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GENERALIZED CLASS OF BIVARIATE HERMITE AND LAGUERRE POLYNOMIALS

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Abstract. This paper introduces and studies a new class of bivariate Hermite-Laguerre polynomials that unify and extend several well-known families of two-dimensional Hermite and Laguerre polynomials. We establish their fundamental properties and demonstrate that these polynomials provide explicit solutions to both fractional and classical heat equations in one and two dimensions, highlighting their analytical richness and potential for diverse applications.

1. Introduction

In [13] Dattoli et. al. introduced two incomplete Hermite polynomials of two variables of the forms:

$$h_{m,n}(x,y|\tau) = m!n! \sum_{r=0}^{\min\{m,n\}} \frac{\tau^r\, x^{m-r}\, y^{n-r}}{r!(m-r)!\,(n-r)!}, \tag{1}$$

and

$$h_{m,n}(x,y) = m!n! \sum_{r=0}^{\min\{m,n\}} \frac{x^{m-r}\, y^{n-r}}{r!(m-r)!\,(n-r)!}. \tag{2}$$

It's crucial to remember that the incomplete 2D-Hermite polynomials $h_{m,n}(x,y|\tau)$ are specific instances of the more general family of complete multi-dimensional Hermite polynomials [12]. These polynomials are frequently used in applications involving entangled harmonic oscillator states [28] or to create sets of bi-orthogonal basis [9, 12, 21].

The generating function may also define the 2D-Hermite polynomials that are incomplete: $h_{m,n}(x,y|\tau)$ [13]:

$$\sum_{m,n=0}^{\infty} h_{m,n}(x,y|\tau)\frac{u^m v^n}{m!n!} = \exp(xu + yv + \tau uv). \tag{3}$$

The Laguerre polynomials $L_n(x,y)$ with two variables are another intriguing class of polynomials [10]:

$$L_n(x,y) = n! \sum_{k=0}^{n} \frac{(-1)^k x^k y^{n-k}}{(k!)^2 (n-k)!}. \tag{4}$$

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The 2D-Hermite polynomials:

$$H_{m,n}(x,y) = \sum_{k=0}^{\min(m,n)} (-1)^k k! \binom{m}{k} \binom{n}{k} x^{m-k}\, y^{n-k}, \tag{5}$$

were introduced by Itô in [17] and have many applications to physical problems, (see, for example [1, 2, 7, 13], [25]–[28]). Mathematical properties of these polynomials have been developed in [2]–[4] and [14, 16]. 2D-Laguerre polynomials are introduced by Ismail and Zhang (see, for example [15] or [16]) and defined for $m \geqslant n$ by the explicit formula:

$$Z_{m,n}^{(\beta)}(x,y) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \binom{\beta+m}{k} (-1)^k k! x^{m-k} y^{n-k}. \tag{6}$$

Indeed, the 2D-Laguerre polynomials $Z_{m,n}^{(\beta)}(x,y)$ are one parameter generalization of the 2D-Hermite polynomials in (1.5).

Very recently, Shahwan and Bin-Saad [22] exploited operational techniques combined with the monomial principle to introduce and discuss a new extended Laguerre two-dimensional polynomials in the form:

$$L_{m,n}^{(\beta)}(x,y;\tau) = \sum_{k=0}^{n} \binom{m+\beta}{k} \binom{n}{k} (-1)^k \tau^k x^{m-k} y^{n-k}, \tag{7}$$

which provide operational relations with several known polynomials. The present work is a sequel to the papers [3] and [5] and it aims to introduce and investigate a new kind of bivariate Hermite-Laguerre polynomials defined in (10) and denoted by $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$. This generalized class not only contains the 2D-Laguerre polynomials in (7) but it also contains several known polynomials (see Eqs. (11)–(19)). The $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ polynomials will be introduced and formulated in Section 2. The rest of this paper is organized as follows. In Section 3, we follow the approach of Burchnall [6] and obtain operational representations for the polynomials $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$. Section 4 is devoted to deriving some raising and lowering operational relations. In Section 5, we show how readily new generating relations for the polynomials $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ can be derived from the operational representations obtained in Section 2 and 3. Section 6 concludes with some expansions for the polynomials $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$. In addition, we provide expansion of the product of two $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ polynomials as a sum involving Appelś series in two variables F_2. In Section 7, we show that these polynomials yield explicit solutions to both fractional and classical heat equations in one- and two-dimensional settings, thereby underscoring their analytical depth and broad applicability.

2. Generalized bivariate Hermite-Laguerre polynomials

In this section, we introduce a novel class of bivariate Hermite-Laguerre polynomials, denoted by $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$, where x and y are complex variables, and $\alpha,\beta,\tau \in \mathbb{R}$. Their construction is based on a combination of the series rearrangement method and the operational technique (see [24], Chapters 2 and 4; also see [25]). Let us consider the generating relation

$$f^{(\alpha,\beta)}(x,y;\tau|t) = \frac{(tx)^{-\beta}y^{-\alpha}}{(1-tx)} \exp\left[\frac{-\tau t\frac{\partial}{\partial y}}{(1-tx)}\right]\left\{y^{\alpha+n}\right\}, \tag{8}$$

where $n = 1,2,\ldots$; $\frac{\partial}{\partial y}$ denotes the derivative operator, $\alpha,\beta \in \mathbb{R}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $|xt| < 1$. Expressing the exponential function in series and applying the operator (see e.g. [24], Chapter 5):

$$\left(\frac{\partial}{\partial x}\right)^n x^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}x^{\alpha-n}, \quad \alpha \geqslant 0,\ n \in \mathbb{Z},$$

we can conclude that

$$f^{(\alpha,\beta)}(x,y;\tau|t) = \sum_{m=0}^{\infty}\left\{\sum_{k=0}^{n}\frac{(\alpha+n)!(-1)^k\tau^k x^{\beta+m}y^{n-k}(k+1)_m}{m!k!(n-k)!}\right\}t^{m+k-\beta}.$$

