Journal of Classical Analysis
Volume 27, Number 2 (2026), 197–208 doi:10.7153/jca-2026-27-14

ON COMPLEX INTERPOLATION IN VARIABLE LEBESGUE SPACES WITH MIXED NORM

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Abstract. In this paper, we prove a multiplicative interpolation inequality in variable Lebesgue spaces with mixed norm. By the method of complex interpolation, we define an interpolation spaces between variable Lebesgue spaces with mixed norm. As an application of the main result we establish an analog of Riesz-Thorin interpolation theorem in variable Lebesgue spaces with mixed norm.

1. Introduction

Interpolation is a useful tool for linear, bounded operators on Banach spaces. The idea is the following: if S is a linear operator which is bounded as an operator from X_0 to Y_0 and as an operator from X_1 to Y_1, then automatically S is linear and bounded as an operator from $X_{[\theta]}$ to $Y_{[\theta]}$ for all $0 < \theta < 1$, where $X_{[\theta]}$ and $Y_{[\theta]}$ are the complex interpolation spaces of the Banach couples (X_0, X_1) and (Y_0, Y_1). Here Banach couple means that both spaces are continuously embedded into some Hausdorff topological vector space. We refer to Bergh and Löfström [10] for an exposition on interpolation theory.

It is well known that the variable Lebesgue space in the literature for the first time was studied by Orlicz [27] in 1931. In [27], Hölders inequality for variable discrete Lebesgue space was proved. Orlicz also considered the variable Lebesgue space on the real line, and proved the Hölder inequality in this setting. However, this paper is essentially the only contribution of Orlicz to the study of the variable Lebesgue spaces (see also [24]). The next step in the development of the variable Lebesgue spaces came two decades later in the work of Nakano [25], [26]. Somewhat later, a more explicit version of such spaces, namely modular function spaces, were investigated by Musielak and others Polish mathematicians (see [23]). In particular, the variable Lebesgue spaces were objects of interest during the last three decades (see [14, 15, 19, 20]). The modern period in the study of variable Lebesgue spaces begun with the foundational papers of Sharapudinov [31] from 1979 and Kovacik and Rakosnik [21] from 1991. Interest in the variable Lebesgue spaces has increased since the 1990's because of their use in a variety of applications. Foremost among these is the mathematical modeling of electrorheological fluids, namely, fluids whose viscosity changes when exposed to an electric field:

Mathematics subject classification (2020): 46B70, 46E30.
Keywords and phrases: Variable Lebesgue space with mixed norm, interpolation inequality, interpolation space, complex interpolation method, Riesz-Thorin interpolation theorem, embedding theorems.
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Diening and Růžička [16]–[18], [29], [30], Butakin et al. in [12], Makharadze et al. in [22], Ouaziz et al. in [28], Acerbi and Mingione [1, 2] and so on. The variable Lebesgue spaces have also been used to model the behavior of other physical problems. Some examples include quasi-Newtonian fluids [32], the thermistor problem [33], fluid flow in porous media, magnetostatics [11], and the study of image processing [13] (see, also [8]).

It is well known that the structural properties of Lebesgue spaces with mixed norm and with constant exponent was studied in [9]. The variable Lebesgue space with mixed norm was introduced in [5]. Also, some structural properties of the variable Lebesgue spaces are studied in [3], [5], [6] and so on. Very recently, in [4] the variable Lebesgue space with mixed norm for exponents depending on all variables is defined. Also, in [4] embedding theorem for the function spaces with variable smoothness is proved. We observe that the embedding theorems between variable Lebesgue spaces with measures was proved in [7] and [15].

In this paper, we established some interpolation estimate in variable Lebesgue spaces with mixed norm. By the complex interpolation method we define an interpolation space between two variable Lebesgue spaces with mixed norm. Finally, we proved an analog of Riesz-Thorin interpolation theorem in variable Lebesgue spaces with mixed norm.

The paper is organized as follows. Section 2 contains some preliminaries using in the definition of the variable Lebesgue space with mixed norm. In Section 3 we proved some interpolation inequality in variable Lebesgue spaces with mixed norm. Also, we give an embedding theorem between intersection of two variable Lebesgue spaces with mixed norm and interpolation spaces. In Section 4, using the complex interpolation method, we define interpolation spaces between two variable Lebesgue spaces with mixed norm.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x=(x_1,\ldots,x_n)$ and Ω be a Lebesgue measurable subset of $\mathbb{R}^n$. Suppose that $\mathbf{p}(x) = (p_1(x_1,\ldots,x_n),$ $p_2(0,x_2,\ldots,x_n),\ldots,p_n(0,\ldots,0,x_n))$ is a vector function defined on $\mathbb{R}^n$ with Lebesgue measurable components $p_i\left(x^{(i)}\right)$ such that $1 \leqslant p_i\left(x^{(i)}\right) < \infty$ and $x^{(i)} = (0,\ldots,0,x_i,\ldots,$ $x_n)$ $(i=1,\ldots,n)$. In this paper, all sets and functions are supposed to be Lebesgue measurable and $x^{(1)}=x$, $x^{(n)}=(0,\ldots,0,x_n)$.

