sin\left(\frac{\pi\nu}{2}\right)(-1)^k}{2^{2k}\Gamma\left(k+\frac{\nu}{2}+\frac{3}{2}\right)\Gamma\left(k-\frac{\nu}{2}+\frac{3}{2}\right)}=2\nu\beta_k(\nu)\end{aligned}$$

and similarly, $\beta_k(\nu-1)+\beta_k(\nu+1)=2\nu\alpha_{k+1}(\nu)$. Thus, using these formulas and reindexing, we obtain

$$\begin{aligned}t\mathbf{J}_{\nu-1}(t-1)+t\mathbf{J}_{\nu+1}(t-1)&=2\nu\sum_{k=0}^{\infty}\beta_k(\nu)t^{\underline{2k+1}}+2\nu\sum_{k=0}^{\infty}\alpha_{k+1}(\nu)t^{\underline{2(k+1)}}\\&=-2\nu\alpha_0(\nu)t+2\nu\mathbf{J}_\nu(t).\end{aligned}$$

By (9), the double angle identity for sin yields

$$-2\nu\alpha_0(\nu)=\frac{-2\nu\cos\left(\frac{\pi\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}+1\right)\Gamma\left(-\frac{\nu}{2}+1\right)}=\frac{-2\nu\cos\left(\frac{\pi\nu}{2}\right)}{\frac{\nu}{2}\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(1-\frac{\nu}{2}\right)}=-\frac{2\sin(\pi\nu)}{\pi},$$

completing the proof. □

We now prove the discrete analogue of (18).

THEOREM 4. *The* $\mathbf{J}_\nu$ *satisfy the recurrence*

$$\boldsymbol{J}_{\nu-1}(t)+\boldsymbol{J}_{\nu+1}(t)=2\Delta\boldsymbol{J}_\nu(t).$$

Proof. Let $\alpha_k(\nu)$ and $\beta_k(\nu)$ be as in the proof of Theorem 1. First compute

$$\Delta\mathbf{J}_\nu(t)=\sum_{k=1}^{\infty}(2k)\alpha_k(\nu)t^{\underline{2k-1}}+\sum_{k=0}^{\infty}(2k+1)\beta_k(\nu)t^{\underline{2k}}.$$

Now compute

$$\alpha_k(\nu-1)-\alpha_k(\nu+1)=2(2k+1)\beta_k(\nu)$$

and

$$\beta_k(\nu-1)-\beta_k(\nu+1)=4(k+1)\alpha_{k+1}(\nu).$$

Thus,

$$\begin{aligned}\mathbf{J}_{\nu-1}(t)-\mathbf{J}_{\nu+1}(t)&=2\sum_{k=0}^{\infty}(2k+1)\beta_k(\nu)t^{\underline{2k}}+4\sum_{k=0}^{\infty}(k+1)\alpha_{k+1}(\nu)t^{\underline{2k+1}}\\&=2\Delta\sum_{k=0}^{\infty}\beta_k(\nu)t^{\underline{2k+1}}+2\sum_{k=1}^{\infty}(2k)\alpha_k(\nu)t^{\underline{2k-1}}\\&=2\Delta\sum_{k=0}^{\infty}\beta_k(\nu)t^{\underline{2k+1}}+2\Delta\sum_{k=1}^{\infty}\alpha_k(\nu)t^{\underline{2k}},\end{aligned}$$

completing the proof. □

4. Oscillation

It is known [6, Theorem 3] that the left-hand side of (1) can be put into Sturm–Liouville form (4), where

$$p(t)=\frac{2^{\nu-t}(t+1)}{\binom{t+1}{\nu}\binom{t+1}{\nu+1}}\binom{1+\nu+t}{t-\nu},\qquad q(t)=\frac{(t+2)(t+1)-\nu^2}{2(t+2)(t+1)}p(t).\tag{22}$$

This form was crucial to the result [6, Theorem 12], which proved that the discrete Bessel functions J_ν are oscillatory. There, the discrete Leighton–Wintner Theorem [5, Theorem 4.64] was used to establish oscillation, but that theorem assumes the difference equation is homogeneous, which does not apply to (1).

The forcing function in (1) is linear, and so results like [1, Corollary 1.15.6] and [1, Theorem 1.15.16] which assume f eventually continually oscillates do not apply. One promising and simple to compute method includes the following theorem.

THEOREM 5. [8, Corollary 2], [1, Corollary 1.15.4] *If* (4) *is oscillatory and has a solution x and T is an integer such that*

$$\sum_{k=T}^{\infty}f(k)x(k+1)=\infty\qquad \text{or}\qquad \sum_{k=T}^{\infty}f(k)x(k+1)=-\infty,$$

then (5) *is oscillatory.*

For us, consider $x(t)=J_\nu(t)$ and the $f(t)=\dfrac{(t-\nu)\sin(\pi\nu)}{\pi}$ is the right-hand side of (1). To apply Theorem 5, it must be shown that the function

$$g_\nu(m):=\frac{\sin(\pi\nu)}{\pi}\sum_{k=0}^{m}(k-\nu)J_\nu(k+1)$$

diverges to either ∞ or $-\infty$. However, Figure 2 suggests that the sum will diverge due to oscillation suggesting that this approach will not be successful.