Finally, the change of index $m = m+\beta-k$ leads to

$$\frac{(xt)^{-\beta}y^{-\alpha}}{(1-xt)} \exp\left[\frac{-\tau t\frac{\partial}{\partial y}}{(1-xt)}\right]\left\{y^{\alpha+n}\right\} = \sum_{m=0}^{\infty}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\,t^m, \tag{9}$$

where $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ is the generalized class of bivariate Hermite-Laguerre polynomials defined by the explicit formula:

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau) = \sum_{k=0}^{\min(m,n)}\binom{n+\alpha}{k}\binom{m+\beta}{k}(-1)^k\tau^k x^{m-k}y^{n-k} \quad (m,n \in \mathbb{N}_0). \tag{10}$$

It may be of interest to point out that the series representation (10) involves the following important particular cases:

$$\mathfrak{S}_{m,n}^{(0,\beta)}(x,y;\tau) = L_{m,n}^{(\beta)}(x,y;\tau), \tag{11}$$

$$\mathfrak{S}_{m,n}^{(0,0)}\left(x,y;\tau\frac{\partial}{\partial\tau}\tau\right) = h_{m,n}(x,y|\tau), \tag{12}$$

$$\mathfrak{S}_{m,n}^{(0,\beta)}\left(x,y;\frac{\partial}{\partial\tau}\tau\right) = Z_{m,n}^{(\beta)}(x,y), \tag{13}$$

$$\mathfrak{S}_{m,n}^{(0,0)}\left(x,y;\frac{\partial}{\partial\tau}\tau\right) = H_{m,n}(x,y). \tag{14}$$

Inversely, comparing the series representation (10) with the polynomials (1), (4), (5), (6) and (7), we find that:

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau) = h_{m+\beta,n+\alpha}\left(x,y \mid -\left(\frac{\partial}{\partial\tau}\right)^{-1}\right)\{x^{-\beta}y^{-\alpha}\}, \tag{15}$$

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau) = \frac{(\alpha+n)!}{n!}y^{-\alpha}\left(\frac{\partial}{\partial y}\right)^{\alpha} x^{\beta}L_n\left(\tau\frac{\partial}{\partial x},y\right)\{x^{m+\beta}\}, \tag{16}$$

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau) = (\partial_\tau)^{-\frac{1}{2}(m+n)}x^{-\beta}y^{-\alpha}H_{m+\beta,n+\alpha}(x\sqrt{(\partial_\tau)},y(\partial_\tau)), \tag{17}$$

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau) = (\partial_\tau)^{-\frac{1}{2}(m+n)}y^{-\alpha}Z_{m,n+\alpha}(x\sqrt{(\partial_\tau)},y(\partial_\tau)), \tag{18}$$

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau) = y^{-\alpha}L_{m,n+\alpha}^{\beta}(x,y;\tau). \tag{19}$$

The significance of the above operational connections lies in their ability to facilitate the derivation of several essential properties for the polynomials $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$, in close analogy with those established for the polynomials in (1), (4)–(7).

3. Operational representations

By the operators (see [24], Chapter 5):

$$\left(\frac{\partial}{\partial x}\right)^{r} x^{\beta+m} = \frac{(\beta+m)!}{(\beta+m-r)!}\, x^{\beta+m-k}, \tag{20}$$

and

$$\left(\frac{\partial}{\partial y}\right)^{r} y^{\alpha+n} = \frac{(\alpha+n)!}{(\alpha+n-r)!}\, y^{\alpha+n-r}, \tag{21}$$

we infer from the operational representation (10) the following results:

THEOREM 1. *For the polynomials* $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$, *we have the following operational representation:*

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau) = \left\{x^{-\beta}y^{-\alpha}\right\}{}_0F_1\left[\begin{matrix} -; \\ & -\tau\frac{\partial^2}{\partial x\partial y} \\ 1; \end{matrix}\right]\left\{x^{\beta+m}y^{\alpha+n}\right\}. \tag{22}$$

Proof. The proof is the direct use of the formulas (20), (21), and the definition (10). □

THEOREM 2. *For the polynomials* $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$, *we have the following operational representation:*

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau) = \left\{x^{-\beta}y^{-\alpha}\exp\left[-\left(\frac{\partial}{\partial\tau}\right)^{-1}\left(\frac{\partial^2}{\partial x\partial y}\right)\right]x^{\beta+m}y^{\alpha+n}\right\}. \tag{23}$$

Proof. By direct use of the identity

$$\left(\frac{\partial}{\partial \tau}\right)^{-r} = \frac{\tau^r}{r!}, \tag{24}$$

in assertion (22), we can infer the desired result (23). □

REMARK 1. The operational rule

$$\exp\left[\left(\frac{\partial}{\partial \tau}\right)^{-1}\left(\frac{\partial^2}{\partial x \partial y}\right)\right]\left\{x^{\beta} y^{\alpha}\right\} \mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;.\tau) = \{x^{\beta+m} y^{\alpha+n}\}, \tag{25}$$

provide the inverse of the assertion (23).