Throughout this paper $\overline{p}_i = \underset{x^{(i)}\in\mathbb{R}^{n-i+1}}{\operatorname{ess\,sup}}\, p_i(x^{(i)})$, $\underline{p}_i = \underset{x^{(i)}\in\mathbb{R}^{n-i+1}}{\operatorname{ess\,inf}}\, p_i(x^{(i)})$, and $p_n\left(x^{(n)}\right)$ $= p_n(x_n)$. We denote by $\mathbf{p}'(x) = \left(p_1'(x), p_2'\left(x^{(2)}\right),\ldots,p_n'\left(x^{(n)}\right)\right)$ the conjugate exponent vector-function defined by $\dfrac{1}{\mathbf{p}(x)} + \dfrac{1}{\mathbf{p}'(x)} = \mathbf{1}$, $x\in\mathbb{R}^n$, i.e., $\dfrac{1}{p_i\left(x^{(i)}\right)} + \dfrac{1}{p_i'\left(x^{(i)}\right)} = 1$, $i=1,\ldots,n$ and $\mathbf{1}=(1,\ldots,1)$.

By $L_{(p_1(x),x_1)}(\mathbb{R}^n)$ we denote the space of all measurable functions on $\mathbb{R}^n$ such that for some $\lambda_1 > 0$

$$\left(I_{p_1,x_1} f\right)(x_2,\dots,x_n) = \int_{\mathbb{R}} \left(\frac{|f(x)|}{\lambda_1}\right)^{p_1(x)} dx_1 < \infty.$$

The expression

$$\|f\|_{L_{(p_1(x),x_1)}(\mathbb{R}^n)} = \|f\|_{p_1(\cdot),x_1} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}} \left(\frac{|f(x)|}{\lambda}\right)^{p_1(x)} dx_1 \leqslant 1\right\}$$

is a norm in $L_{(p_1(x),x_1)}(\mathbb{R}^n)$ with respect to the variable x_1. It is obvious that the result is a function of variables $x_2,\dots,x_n$. So, $\|f\|_{p_1(\cdot),x_1} = \|f\|_{p_1(\cdot),x_1}(x_2,\dots,x_n)$.

Further, by $L_{\left(p_1(x),\, p_2\left(x^{(2)}\right),x_1,x_2\right)}(\mathbb{R}^n)$ we denote the space of all measurable functions on $\mathbb{R}^n$ such that for some $\lambda_2 > 0$

$$\left(I_{p_2,x_2} f\right)(x_3,\dots,x_n) = \int_{\mathbb{R}} \left(\frac{\|f\|_{p_1(\cdot),x_1}(x_2,\dots,x_n)}{\lambda_2}\right)^{p_2\left(x^{(2)}\right)} dx_2 < \infty.$$

The expression

$$\|f\|_{L_{\left(p_1(x),\, p_2\left(x^{(2)}\right),x_1,x_2\right)}(\mathbb{R}^n)} = \left\|\|f\|_{p_1(\cdot),x_1}\right\|_{p_2(\cdot),x_2}$$

$$= \inf\left\{\mu > 0 : \int_{\mathbb{R}} \left(\frac{\|f\|_{p_1(\cdot),x_1}}{\mu}\right)^{p_2\left(x^{(2)}\right)} dx_2 \leqslant 1\right\}$$

is a norm in $L_{\left(p_1(x),\, p_2\left(x^{(2)}\right),x_1,x_2\right)}(\mathbb{R}^n)$ with respect to the variable x_2. It is obvious that the result is a function of variables $x_3,\dots,x_n$. So,

$$\left\|\|f\|_{p_1(\cdot),x_1}\right\|_{p_2(\cdot),x_2} = \left\|\|f\|_{p_1(\cdot),x_1}\right\|_{p_2(\cdot),x_2}(x_3,\dots,x_n).$$

Next, by $L_{\left(p_1(x),\, p_2\left(x^{(2)}\right),\dots,p_n(x_n),x_1,x_2,\dots,x_n\right)}(\mathbb{R}^n)$ we denote the space of all measurable functions on $\mathbb{R}^n$ such that for some $\lambda_{\mathrm{n}-1}>0$

$$\left(I_{p_n,x_n} f\right)(x_n) = \int_{\mathbb{R}} \left(\frac{\|\dots\|\,\|f\|_{p_1(\cdot),x_1}\|_{p_2(\cdot),x_2}\cdots\|_{p_{n-1}(\cdot),x_{n-1}}}{\lambda_{n-1}}\right)^{p_n(x_n)} dx_n < \infty.$$

The expression

$$\|f\|_{L_{\mathbf{p}(\cdot)}(\mathbb{R}^n)} = \|f\|_{\mathbf{p}(\cdot)} = \|\dots\|\,\|f\|_{p_1(\cdot),x_1}\|_{p_2(\cdot),x_2}\cdots\|_{p_n(\cdot),x_n}$$

$$= \inf\left\{\nu > 0 : \int_{\mathbb{R}} \left(\frac{\|\dots\|\,\|f\|_{p_1(\cdot),x_1}\|_{p_2(\cdot),x_2}\cdots\|_{p_{n-1}(\cdot),x_{n-1}}}{\nu}\right)^{p_n(x_n)} dx_n \leqslant 1\right\}$$

defines a norm in $L_{\mathbf{p}(x)}(\mathbb{R}^n)$.