Another theorem approaches the question using the Wronskian of functions f and g defined by [1, (1.15.3)]

$$W(f,g)(k) = p(k)\left[f(k+1)g(k) - f(k)g(k+1)\right],$$

where p comes from (4) or (5).

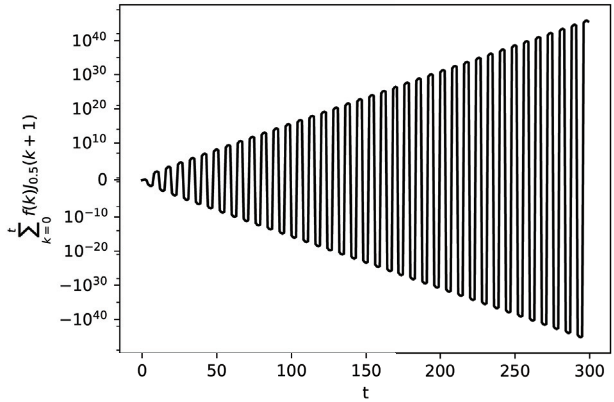

Figure 2: *The function $g_{0.5}$ plotted on $[0,300]\cap\mathbb{Z}$ with connecting lines between the function values.*

THEOREM 6. [1, Theorem 1.15.2] *For some solution x of* (4) *and some solution u of* (5)*, suppose $W(x,u)(k)$ is eventually of one sign. Then* (4) *is oscillatory if and only if* (5) *is oscillatory.*

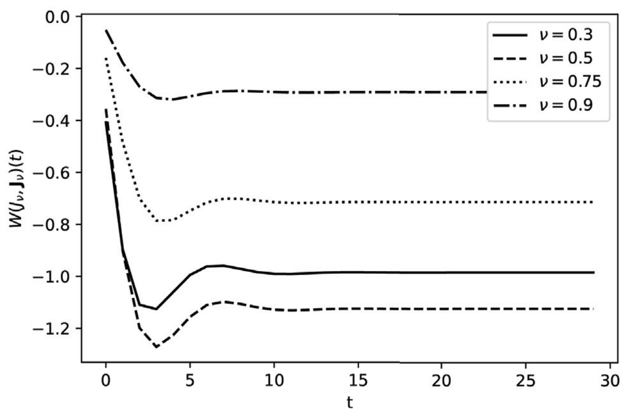

Figure 3: *The Wronskian of the Bessel and Anger functions at various values of the parameter ν.*

It is difficult to estimate the Wronskian symbolically in our case, but we provide numerical evidence in Figure 3 that the Wronskian seems to tend to a finite limit, suggesting that it is eventually of one sign. The numerical evidence suggests the following conjecture.

CONJECTURE 7. The discrete Anger functions $\mathbf{J}_\nu$ are oscillatory.

5. Conclusion

We have defined and explored the discrete Anger functions, particularly, its difference equation, some of its recurrences, its hypergeometric representation, and its Laplace transform. We have presented numerical evidence that these Anger functions are oscillatory.

Future work to build on these results could include a proof of the oscillation property. This could be accomplished by generalizing the Leighton–Wintner Theorem and using it to prove that the discrete Anger functions are oscillatory, mimicking the direct approach taken in [6]. Alternatively, a careful estimate of the Wronskian of the discrete Anger function and discrete Bessel function in conjunction with Theorem 6 could also establish oscillation of these functions.

The difference equation (1) has a second solution linearly independent from the discrete Anger functions, which could be called a "discrete Weber function". Other future work could include finding multivariable fully discrete or semidiscrete equations that the Anger functions solve.

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Megan Benkendorf
Department of Mathematics and Statistics
Missouri University of Science and Technology
Rolla, MO 65409
e-mail: meb2nc@mst.edu

Tom Cuchta
Department of Mathematics and Physics
Marshall University
1 John Marshall Drive, Huntington, WV 25755
e-mail: cuchta@marshall.edu

Tim Lund
Mathematical Sciences
Houghton University
1 Willard Avenue Houghton, NY 14744
and
Mathematical Sciences
Rensselaer Polytechnic Institute
110 Eighth Street, Troy, NY USA 12180
e-mail: lundt@rpi.edu

Gauravpreet Singh
Department of Mathematics
Washington & Jefferson College
60 S Lincoln St., Washington, PA 15301
e-mail: gauravpreets2002@gmail.com