THEOREM 3. *For the polynomials* $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$, *we have*

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau) = \{x^m y^n\}\, {}_2F_1\left[\begin{matrix} -\beta-m, -\alpha-n; & \\ & -\tau/xy \\ 1; & \end{matrix}\right]. \tag{26}$$

Proof. By using the result (see e.g., [24], p.23(23); see also [25])

$$(-m)_n = \begin{cases} \frac{(-1)^n m!}{(m-n)!}, & 0 \leqslant n \leqslant m, \\ 0, & n > m. \end{cases} \tag{27}$$

definition (10) can be written in the form of Gaussian hypergeometric series ${}_2F_1$ (see [21]) as follows. □

THEOREM 4. *For the polynomials* $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$, *we have the following operational representations*

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau) = y^{-\alpha}\left(1 - x^{-1}\left(\frac{\partial}{\partial \tau}\right)^{-1}\frac{\partial}{\partial y}\right)^{\beta+m}\{x^m y^{\alpha+n}\}, \tag{28}$$

or equivalently

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau) = x^{-\beta}\left(1 - y^{-1}\left(\frac{\partial}{\partial \tau}\right)^{-1}\frac{\partial}{\partial x}\right)^{\alpha+n}\{x^{\beta+m} y^{n}\}. \tag{29}$$

Proof. The assertions (28) and (29) follow from the binomial theorem and the differential operators (21) and (23) respectively. □

4. Quasi-monomials approach

The purpose of this section is to exploit the theory of quasi-monomials principles to construct derivative and multiplicative operators, which are used to establish a new property for the 2D-Laguerre polynomials $\mathfrak{S}^{(\alpha,\beta)}(x,y;\tau)$. The two-variable monomials principle is defined as follows (see, for example [8], [10]–[12]).

DEFINITION 1. Let $p_{m,n}(x,y)_{(m,n)\in\mathbb{N}_0\times N_0}$, be a two-variable two-index polynomial set with deg $p_{m,n}(.,y)=n$ and deg $p_{m,n}(x,.)=m$. $p_{m,n}(x,y)=p_{m,n}(x,y)_{(m,n)\in\mathbb{N}_0\times\mathbb{N}_0}$ is said to be quasi-monomial if four operators, not depending on n and m, denoted by $\hat{P}_x,\hat{P}_y,\hat{M}_x,\hat{M}_y$, exist in such a way that

$$\hat{P}_x p_{m,n}(x,y)=mp_{m-1,n}(x,y), \tag{30}$$

$$\hat{P}_y p_{m,n}(x,y)=np_{m,n-1}(x,y), \tag{31}$$

$$\hat{M}_x p_{m,n}(x,y)=p_{m+1,n}(x,y), \tag{32}$$

$$\hat{M}_x p_{m,n}(x,y)=p_{m,n+1}(x,y). \tag{33}$$

Note that the commutation properties

$$[\hat{P}_x,\hat{M}_x]=1=[\hat{P}_y,\hat{M}_y], \tag{34}$$

follow from equations (30) to (33), so that the above operators display Weyl group structures. The operational identity (23) can be taken as the starting point of the theory of quasi-monomial for $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$, polynomials. Using (23) we can easily establish that:

$$\left\{x^{-\beta}y^{-\alpha}\right\}\exp\left[-\left(\frac{\partial}{\partial\tau}\right)^{-1}\left(\frac{\partial^2}{\partial x\partial y}\right)\right]\left\{\frac{\partial}{\partial x}x^{\beta+m}y^{\alpha+n}\right\}=\mathfrak{S}_{m-1,n}^{(\alpha,\beta)}(x,y;\tau), \tag{35}$$

$$\left\{x^{-\beta}y^{-\alpha}\right\}\exp\left[-\left(\frac{\partial}{\partial\tau}\right)^{-1}\left(\frac{\partial^2}{\partial x\partial y}\right)\right]\left\{\frac{\partial}{\partial y}x^{\beta+m}y^{\alpha+n}\right\}=\mathfrak{S}_{m,n-1}^{(\alpha,\beta)}(x,y;\tau), \tag{36}$$

$$\left\{x^{-\beta}y^{-\alpha}\right\}\exp\left[-\left(\frac{\partial}{\partial\tau}\right)^{-1}\left(\frac{\partial^2}{\partial x\partial y}\right)\right]\left\{x\,x^{\beta+m}y^{\alpha+n}\right\}=\mathfrak{S}_{m+1,n}^{(\alpha,\beta)}(x,y;\tau), \tag{37}$$

and

$$\left\{x^{-\beta}y^{-\alpha}\right\}\exp\left[-\left(\frac{\partial}{\partial\tau}\right)^{-1}\left(\frac{\partial^2}{\partial x\partial y}\right)\right]\left\{y\,x^{\beta+m}y^{\alpha+n}\right\}=\mathfrak{S}_{m,n+1}^{(\alpha,\beta)}(x,y;\tau). \tag{38}$$

An analogous way, we find from (10) that

$$\left\{x^{-\beta}y^{-\alpha}\frac{\partial^2}{\partial x\partial y}x^{\beta}y^{\alpha}\right\}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)=(\beta+m)(\alpha+n)\mathfrak{S}_{m-1,n-1}^{(\alpha,\beta)}(x,y;\tau), \tag{39}$$

$$\left\{x^{-\beta}y^{-\alpha}\left\{\frac{\partial}{\partial x}\right)^{-1}\left(\frac{\partial}{\partial y}\right)^{-1}x^{-\beta}y^{\alpha}\right\}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$$
$$=\frac{1}{(\beta+m+1)(\alpha+n+1)}\mathfrak{S}_{m+1,n+1}^{(\alpha,\beta)}(x,y;\tau), \tag{40}$$

$$-\frac{\partial}{\partial\tau}\tau\frac{\partial}{\partial\tau}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)=(\beta+m)(\alpha+n)\mathfrak{S}_{m-1,n-1}^{(\alpha,\beta)}(x,y;\tau), \tag{41}$$

$$\left(-\frac{\partial}{\partial\tau}\tau\frac{\partial}{\partial\tau}\right)^{-1}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)=\frac{1}{(\beta+m+1)(\alpha+n+1)}\mathfrak{S}_{m+1,n+1}^{(\alpha,\beta)}(x,y;\tau). \tag{42}$$