REMARK 1. Let $\mathbf{p}(x)=(p_1,\dots,p_n)=\mathbf{p}\geqslant 1$, i.e. $1\leqslant p_i\leqslant\infty$, $i=1,\dots,n$. It is well known that constant exponent Lebesgue spaces with mixed norm were introduced and studied in [9]. The variable Lebesgue spaces with mixed norm were introduced and studied in [5] and [6].

Suppose that $\Omega\subset\mathbb{R}^n$ is a measurable set and $f:\Omega\mapsto\mathbb{R}$. We define the norm in the space $L_{\mathbf{p}(x)}(\Omega)$ by

$$\|f\|_{L_{\mathbf{p}(\cdot)}(\Omega)}=\|f\chi_\Omega\|_{L_{\mathbf{p}(\cdot)}(\mathbb{R}^n)},$$

where $\chi_\Omega(x)$ is the characteristic function of Ω.

REMARK 2. Let $\mathbf{p}(x)=(p_1,\dots,p_n)=\mathbf{p}\geqslant 1$. Then $L_{\mathbf{p}(x)}(\mathbb{R}^n)$ coincides with the usual constant exponent Lebesgue spaces with mixed norm.

REMARK 3. Let $p_1(x_1,\dots,x_n)=p_2\left(x^{(2)}\right)=\dots=p_n\left(x^{(n)}\right)=p(x_n)$, i.e. $\mathbf{p}(x)=(p(x_n),\dots,p(x_n))$. Then $L_{\mathbf{p}(x)}(\mathbb{R}^n)=L_{p(x_n)}(\mathbb{R}^n)$.

3. Interpolation estimate in variable Lebesgue space with mixed norm

In this section, we give the multiplicative interpolation inequality in a variable Lebesgue space with mixed norm.

In first, we need some necessary results using in the proving of some interpolation estimate in a variable Lebesgue space with mixed norm.

THEOREM 1. [5] *Let* $\boldsymbol{p}(x)=(p_1(x),\dots,p_n(x_n))$, $\boldsymbol{q}(x)=(q_1(x),\dots,q_n(x_n))$ *and* $\boldsymbol{r}(x)=(r_1(x),\dots,r_n(x_n))$. *Suppose that* $1\leqslant p_i\left(x^{(i)}\right),q_i\left(x^{(i)}\right),r_i\left(x^{(i)}\right)<\infty$ *with* $\dfrac{1}{r_i\left(x^{(i)}\right)}=\dfrac{1}{p_i\left(x^{(i)}\right)}+\dfrac{1}{q_i\left(x^{(i)}\right)}$, $i=1,\dots,n$. *Then the inequality*

$$\|fg\|_{L_{\boldsymbol{r}(\cdot)}(\Omega)}\leqslant\prod_{i=1}^{n}\left(A_i+B_i+\left\|\chi_{\Omega_{2,i}}\right\|_{L_\infty(\Omega)}\right)^{1/\underline{r}_i}\|f\|_{L_{\boldsymbol{p}(\cdot)}(\Omega)}\|g\|_{L_{\boldsymbol{q}(\cdot)}(\Omega)}\tag{1}$$

holds for every $f\in L_{\boldsymbol{p}(x)}(\Omega)$, $g\in L_{\boldsymbol{q}(x)}(\Omega)$, *where*

$$A_i=\operatorname*{ess\,sup}_{x\in\Omega_{1,i}}\frac{r_i\left(x^{(i)}\right)}{p_i\left(x^{(i)}\right)}\quad\text{and}\quad B_i=\operatorname*{ess\,sup}_{x\in\Omega_{1,i}}\frac{p_i\left(x^{(i)}\right)-r_i\left(x^{(i)}\right)}{p_i\left(x^{(i)}\right)}$$

$$\Omega_{1,i}=\left\{x\in\Omega:\ r_i\left(x^{(i)}\right)<p_i\left(x^{(i)}\right)\right\},\ \text{and}\ \Omega_{2,i}=\left\{x\in\Omega:\ p_i\left(x^{(i)}\right)=r_i\left(x^{(i)}\right)\right\}.$$