The generalized class $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ are quasi-monomials under the action of the multiplicative operator (see e.g. [11]):

$$\hat{M}_1=\left(x-\left(\frac{\partial}{\partial\tau}\right)^{-1}y^{-\alpha}\left(\frac{\partial}{\partial y}\right)y^{\alpha}\right), \tag{43}$$

$$\hat{M}_2=\left(y-\left(\frac{\partial}{\partial\tau}\right)^{-1}x^{-\beta}\left(\frac{\partial}{\partial x}\right)x^{\beta}\right), \tag{44}$$

and the derivatives operators

$$\hat{P}_1=x^{-\beta}\left(\frac{\partial}{\partial x}\right)x^{\beta}, \tag{45}$$

$$\hat{P}_2=y^{-\alpha}\left(\frac{\partial}{\partial y}\right)y^{\alpha}. \tag{46}$$

According to the quasi-monomiality properties, we have

$$\hat{M}_1\left\{\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\right\}=\mathfrak{S}_{m+1,n}^{(\alpha,\beta)}(x,y;\tau), \tag{47}$$

$$\hat{M}_2\left\{\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\right\}=\mathfrak{S}_{m,n+1}^{(\alpha,\beta)}(x,y;\tau), \tag{48}$$

$$\hat{P}_1\left\{\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\right\}=(\beta+m)\mathfrak{S}_{m-1,n}^{(\alpha,\beta)}(x,y;\tau), \tag{49}$$

$$\hat{P}_2\left\{\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\right\}=(\alpha+n)\mathfrak{S}_{m,n-1}^{(\alpha,\beta)}(x,y;\tau). \tag{50}$$

Therefore, the identity

$$\left(\hat{M}_1\hat{P}_1+\hat{M}_2\hat{P}_2\right)=(\beta+\alpha+m+n)\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau), \tag{51}$$

in differential forms give us

$$\left[x^{1-\beta}\frac{\partial}{\partial x}y^{\alpha}+y^{1-\alpha}\frac{\partial}{\partial y}x^{-\beta}-2\left(\frac{\partial}{\partial\tau}\right)^{-1}y^{-\alpha}x^{-\beta}\frac{\partial^2}{\partial x\partial y}\right]y^{\alpha}x^{\beta}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$$
$$=(\beta+\alpha+m+n)\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau). \tag{52}$$

Moreover, the operational identities in (40)–(43) can be handled to get the new differential relations

$$\left[y^{-\alpha}x^{-\beta}\frac{\partial^2}{\partial x\partial y}y^{\alpha}x^{\beta}+\frac{\partial}{\partial\tau}\tau\frac{\partial}{\partial\tau}\right]\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)=0, \tag{53}$$

$$\left[y^{-\alpha}x^{-\beta}\left(\frac{\partial}{\partial x}\right)^{-1}\left(\frac{\partial}{\partial y}\right)^{-1}y^{\alpha}x^{\beta}+\left(-\frac{\partial}{\partial\tau}\tau\frac{\partial}{\partial\tau}\right)^{-1}\right]\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)=0. \tag{54}$$

5. Generating relations

In this section, we show how readily new generating functions for the polynomials $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ can be derived from their operational representations.

First, in the identity (16) multiply throughout by $\frac{t^n}{(\alpha+n)!}$ sum and then employ the well-known generating function (see e.g., [25]):

$$e^{yt}{}_0F_1\left[-;1;-xt\right]=\sum_{n=0}^{\infty}L_n(x,y)\frac{t^n}{n!}, \tag{55}$$

to get

$$\left\{x^{-\beta}(y\partial_y)^{-\alpha}\right\}\,e^{yt}{}_0F_1\left[-;1;-\tau\frac{\partial}{\partial x}\right]\left\{x^{\beta+m}\right\}=\sum_{n=0}^{\infty}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\frac{t^n}{n!}. \tag{56}$$

In the same manner, from the operational identity in (15) and (3), one can derive the following unilateral generating function

$$e^{xu+yv-(\frac{\partial}{\partial\tau})^{-1}uv}=\left\{x^{\beta}y^{\alpha}\right\}\sum_{m,n=0}^{\infty}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\frac{u^m}{m!}\frac{v^n}{n!}. \tag{57}$$

Again, by starting from Equation (28) multiplying throughout by $\frac{u^m}{m!}\frac{v^n}{n!}$, and exploiting the previously outlined method, we can show that

$$\left\{y^{-\alpha}\right\}\left(1-x^{-1}\left(\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial y}\right)^{\beta}\,e^{\left[\left(xu-u\left(\frac{\partial}{\partial\tau}\right)^{-1}\frac{\partial}{\partial y}+yv\right]}\left\{y^{\alpha}\right\}$$
$$=\sum_{m,n=0}^{\infty}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\frac{u^m}{m!}\frac{v^n}{n!}. \tag{58}$$

The previously outlined procedure offers a useful tool for the derivation of other families of generating functions for the polynomials $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$. For instance, let us consider the generating relation:

$$f\left(x,y,w,z;\lambda,\tau|u,v\right)=\sum_{m,n=0}^{\infty}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\lambda)\times\mathfrak{S}_{m,n}^{(\alpha,\beta)}(w,z;\tau)\frac{u^m}{m!}\frac{v^n}{n!}, \tag{59}$$

which according to Equations (28) yields the following bilinear-generating function:

$$
\begin{aligned}
&(yw)^{-\alpha}\left[\left(1-x^{-1}\left(\frac{\partial}{\partial\lambda}\right)\frac{\partial}{\partial y}\right)\left(1-z^{-1}\left(\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial w}\right)\right]^{\beta}\\
&\quad\times e^{\left[\left(1-x^{-1}\left(\frac{\partial}{\partial\lambda}\right)\frac{\partial}{\partial y}\right)\left(1-z^{-1}\left(\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial w}\right)xzu+ywv\right]}(yw)^{\alpha}\\
&=\sum_{m,n=0}^{\infty}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\lambda)\times\mathfrak{S}_{m,n}^{(\alpha,\beta)}(w,z;\tau)\frac{u^m}{m!}\frac{v^n}{n!}.
\end{aligned}
\tag{60}
$$

In [3] the following 2D-Laguerre-Konhauser polynomials have been introduced

$$
{}_kL_n^{(\alpha,\beta)}(x,y)=n!\sum_{s}^{n}\sum_{r}^{n-s}\frac{(-1)^{s+r}x^{\alpha+r}y^{\beta+ks}}{s!r!(n-s-r)!\Gamma(\alpha+r+1)\Gamma(\beta+ks+1)},\tag{61}
$$

where $k=1,2,\cdots$, together with the operational identity

$$
\left(1-\left(\frac{\partial}{\partial z}\right)^{-1}-\left(\frac{\partial}{\partial w}\right)^{-k}\right)^{n}\left\{\frac{z^{\gamma}w^{\delta}}{\Gamma(\gamma+1)\Gamma(\delta+1)}\right\}={}_kL_n^{(\gamma,\delta)}(z,w).\tag{62}
$$

Let us consider the generating relation

$$
f(x,y,w,z|t)=\sum_{n=0}^{\infty}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\times{}_kL_n^{(\gamma,\delta)}(z,w)\frac{t^n}{n!}.\tag{63}
$$

Now, directly from (29) and (53) by employing the previously outlined method leading to the bilinear generating function, we obtain from (63) the following bilateral generating function:

$$
\begin{aligned}
&x^{-\beta}\left(1-x^{-1}\left(\frac{\partial}{\partial\lambda}\right)^{-1}\frac{\partial}{\partial y}\right)^{\alpha}e^{\left(1-x^{-1}\left(\frac{\partial}{\partial\tau}\right)^{-1}\frac{\partial}{\partial y}\right)\left(1-\left(\frac{\partial}{\partial z}\right)^{-1}-\left(\frac{\partial}{\partial w}\right)^{-1}\right)yt}\\
&\qquad\times\left\{\frac{z^{\gamma}w^{\delta}x^{\beta+m}y^{\alpha}}{\Gamma(\gamma+1)\Gamma(\delta+1)}\right\}\\
&=\sum_{n=0}^{\infty}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\times{}_kL_n^{(\gamma,\delta)}(z,w)\frac{t^n}{n!}.
\end{aligned}
\tag{64}
$$

6. Series expansions

In this section, we establish expansion formulas for the polynomials $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$. In addition, we provide expansion of the product of two $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ polynomials as a sum involving Appellś series in two variables.

Let

$$
\hat{H}=1-x^{-1}\left(\frac{\partial}{\partial\tau}\right)^{-1}\frac{\partial}{\partial y}.
$$

Then from (28) we can state that:

$$\left\{y^{-\alpha}\right\}\left[1-\hat{H}\right]^{n}\left\{x^{m}y^{\alpha+n}\right\}=\sum_{s=0}^{n}\frac{(-n)_{s}x^{m-s}}{s!}\mathfrak{S}_{s,n}^{(\alpha,\beta)}(x,y;\tau). \tag{65}$$

Alternatively, if we let

$$\hat{H}=z-1+x^{-1}\left(\frac{\partial}{\partial\tau}\right)^{-1}\frac{\partial}{\partial y},$$

and applying the same method leading to (65), we find

$$\mathfrak{S}_{n,m}^{(\alpha,\beta)}(x,y;\tau)=\sum_{s=0}^{n}\frac{(-n)_{s}z^{n-s}x^{m-s}}{s!}\mathfrak{S}_{s,n}^{(\alpha,\beta)}(x(1-z),y;\tau). \tag{66}$$

Further, the generalized $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ polynomials (10), given Taylor's formulas:

$$\sum_{s=0}^{\infty}\frac{z^{s}}{s!}\left(\frac{\partial}{\partial x}\right)^{s}f(x)=f(x+z),$$

$$\sum_{s=0}^{\infty}\frac{[(z-1)x]^{s}}{s!}\left(\frac{\partial}{\partial x}\right)^{s}f(x)=f(xz),$$

yields the following interesting addition and multiplication formulas:

$$e^{z\frac{\partial}{\partial\tau}}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)=\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau+z), \tag{67}$$

$$e^{z\frac{\partial}{\partial x}}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)=\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x+z,y;\tau), \tag{68}$$

$$e^{z\frac{\partial}{\partial y}}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)=\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y+z;\tau), \tag{69}$$

$$e^{(z-1)\tau\frac{\partial}{\partial\tau}}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)=\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau z), \tag{70}$$

$$e^{(z-1)x\frac{\partial}{\partial x}}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)=\mathfrak{S}_{m,n}^{(\alpha,\beta)}(xz,y;\tau), \tag{71}$$

$$e^{(z-1)y\frac{\partial}{\partial y}}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)=\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,yz;\tau). \tag{72}$$

Similarly, the use of the inverse operator $\left(\frac{\partial}{\partial z}\right)^{-1}$ allows to conclude

$$\sum_{s=0}^{\infty}\frac{(\beta+m)!z^{s}}{s!(\beta+m-2s)!}\left(\frac{\partial}{\partial x}\right)^{s}x^{\beta}\left\{\mathfrak{S}_{m-2s,n}^{(\alpha,\beta)}(x,y;\tau)\right\}=(x+z)^{\beta}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x+z,y;\tau),\ (73),$$

$$\sum_{s=0}^{\infty}\frac{(\alpha+n)!z^{s}}{s!(\alpha+n-2s)!}\left(\frac{\partial}{\partial y}\right)^{s}y^{\alpha}\left\{\mathfrak{S}_{m,n-2s}^{(\alpha,\beta)}(x,y;\tau)\right\}=(y+z)^{\alpha}\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x+z,y;\tau). \tag{74}$$

REMARK 2. For $z=-x,\ (z=-y,\ z=-\tau)$, the results (67), (68), (69), (73) and (74) reduce immediately to curious results since $x(y,\tau)$ does not appear on the right-hand side of the equations (67), (68), (69), (73), and (74).