LEMMA 1. *Let* $s\geqslant 1$ *and let* $\boldsymbol{p}(x)=(p_1(x),\dots,p_n(x_n))$ *with* $1\leqslant p_i\left(x^{(i)}\right)<\infty$, $i=1,\dots,n$. *Suppose that* $f\in L_{\boldsymbol{p}(x)}(\mathbb{R}^n)$. *Then*

$$\||f|^s\|_{\boldsymbol{p}(\cdot)}=\|f\|^s_{s\boldsymbol{p}(\cdot)}.\tag{2}$$

Proof. It is obvious that

$$\||f|^s\|_{p_1(\cdot),x_1} = \inf\left\{\lambda > 0 : \int_{-\infty}^{\infty} \left(\frac{|f(x)|^s}{\lambda}\right)^{p_1(x_1,\dots,x_n)} dx_1 \leqslant 1\right\}$$
$$= \inf\left\{\lambda > 0 : \int_{-\infty}^{\infty} \left(\frac{|f(x)|}{\lambda^{\frac{1}{s}}}\right)^{s\,p_1(x_1,\dots,x_n)} dx_1 \leqslant 1\right\}.$$

Let $\mu = \lambda^{\frac{1}{s}}$. One has

$$\||f|^s\|_{p_1(\cdot),x_1} = \inf\left\{\mu^s > 0 : \int_{-\infty}^{\infty} \left(\frac{|f(x)|}{\mu}\right)^{s\,p_1(x_1,\dots,x_n)} dx_1 \leqslant 1\right\} = \|f\|^s_{s\,p_1(\cdot),x_1}.$$

By iteration, we have that

$$\begin{aligned}
\||f|^s\|_{s\mathbf{p}(\cdot)} &= \|\dots\| \||f|^s\|_{p_1(\cdot),x_1} \|_{p_2(\cdot),x_2} \cdots \|_{p_n(\cdot),x_n} \\
&= \|\dots\| \|f\|^s_{s\,p_1(\cdot),x_1} \|_{p_2(\cdot),x_2} \cdots \|_{p_n(\cdot),x_n} \\
&= \|\dots\| \|f\|_{s\,p_1(\cdot),x_1} \|^s_{s\,p_2(\cdot),x_2} \cdots \|_{p_n(\cdot),x_n} \\
&= \|\dots\| \|f\|_{s\,p_1(\cdot),x_1} \|_{s\,p_2(\cdot),x_2} \cdots \|^s_{s\,p_n(\cdot),x_n} \\
&= \|f\|^s_{s\mathbf{p}(\cdot)}.
\end{aligned}$$

This completes the proof. □

By Theorem 1 and Lemma 1 we have the following lemma on interpolation inequality.

LEMMA 2. *Let* $\boldsymbol{p}_j(x) = \left(p_1^j(x), \dots, p_n^j(x_n)\right)$ *with* $1 \leqslant p_i^j\left(x^{(i)}\right) < \infty,\ i = 1, \dots, n,$ *and* $j = 0, 1$. *Suppose that* $\dfrac{1}{\boldsymbol{p}_\theta(x)} = \dfrac{1-\theta}{\boldsymbol{p}_0(x)} + \dfrac{\theta}{\boldsymbol{p}_1(x)}$ *for some* $\theta \in [0,1]$. *Then the following interpolation estimate*

$$\|f\|_{\boldsymbol{p}_\theta(\cdot)} \leqslant \prod_{i=1}^{n} \left(A_i + B_i + \left\|\chi_{\Omega_{2,i}}\right\|_{L_\infty(\Omega)}\right)^{1/\underline{r}_i} \|f\|^{1-\theta}_{\boldsymbol{p}_0(\cdot)} \|f\|^{\theta}_{\boldsymbol{p}_1(\cdot)} \tag{3}$$

holds.

Proof. We observe that in the cases $\theta = 0$ and $\theta = 1$ the inequality (3) is obvious.

Let $0<\theta<1$. By the inequalities (1) and (2), we have

$$\begin{aligned}\|f\|_{\mathbf{p}_\theta(\cdot)} &= \left\| |f|^{1-\theta} |f|^{\theta} \right\|_{\mathbf{p}_\theta(\cdot)} \\ &\leqslant \prod_{i=1}^{n}\left(A_i+B_i+\left\|\chi_{\Omega_{2,i}}\right\|_{L_\infty(\Omega)}\right)^{1/\underline{r}_i} \left\| |f|^{1-\theta}\right\|_{\frac{\mathbf{p}_0(\cdot)}{1-\theta}} \left\| |f|^{\theta}\right\|_{\frac{\mathbf{p}_1(\cdot)}{\theta}} \\ &= \prod_{i=1}^{n}\left(A_i+B_i+\left\|\chi_{\Omega_{2,i}}\right\|_{L_\infty(\Omega)}\right)^{1/\underline{r}_i} \|f\|_{\mathbf{p}_0(\cdot)}^{1-\theta} \|f\|_{\mathbf{p}_1(\cdot)}^{\theta}.\end{aligned}$$

This completes the proof of Lemma 2. □

Let X and Y be Banach spaces and let both X,Y continuously embedded in a Hausdorff topological vector space Z. It is well known that the norm in the space Z is defined as follows

$$\|z\|_Z := \inf\{\|x\|_X+\|y\|_Y:\ z=x+y\}.$$

Then Z is a quotient of the Banach space $X\times Y$ with respect to the subspace $X\bigcap Y$ and hence itself a Banach space.

We observe that the spaces $X\bigcap Y$ equipped with the norm $\|z\|_{X\bigcap Y}=\max\{\|z\|_X;\ \|z\|_Y\}$ or $\|z\|_{X\bigcap Y}=\|z\|_X+\|z\|_Y$.

Let $1<s<\infty$ and let $a,b\geqslant 0$. Suppose that $s'=\dfrac{s}{s-1}$. Then the following Young's inequality holds

$$ab\leqslant \frac{a^s}{s}+\frac{b^{s'}}{s'}.$$

Let $h(\theta)=\max\{1-\theta;\theta\},\ 0\leqslant\theta\leqslant 1$. Suppose that $s=\dfrac{1}{1-\theta}$. At $\theta=1$ we use the convention $s=\infty$. By (3) and by Young's inequality we have that

$$\begin{aligned}\|f\|_{\mathbf{p}_\theta(\cdot)} &\leqslant \prod_{i=1}^{n}\left(A_i+B_i+\left\|\chi_{\Omega_{2,i}}\right\|_{L_\infty(\Omega)}\right)^{1/\underline{r}_i} \|f\|_{\mathbf{p}_0(\cdot)}^{1-\theta} \|f\|_{\mathbf{p}_1(\cdot)}^{\theta} \\ &\leqslant \prod_{i=1}^{n}\left(A_i+B_i+\left\|\chi_{\Omega_{2,i}}\right\|_{L_\infty(\Omega)}\right)^{1/\underline{r}_i} h(\theta)\left(\|f\|_{\mathbf{p}_0(\cdot)}+\|f\|_{\mathbf{p}_1(\cdot)}\right). \qquad (4)\end{aligned}$$

Thus, $L_{\mathbf{p}_0(\cdot)}\bigcap L_{\mathbf{p}_1(\cdot)}\subset L_{\mathbf{p}_\theta(\cdot)}$.

Now let us prove the following Proposition.

PROPOSITION 1. *Let* $\boldsymbol{p}_j(x)=\left(p_1^j(x),\dots,p_n^j(x_n)\right)$ *with* $1\leqslant p_i^j\left(x^{(i)}\right)<\infty,\ i=1,\dots,n,$ *and* $j=0,1$. *Suppose that* $\mathbf{p}_\theta(x)=\left(p_\theta^{(1)}(x),\dots,p_\theta^{(n)}(x_n)\right)$ *is the variable exponent defined in Lemma 2 and let* $\mathbf{p}_0(x)\leqslant\mathbf{p}_\theta(x)\leqslant\mathbf{p}_1(x)$. *Then the following embeddings hold:*

$$L_{\boldsymbol{p}_0(\cdot)}\bigcap L_{\boldsymbol{p}_1(\cdot)}\hookrightarrow L_{\boldsymbol{p}_\theta(\cdot)}\hookrightarrow L_{\boldsymbol{p}_0(\cdot)}+L_{\boldsymbol{p}_1(\cdot)}.$$

Proof. We proceed along the line [15]. By inequality (4), we have that $L_{\mathbf{p}_0(\cdot)} \bigcap L_{\mathbf{p}_1(\cdot)} \subset L_{\mathbf{p}_\theta(\cdot)}$. Let $f_0 = \operatorname{sgn} f \max\{|f|-1,\, 0\}$ and $f_1 = \operatorname{sgn} f \min\{|f|,\, 1\}$. It is obvious that $f = f_0 + f_1$. We suppose that $f \in L_{\left(p_\theta^{(1)}(x),\, x_1\right)}(\mathbb{R}^n)$, where $p_1^0(x) \leqslant p_\theta^{(1)}(x) \leqslant p_1^1(x)$. Now, let $\left(I_{p_\theta^{(1)},x_1} f\right)(x_2,\ldots,x_n) \leqslant 1$ for fixed variables $x_2,\ldots,x_n$. Since $p_1^0(x) \leqslant p_\theta^{(1)}(x)$, we get

$$\begin{aligned}
\left(I_{p_\theta^{(1)},x_1} f\right)(x_2,\ldots,x_n) &= \int\limits_{\mathbb{R}} |f(x)|^{p_\theta^{(1)}(x)}\, dx_1 \geqslant \int\limits_{\{x_1:\, |f(x_1,\ldots,x_n)|>1\}} |f(x)|^{p_\theta^{(1)}(x)}\, dx_1 \\
&= \int\limits_{\{x_1:\, |f(x_1,\ldots,x_n)|>1\}} \left[(1+(|f(x)|-1))^{\frac{p_\theta^{(1)}(x)}{p_1^0(x)}}\right]^{p_1^0(x)} dx_1 \\
&\geqslant \int\limits_{\{x_1:\, |f(x_1,\ldots,x_n)|>1\}} \left(\frac{p_\theta^{(1)}(x)}{p_1^0(x)}\right)^{p_1^0(x)} (|f(x)|-1)^{p_1^0(x)}\, dx_1 \\
&\geqslant \int\limits_{\{x_1:\, |f(x_1,\ldots,x_n)|>1\}} (|f(x)|-1)^{p_1^0(x)}\, dx_1 = \int\limits_{\mathbb{R}} |f_0(x)|^{p_1^0(x)}\, dx_1 \\
&= \left(I_{p_1^0,x_1} f_0\right)(x_2,\ldots,x_n).
\end{aligned}$$