Finally, we establish the expansion of the product

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\,\mathfrak{S}_{m,k}^{(\alpha,\beta)}(z,w;\lambda).$$

In light of relation (28), we now consider the following identity:

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\times\mathfrak{S}_{m,k}^{(\alpha,\beta)}(z,w;\lambda)$$

$$=(yw)^{-\alpha}\left(1-x^{-1}\left(\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial y}\right)^{\beta+m}\times\left(1-z^{-1}\left(\frac{\partial}{\partial\lambda}\right)\frac{\partial}{\partial w}\right)^{\beta+m}\left\{x^m y^{\alpha+n}z^m w^{\alpha+k}\right\}$$

$$=(yw)^{-\alpha}\left[1-x^{-1}\left(\frac{\partial}{\partial\tau}\right)\frac{\partial}{\partial y}-z^{-1}\left(\frac{\partial}{\partial\lambda}\right)\frac{\partial}{\partial w}+(xz)^{-1}\left(\frac{\partial^2}{\partial\tau\partial\lambda}\right)^{-1}\frac{\partial^2}{\partial y\partial w}\right]^{\beta+m}$$

$$\times\left\{x^m y^{\alpha+n}z^m w^{\alpha+k}\right\}.$$

Using the multinomial theorem

$$(1-z_1-\ldots z_k)^{\gamma}=\sum_{n_1\ldots n_k=0}^{\infty}(-\gamma)_{n_1+\ldots+n_k}\frac{z_1^{n_1}}{n_1!}\cdots\frac{z_k^{n_k}}{n_k!},$$

one obtains, after routine calculations, the following identity:

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\times\mathfrak{S}_{m,k}^{(\alpha,\beta)}(z,w;\lambda)$$

$$=\sum_{p,q,r=0}^{\infty}\frac{(-1)^r(-(\beta+m))_{r+p+q}(\alpha+n)!(\alpha+k)!x^{m-r-p}z^{m-r-q}y^{n-r-p}w^{k-r-q}\lambda^{r+p}\tau^{r+q}}{r!p!q!(r+p)!(r+q)!(\alpha+n-r-p)!(\alpha+k-r-q)!},$$

On using the results [24]:

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}\quad\text{and}\quad(a)_{-n}=\frac{(-1)^n}{(1-a)_n},$$

employing the definition of Applls̀ function of two variables F_2 [24]:

$$F_2(a,b_1,b_2;c_1,c_2;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{m!n!(c_1)_m(c_2)_m}x^m y^n,$$

we obtain the expansion:

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)\times\mathfrak{S}_{m,k}^{(\alpha,\beta)}(z,w;\lambda)$$

$$=(xz)^m y^n w^k\sum_{r=0}^{\infty}\frac{(-\beta-m)_r(-\alpha-n)_r(-\alpha-k)_r}{(r!)^3}\left(\frac{-\tau\lambda}{xyzw}\right)^r$$

$$\times F_2\left[r-\beta-m,r-n-\alpha,r-k-\alpha;r+1,r+1;\frac{-\tau}{xy},\frac{-\lambda}{zw}\right].\qquad(75)$$

7. Applications to heat-type equations

In this section, as applications of our Hermite polynomials $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$, we look at the solutions to two types of heat equations. These applications demonstrate the real-world relevance and potential impact of novel definition $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ and establishes their role as fundamental building blocks in diffusion processes and analytic function theory. The standard Hermite polynomials are classical solutions of the heat equation, which in one dimension takes the form

$$\frac{\partial u}{\partial t}=\kappa\frac{\partial^2 u}{\partial x^2}=\kappa\Delta_x,\ x\in\mathbb{R},\ t>0,$$

where Δ_x is the Laplace operator in one dimension given by $\Delta_x f=\frac{\partial^2 f}{\partial x^2}$.

In two dimensions, the corresponding equations are

$$\frac{\partial u}{\partial t}=\kappa\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\kappa\Delta_{x,y},\ (x,y)\in\mathbb{R}^2,$$

$\Delta_{x,y}$ is the Laplace operator in tow dimension given by $\Delta_x f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$.

More generally, in the fractional diffusion setting one considers the equations

$$\frac{\partial u}{\partial t}=-\kappa(-\Delta)^{\nu/2}u,\qquad 0<\nu\leqslant 2,$$

where $(-\Delta)^{\nu/2}$ denotes the fractional Laplacian.

First, we demonstrate that the generalized complex Hermite polynomials $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ arise naturally as explicit solutions of standard heat-type equations in one and two dimensions. For this seek, let us consider an helpful generating function in the form:

THEOREM 5. *For the polynomials $\mathfrak{S}_{m,n}^{\alpha,\beta}(x,y;\tau)$ with $\tau=-\tau$, we have the following generating function:*

$$U(x,y;\tau)=\sum_{m,n=0}^{\infty}\mathfrak{S}_{m,n}^{\alpha,\beta}(x,y;\tau)\frac{t^m s^n}{(\beta+m)!(\alpha+n!)!}=(xt)^{-\beta}(ys)^{-\alpha}e^{xt+ys+(\partial_\tau)^{-1}ts}. \tag{76}$$

Proof. Using

$$(xt)^{-\beta}(ys)^{-\alpha}e^{xt+ys+(\partial_\tau)^{-1}ts}=(xt)^{-\beta}(ys)^{-\alpha}e^{xt}e^{ys}e^{(\partial_\tau)^{-1}ts}$$

and the Taylor expansion series for the exponential function; one can easily derive (76). □

REMARK 3. In the generalized function (76) the exponential part $xt+ys$ is the free generating term, giving uncoupled variables. The term $-(\partial_\tau)^{-1}ts$ introduces coupling between t, s via the operator $(\partial_\tau)^{-1}$. Mathematically, $(\partial_\tau)^{-1}$ is the inverse of derivative operator ∂_τ. That is, if $(\partial_\tau)G(\tau)=f(\tau)$ then $G(\tau)=(\partial_\tau)^{-1}f(\tau)$. Indeed, for r $\in\mathbb{N}_0$, we have $(\partial_\tau)^{-r}=\frac{\tau^r}{r!}$.