So, $\left(I_{p_1^0,x_1} f_0\right)(x_2,\ldots,x_n) \leqslant 1$ and by the definition of $L_{\left(p_1^0(x),x_1\right)}(\mathbb{R}^n)$ it follows that $\|f_0\|_{p_1^0(\cdot),x_1} \leqslant 1$. Next, we have

$$\begin{aligned}
&\left(I_{p_1^1,x_1} f_1\right)(x_2,\ldots,x_n) \\
&= \int\limits_{\mathbb{R}} |f_1(x)|^{p_1^1(x)}\, dx_1 \\
&= \int\limits_{\{x_1:\, |f(x_1,\ldots,x_n)|\leqslant 1\}} |f(x)|^{p_1^1(x)}\, dx_1 + \int\limits_{\{x_1:\, |f(x_1,\ldots,x_n)|>1\}} dx_1 \\
&\leqslant \int\limits_{\{x_1:\, |f(x_1,\ldots,x_n)|\leqslant 1\}} |f(x)|^{p_\theta^{(1)}(x)}\, dx_1 + \int\limits_{\{x_1:\, |f(x_1,\ldots,x_n)|>1\}} |f(x)|^{p_\theta^{(1)}(x)}\, dx_1 \\
&= \left(I_{p_\theta^{(1)},x_1} f\right)(x_2,\ldots,x_n).
\end{aligned}$$

So, $\left(I_{p_1^1,x_1} f_1\right)(x_2,\ldots,x_n) \leqslant 1$ and by the definition of $L_{\left(p_1^1(x),x_1\right)}(\mathbb{R}^n)$ it follows that

$\|f_1\|_{p_1^1(\cdot),x_1} \leqslant 1$. In particular, one has

$$\|f\|_{L_{\left(p_1^0(\cdot),x_1\right)}+L_{\left(p_1^1(\cdot),x_1\right)}} \leqslant \|f_0\|_{p_1^0(\cdot),x_1} + \|f_1\|_{p_1^1(\cdot),x_1} \leqslant 2.$$

The scaling argument proves that $\|f\|_{L_{\left(p_1^0(\cdot),x_1\right)}+L_{\left(p_1^1(\cdot),x_1\right)}} \leqslant 2\|f\|_{L_{\left(p_\theta^{(1)}(\cdot),x_1\right)}}$. So, $L_{\left(p_\theta^{(1)}(x),x_1\right)}(\mathbb{R}^n) \hookrightarrow L_{\left(p_1^0(\cdot),x_1\right)}(\mathbb{R}^n)+L_{\left(p_1^1(\cdot),x_1\right)}(\mathbb{R}^n)$.

By induction, we complete the proof of Proposition 1. $\square$

4. Main results

Now we recall the definition of the norm in the interpolation space $\left[L_{\mathbf{p}(\cdot)}, L_{\mathbf{q}(\cdot)}\right]_\theta$. Let $S := \{z \in \mathbb{C} :\ 0 < Re\, z < 1\}$, so that $\overline{S} = \{z \in \mathbb{C} : 0 \leqslant Re\, z \leqslant 1\}$, where $Re\, z$ is the real part of z. Let $\mathscr{F}$ be the space of functions on $\overline{S}$ with values in $L_{\mathbf{p}_0(\cdot)} + L_{\mathbf{p}_1(\cdot)}$ which are analytic on S and bounded and continuous on $\overline{S}$ such that $F(it)$ and $F(1+it)$ tend to zero for $|t| \to \infty$. For $F \in \mathscr{F}$ we set

$$\|F\|_{\mathscr{F}} := \sup_{t\in\mathbb{R}} \max\left\{\|F(it)\|_{\mathbf{p}_0(\cdot)},\ \|F(1+it)\|_{\mathbf{p}_1(\cdot)}\right\}.$$

Then we define the following quantity:

$$\|f\|_{[\theta]} := \inf\{\|F\|_{\mathscr{F}} :\ F \in \mathscr{F} \text{ and } f = F(\theta)\}.$$

We give the main results of the paper.

THEOREM 2. *Let* $x \in \mathbb{R}^n$, $1 \leqslant \mathbf{p}_0(x), \mathbf{p}_1(x) < \infty$ *and let* $\theta \in (0,1)$. *Suppose that* $\mathbf{p}_\theta(x) = \left(p_\theta^{(1)}(x), \ldots, p_\theta^{(n)}(x_n)\right)$ *is the variable exponent defined in Lemma 2 and let* $\mathbf{p}_0(x) \leqslant \mathbf{p}_\theta(x) \leqslant \mathbf{p}_1(x)$.. *Then*

$$\left[L_{\mathbf{p}_0(\cdot)}, L_{\mathbf{p}_1(\cdot)}\right]_\theta \cong L_{\mathbf{p}_\theta(\cdot)}.$$