THEOREM 6. (2D-Heat-type equation). *The polynomials* $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ *satisfy the one-dimensional heat-like equations:*

$$\frac{\partial U}{\partial\tau}=\frac{ts}{\left(\frac{\beta(\beta+1)}{x^2}-\frac{2t\beta}{x}+t^2\right)}\frac{\partial^2 U}{\partial x^2},\quad\left(\frac{\beta(\beta+1)}{x^2}-\frac{2t\beta}{x}+t^2\right)>0 \tag{77}$$

$$\frac{\partial U}{\partial\tau}=\frac{ts}{\left(\frac{\alpha(\alpha+1)}{y^2}-\frac{2s\alpha}{y}+s^2\right)}\frac{\partial^2 U}{\partial y^2},\quad\left(\frac{\alpha(\alpha+1)}{y^2}-\frac{2s\alpha}{y}+s^2\right)>o. \tag{78}$$

In particular, we have the following one-dimensional heat-type equations:

$$\frac{\partial U}{\partial\tau}=\left(\frac{s}{t}\right)\frac{\partial^2 U}{\partial x^2},\quad\left(\frac{s}{t}\right)>0, \tag{79}$$

$$\frac{\partial U}{\partial\tau}=\left(\frac{t}{s}\right)\frac{\partial^2 U}{\partial y^2},\quad\left(\frac{t}{s}\right)>0. \tag{80}$$

Proof. Using (76), we can compute the following derivations:

$$\frac{\partial U}{\partial\tau}=tsU, \tag{81}$$

$$\frac{\partial^2 U}{\partial x^2}=\left(\frac{\beta(\beta+1)}{x^2}-\frac{2t\beta}{x}+t^2\right)U, \tag{82}$$

$$\frac{\partial^2 U}{\partial y^2}=\left(\frac{\alpha(\alpha+1)}{y^2}-\frac{2s\alpha}{y}+s^2\right)U. \tag{83}$$

Combining the identity (81) with (82) and the identity (81) with (83), we obtain the 1D heat-like equations (77) and (79) respectively. whereas letting $\alpha=\beta=0$ in (77) and (78), give as the heat-type equations (79) and (80) respectively. □

As generalization of the assertions (77) to (80), we present the following result.

THEOREM 7. (2D-Heat-type equation). *The polynomials* $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ *satisfy the two-dimensional heat-like equations:*

$$\frac{\partial U}{\partial\tau}=\frac{ts}{\left(\frac{\beta(\beta+1)}{x^2}+\frac{\alpha(\alpha+1)}{y^2}-\frac{2t\beta}{x}-\frac{2s\alpha}{y}+t^2+s^2\right)}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right). \tag{84}$$

In particular, we have the following two-dimensional heat-type equations:

$$\frac{\partial U}{\partial\tau}=\frac{ts}{t^2+s^2}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right),\quad\left(\frac{ts}{t^2+s^2}\right)>0. \tag{85}$$

Proof. For the proof of (84), we refer to the proof of Theorem 1. Letting $\alpha = \beta = 0$ in (84), we infer the result (85). $\square$

REMARK 4. If in (85) $t^2 + s^2 = 0$ then $t = s = 0$, and both sides vanish, so the identity still holds.

REMARK 5. Since $(-\Delta)^{\alpha/2}$ is a Fourier multiplier, we can compute the Fourier transform U only in the case when U is exponential function. Because, only the exponentials diagonalize the operator in Fourier space. Unfortunately, the generating function (76) involves the following product forms: $(xt)^{-\beta} e^{xt}, (ys)^{-\alpha} e^{ys}$. Thus, to derive the fractional heat-type equation, we should let $\alpha = \beta = 0$. In this case, we get the modified polynomials:

$$\mathfrak{S}_{m,n}(x,y,\tau) = \sum_{k=0}^{\infty} \binom{n}{k} \binom{m}{k} (-1)^k \tau^k x^{m-k} y^{n-k} \quad (m,n \in \mathbb{N}_0). \tag{86}$$

Their generating function is given by:

$$U(x,y;\tau) = \sum_{m,n=0}^{\infty} \mathfrak{S}_{m,n}(x,y;\tau) \frac{t^m s^n}{m!n!} = e^{xt+ys+(\partial_\tau)^{-1} ts}. \tag{87}$$

THEOREM 8. (Fractional heat-type equation). *The polynomials $\mathfrak{S}_{m,n}(x,y,\tau)$ satisfy the following two-dimensional heat-type equation involving the fractional Laplacian:*

$$\frac{\partial U}{\partial g} = \frac{ts}{(t^2+s^2)^{\nu/2}} (-\Delta)^{\nu/2} U\,, \tag{88}$$

where $1 < \nu \leqslant 2$ and $(-\Delta)^{\nu/2}$ is the fractional Laplacian.

Proof. We have already computed $\frac{\partial U}{\partial \tau}$ in (81). So, we need only to compute $(-\Delta)^{\nu/2} U$.