Moreover,

$$\|f\|_{[\theta]} \leqslant \|f\|_{\mathbf{p}_\theta(\cdot)} \leqslant 4^n \|f\|_{[\theta]}.$$

Proof. We proceed along the line [10]. Let $z = \theta + iv$ and for $z \in \overline{S}$ define the following function

$$\varphi_{[z]}^{-1}(x,t) = t^{\frac{1-z}{p_1^0(x)}+\frac{z}{p_1^1(x)}} = t^{\frac{1-\theta}{p_1^0(x)}+\frac{\theta}{p_1^1(x)}}\, e^{iv\left(\frac{1}{p_1^1(x)}-\frac{1}{p_1^0(x)}\right)\ln t}.$$

It is obvious that $\mathrm{Re}\,\varphi_{[z]}^{-1}(x,t) = t^{\frac{1-\theta}{p_1^0(x)}+\frac{\theta}{p_1^1(x)}} \cos\left(v\left(\frac{1}{p_1^1(x)} - \frac{1}{p_1^0(x)}\right)\ln t\right)$ and $\mathrm{Im}\,\varphi_{[z]}^{-1}(x,t) = t^{\frac{1-\theta}{p_1^0(x)}+\frac{\theta}{p_1^1(x)}} \sin\left(v\left(\frac{1}{p_1^1(x)} - \frac{1}{p_1^0(x)}\right)\ln t\right)$. Then $z \mapsto \varphi_{[z]}^{-1}(x,t)$ is analytic on S and continuous on $\overline{S}$. We observe that $\varphi_{[\theta]}^{-1}(x,t) = t^{\frac{1-\theta}{p_1^0(x)}+\frac{\theta}{p_1^1(x)}}$ $= t^{\frac{1}{p_\theta^{(1)}(x)}}$.

We suppose that $\theta \in (0,1)$ and let $f \in L_{\left(p_\theta^{(1)}(x),\, x_1\right)}(\mathbb{R}^n)$ with $\left(I_{p_\theta^{(1)},x_1} f\right)(x_2,\ldots,x_n) \leqslant 1$ for fixed variables $x_2,\ldots,x_n$. So, the function $|f(x)|^{p_\theta^{(1)}(x)}$ is finite almost everywhere $x \in \mathbb{R}^n$. For $\varepsilon > 0$, $z \in \overline{S}$ and $x \in \mathbb{R}^n$, we define

$$F_\varepsilon(z,x) = e^{-\varepsilon+\varepsilon z^2-\varepsilon\theta^2}\varphi_{[z]}^{-1}\left(x, |f(x)|^{p_\theta^{(1)}(x)}\right) sgnf(x).$$

Then $F_\varepsilon(\theta,x) = e^{-\varepsilon} f(x)$. Obviously, $F_\varepsilon(z,x)$ is analytic in z for a.e. $x \in \mathbb{R}^n$. Therefore, $\dfrac{dF_\varepsilon}{d\overline{z}} = 0$ in $L_{\left(p_1^0(\cdot),x_1\right)}(\mathbb{R}^n) + L_{\left(p_1^1(\cdot),x_1\right)}(\mathbb{R}^n)$. We observe that

$$\left|e^{-\varepsilon+\varepsilon(it)^2-\varepsilon\theta^2}\right| = e^{-\varepsilon\left(1+t^2+\theta^2\right)} \leqslant 1$$

$$\left|e^{-\varepsilon+\varepsilon(1+it)^2-\varepsilon\theta^2}\right| = \left|e^{-\varepsilon\left(t^2+\theta^2\right)-2\varepsilon t i}\right| = \left|e^{-\varepsilon\left(t^2+\theta^2\right)}\right| \leqslant 1$$

for all $t \in \mathbb{R}$. Thus,

$$|F_\varepsilon(it,x)| \leqslant \left| |f(x)|^{p_\theta^{(1)}(x)\left(\frac{1-it}{p_1^0(x)}+\frac{it}{p_1^1(x)}\right)} \right| = |f(x)|^{\frac{p_\theta^{(1)}(x)}{p_1^0(x)}}$$

$$|F_\varepsilon(1+it,x)| \leqslant \left| |f(x)|^{p_\theta^{(1)}(x)\left(\frac{-it}{p_1^0(x)}+\frac{1+it}{p_1^1(x)}\right)} \right| = |f(x)|^{\frac{p_\theta^{(1)}(x)}{p_1^1(x)}}.$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Next, for fixed variables $x_2,\ldots,x_n$, we have

$$\begin{aligned}\left(I_{p_1^0,x_1} F_\varepsilon\right)(x_2,\ldots,x_n) &= \int_{\mathbb{R}} |F_\varepsilon(it,x)|^{p_1^0(x)}\,dx_1 \leqslant \int_{\mathbb{R}} |f(x)|^{p_\theta^{(1)}(x)}\,dx_1 \\ &= \left(I_{p_\theta^{(1)},x_1} f\right)(x_2,\ldots,x_n) \leqslant 1.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\left(I_{p_1^1,x_1} F_\varepsilon\right)(x_2,\ldots,x_n) &= \int_{\mathbb{R}} |F_\varepsilon(1+it,x)|^{p_1^1(x)}\,dx_1 \leqslant \int_{\mathbb{R}} |f(x)|^{p_\theta^{(1)}(x)}\,dx_1 \\ &= \left(I_{p_\theta^{(1)},x_1} f\right)(x_2,\ldots,x_n) \leqslant 1.\end{aligned}$$