Since $(-\Delta)^{\nu/2}$ acts only on (x,y), we can write

$$(-\Delta)^{\nu/2} U(x,y;\tau) = e^{(\partial_\tau)^{-1} ts} (-\Delta)^{\nu/2} e^{xt+ys}.$$

Because $(-\Delta)^{\nu/2}$ is a Fourier multiplier with symbol $|\xi|^\nu$ [20]:

$$\mathscr{F}\left[(-\Delta)^{\nu/2} f\right](\xi) = |\xi|^\nu \hat{f}(\xi), \qquad \xi \in \mathbb{R}^2,$$

and U is an exponential function, we can use the fact that for exponential functions, the fractional Laplacian can be computed by:

$$(-\Delta)^{\nu/2} e^{ax} = |a|^\nu e^{ax},$$

and consequently,

$$\begin{aligned}(-\Delta)^{\nu/2} U = (-\Delta)^{\nu/2} \left[e^{(\partial_\tau)^{-1}\, ts} e^{tx+uy} \right] &= e^{(\partial_\tau)^{-1}\, ts} (-\Delta)^{\nu/2} e^{tx+sy} \\ &= e^{(\partial_\tau)^{-1}\, ts} (t^2+s^2)^{\nu/2} e^{tx+sy}.\end{aligned}$$

But $U = e^{(\partial_\tau)^{-1} ts} e^{tx+sy}$, hence:

$$(-\Delta)^{\nu/2} U = (t^2+s^2)^{\nu/2} U. \tag{89}$$

Combine (81) and (89) to get the fractional 2D-heat-type partial differential equation (88). □

REMARK 6. If $\nu = 2$, $(-\Delta)^{\nu/2} = (-\Delta)^1 = -\Delta$. Then (88) becomes

$$\partial_\tau U = \frac{ts}{(t^2+s^2)}(-\Delta)U,$$

which is equivalent to equation (85). Also, if $t = s = 0$ both sides vanish in (85), so the identity still holds.

THEOREM 9. *For the polynomial $\mathfrak{s}^{m,n}(x,y;\tau)$ we have the following Green-type recursion relation:*

$$\frac{\partial}{\partial \tau} \mathfrak{S}_{m,n}(x,y,\tau) = mnp \mathfrak{S}_{m-1,n-1}(x,y,\tau), \;\; m,n \geqslant 1. \tag{90}$$

Proof. Because $\frac{\partial}{\partial \tau} U = (ts)U$, comparing coefficients of $t^m s^n$ in (76) yields the identity (90). □

REMARK 7. The Green-type recursion in (90) provides a systematic tool for generating higher-order polynomials and finds applications in fractional heat-type equations, multidimensional diffusion models, and analytical studies in quantum and statistical physics. For example, in the fractional 2D-heat equation (90) one can represent U as a series of $\mathfrak{S}_{m,n}(x,y,\tau)$ to get,

$$\frac{\partial}{\partial \tau} \mathfrak{S}_{m,n}(x,y,\tau) = \frac{ts}{(t^2+s^2)^{\nu/2}} (-\Delta)^{\nu/2} \mathfrak{S}_{m,n}(x,y,\tau);,$$

and the recursion (90) gives the action of ∂_τ on each term.

8. Concluding remarks

In this work, using the series rearrangement technique (see [23]), we introduce a novel class of Hermite-Laguerre polynomials $\mathfrak{S}m,n^{(\alpha,\beta)}(x,y;\tau)$ and investigate their key properties, including operational identities, quasi-monomials, partial differential equations, generating functions, and series expansions. We also derive an expansion formula for the product of two such polynomials in terms of Appell's two-variable series F_2. Moreover, we show that these polynomials yield explicit solutions to fractional and classical heat equations in one and two dimensions, underscoring their analytical depth and broad applicability. The Hermite polynomials $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ discussed in this paper will be useful for investigators in various disciplines of applied sciences and engineering, for example, in the analysis of fractional differential equations (see e.g., [18, 19]). For instance, the Laguerre 2D polynomials effectively reflect numerous conclusions in quantum optics, such as quasi-probabilities in Fock-state basis, ordering problems, moments, and other fields of physics (see [26, 27]). Two 2D-Laguerre polynomials

$$L_{m,n}(z,z^*|\tau) = m!(-\tau)^m z^{n-m} L_m^{n-m}\left(\frac{zz^*}{\tau}\right), \tag{91}$$

$$L_{m,n}(z,z^*) = (-1)^m m! z^{n-m} L_m^{n-m}(zz^*), \tag{92}$$

where introduced by Wünsche (see e.g., [27, 28, 29]). The Laguerre 2D polynomials in equations (91) and (92) and their adaptations to quantum optics problems are explored in [26] and [27].

In this regard, and to place our definition (2.3) in a form that admits broader applications, comparable to definitions (91) and (92), we will in forthcoming work introduce a modified version of the two-dimensional Laguerre polynomials (2.3), denoted by $L_{m,n}^{(\beta)}(z,z^*;\tau)$. These are defined in terms of the complex variables $(z = x+iy = re^{i\phi}$, $z^* = x-iy = re^{-i\phi})$ and the real parameter τ, in accordance with [26], [27], and [2]:

$$\mathfrak{S}_{m,n}^{(\alpha,\beta)}(z,z^*;\tau) = \sum_{k=0}^{n} \binom{m+\beta}{k}\binom{n}{k}(-\tau)^k z^{m-k} z^{*\,n-k}. \tag{93}$$

Also, in a subsequent work, we will tackle the problem of extending the polynomials $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ to a multivariable and multi-index class of Hermite and Laguerre polynomials. The framework developed for deriving the properties of $\mathfrak{S}_{m,n}^{(\alpha,\beta)}(x,y;\tau)$ can be adapted to establish novel properties for these generalized families.

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