Hence, by the definition of $L_{\left(p_1^0(x),x_1\right)}(\mathbb{R}^n)$ it follows that $\|F_\varepsilon(it,\cdot)\|_{p_1^0(\cdot),x_1} \leqslant 1$ for all $t \in \mathbb{R}$. Similarly, we have that $\|F_\varepsilon(1+it,\cdot)\|_{p_1^1(\cdot),x_1} \leqslant 1$ for all $t \in \mathbb{R}$. Thus, we get

$$\|F_\varepsilon\|_{\mathscr{F}} = \sup_{t\in\mathbb{R}} \max\left\{\|F_\varepsilon(it,\cdot)\|_{p_1^0(\cdot),x_1}\,;\ \|F_\varepsilon(1+it,\cdot)\|_{p_1^1(\cdot),x_1}\right\} \leqslant 1.$$

Since $F_\varepsilon(\theta,x) = e^{-\varepsilon} f(x)$, this implies that $\|e^{-\varepsilon} f\|_{[\theta]} \leqslant 1$. As $\varepsilon > 0$ was arbitrary, we deduce $\|f\|_{[\theta]} \leqslant 1$. A scaling argument yields $\|f\|_{[\theta]} \leqslant \|f\|_{p_\theta^{(1)},x_1}$ for fixed variables $x_2,\ldots,x_n$. By induction we get that $\|f\|_{[\theta]} \leqslant \|f\|_{\mathbf{p}_\theta(\cdot)}$.

In a similar way, the inequality $\|f\|_{\mathbf{p}_\theta(\cdot)} \leqslant 4^n \|f\|_{[\theta]}$ is proved ([15], Theorem 7.1.2).

This completes the proof of Theorem 2. □

Theorem 2 has the following consequence.

COROLLARY 1. *Let $x \in \mathbb{R}^n$, $1 \leqslant \mathbf{p}_0(x), \mathbf{p}_1(x) < \infty$ and let $\theta \in (0,1)$. Suppose that $\mathbf{p}_\theta(x)$ is the variable exponent defined in Lemma 2. Let S be a linear, bounded mapping $S : L_{\mathbf{p}_j(\cdot)}(\mathbb{R}^n) \mapsto L_{\mathbf{p}_j(\cdot)}(\mathbb{R}^n)$ for $j = 0,1$. Moreover,*

$$\|S\|_{L_{\mathbf{p}_\theta(\cdot)}\mapsto L_{\mathbf{p}_\theta(\cdot)}} \leqslant 4^n \|S\|^{1-\theta}_{L_{\mathbf{p}_0(\cdot)}\mapsto L_{\mathbf{p}_0(\cdot)}} \|S\|^{\theta}_{L_{\mathbf{p}_1(\cdot)}\mapsto L_{\mathbf{p}_1(\cdot)}}.$$

Proof. Let $X = \left[L_{\mathbf{p}_0(\cdot)}, L_{\mathbf{p}_1(\cdot)}\right]_\theta$. Then by the complex interpolation theorem of Riesz-Thorin (e.g., [10], Theorem 4.1.2), we get

$$\|S\|_{X\mapsto X} \leqslant \|S\|^{1-\theta}_{L_{\mathbf{p}_0(\cdot)}\mapsto L_{\mathbf{p}_0(\cdot)}} \|S\|^{\theta}_{L_{\mathbf{p}_1(\cdot)}\mapsto L_{\mathbf{p}_1(\cdot)}}.$$

By Theorem 2 we have that

$$\|Sf\|_{\mathbf{p}_\theta(\cdot)} \leqslant 4^n \|Sf\|_{[\theta]} \leqslant 4^n \|S\|^{1-\theta}_{L_{\mathbf{p}_0(\cdot)}\mapsto L_{\mathbf{p}_0(\cdot)}} \|S\|^{\theta}_{L_{\mathbf{p}_1(\cdot)}\mapsto L_{\mathbf{p}_1(\cdot)}} \|f\|_{[\theta]}$$

$$\leqslant 4^n \|S\|^{1-\theta}_{L_{\mathbf{p}_0(\cdot)}\mapsto L_{\mathbf{p}_0(\cdot)}} \|S\|^{\theta}_{L_{\mathbf{p}_1(\cdot)}\mapsto L_{\mathbf{p}_1(\cdot)}} \|f\|_{\mathbf{p}_\theta(\cdot)}.$$

This completes the proof of Corollary 1. □

Acknowledgements. The authors would like to thank the referees for their useful suggestions which have significantly improved the paper.

Conflict of interest. The authors declare no potential conflict of interests.

Funding information. There is no funding.

Author contributions. All authors contributed equally to this article. They read and approved the final manuscript.

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(Received March 17, 2025)

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