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STRONGLY SEPARATELY EQUICONTINUOUS FAMILIES OF FUNCTIONS ON ℓ^2

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Abstract. In this paper, we introduce the notion of a strongly separately equicontinuous family of functions on ℓ^2. We establish several theorems analogous to classical results on equicontinuity in this new setting.

1. Introduction

Equicontinuity is a fundamental concept in the study of function spaces. It was introduced by [1] and [2] in the context of obtaining a result analogous to the Bolzano-Weierstrass theorem in function spaces. Let (X,d), (Y,ρ) be metric spaces and $C(X,Y)$ denote the space of all continuous functions from X to Y. A family $\mathscr{F} \subset C(X,Y)$ is said to be equicontinuous at $x \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in X$ and all $f \in \mathscr{F}$, the condition $d(x,y) < \delta$ implies $\rho(f(x), f(y)) < \varepsilon$. This notion plays a crucial role in the Arzela-Ascoli Theorem, which provides a criterion for a sequence of functions to have a uniformly convergent subsequence.

In [4], the author introduced the notion of a strongly separately continuous function on $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be strongly separately continuous at $\boldsymbol{x^0} = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$ with respect to the k-th variable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the condition $d(\boldsymbol{x}, \boldsymbol{x^0}) < \delta$ implies $|f(\boldsymbol{x}) - f(\boldsymbol{x}')| < \varepsilon$, where $\boldsymbol{x}' = (x_1, \ldots, x_{k-1}, x_k^0, x_{k+1}, \ldots, x_n)$. Based on this, it was shown in [4] that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous if and only if it is strongly separately continuous. Later, the authors of [3] extended this notion to the space ℓ^2 and showed that, in general, strong separate continuity of a function f does not imply its continuity.

Building on the concepts of strong separate continuity and equicontinuity, we introduce the notion of a strongly separately equicontinuous family of functions on ℓ^2. This notion enables the study of how such families behave under a specific convergence criterion.

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2. Strongly separately equicontinuous families of functions on ℓ^2

Let (X,d) be a metric space, $x^0 \in X$, and $\delta > 0$. Denote by $B_X(x^0,\delta)$ the open ball centered at x^0 with radius δ, that is,

$$B_X(x^0,\delta) := \{x \in X \,|\, d(x,x^0) < \delta\}.$$

Let ℓ^2 denote the space consisting of all real sequences $\boldsymbol{x} = (x_j)$ such that $\sum_{j=1}^{\infty} x_j^2 < \infty$, endowed with the metric d defined by

$$d(\boldsymbol{x},\boldsymbol{y}) = \sqrt{\sum_{j=1}^{\infty} (x_j - y_j)^2},$$

for all $\boldsymbol{x} = (x_j)$, $\boldsymbol{y} = (y_j) \in \ell^2$.

We first recall the definition of a strongly separately continuous functions on ℓ^2.

DEFINITION 1. [3, Definition 1.1] A function $f : \ell^2 \to \mathbb{R}$ is said to be strongly separately continuous at a point $\boldsymbol{x^0} = (x_j^0) \in \ell^2$ with respect to the k-th variable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\boldsymbol{x} = (x_j) \in \ell^2$, the condition $d(\boldsymbol{x},\boldsymbol{x^0}) < \delta$ implies

$$|f(\boldsymbol{x}) - f(\boldsymbol{x}')| < \varepsilon,$$

where $\boldsymbol{x}' = (x_1,\dots,x_{k-1},x_k^0,x_{k+1},\dots)$. The function f is said to be strongly separately continuous at $\boldsymbol{x^0}$ if f is strongly separately continuous at $\boldsymbol{x^0}$ with respect to the k-th variable, for all $k \in \mathbb{N}$. Moreover, f is said to be strongly separately continuous on ℓ^2 if it is strongly separately continuous at every point $\boldsymbol{x^0} \in \ell^2$.

For notational convenience, we denote, for any $\boldsymbol{a} = (a_j) \in \ell^2$ and $t \in \mathbb{R}$,

$$\boldsymbol{a}_k(t) = (a_1,a_2,\dots,a_{k-1},t,a_{k+1},\dots).$$

LEMMA 1. [3, Proposition 1.2] *If $f : \ell^2 \to \mathbb{R}$ is continuous at $\boldsymbol{x^0} = (x_j^0) \in \ell^2$, then f is strongly separately continuous at $\boldsymbol{x^0}$.*

The converse of Lemma 1 does not necessarily hold, as illustrated by Theorem 1 in [3], involving a set of type $(P1)$ whose definition is recalled below.

DEFINITION 2. [3] A subset $S \subseteq \ell^2$ is said to be a set of type $(P1)$ if the following condition holds: if $\boldsymbol{x} = (x_j) \in S$, $\boldsymbol{y} = (y_j) \in \ell^2$, and the set $\{j \in \mathbb{N} \,|\, x_j \neq y_j\}$ contains at most one element, then $\boldsymbol{y} \in S$.

We now introduce the precise definition of a strongly separately equicontinuous family, an essential concept in our paper.

DEFINITION 3. Let $\mathscr{F}$ be a family of functions $f : \ell^2 \to \mathbb{R}$ and $k \in \mathbb{N}$. A family $\mathscr{F}$ is said to be strongly separately equicontinuous at $\boldsymbol{x^0} = (x_j^0) \in \ell^2$ with respect to the k-th variable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\boldsymbol{x} = (x_j) \in \ell^2$ and all $f \in \mathscr{F}$, the condition $d(\boldsymbol{x^0},\boldsymbol{x}) < \delta$ implies

$$|f(\boldsymbol{x}) - f(\boldsymbol{x}_k(x_k^0))| < \varepsilon.$$

Moreover, $\mathscr{F}$ is said to be strongly separately equicontinuous at $\boldsymbol{x^0}$ if $\mathscr{F}$ is strongly separately equicontinuous at $\boldsymbol{x^0}$ with respect to the k-th variable, for all $k \in \mathbb{N}$. The family $\mathscr{F}$ is said to be strongly separately equicontinuous if it is strongly separately equicontinuous at every point $\boldsymbol{x^0} \in \ell^2$.

It is easy to see that if $\mathscr{F}$ is strongly separately equicontinuous, then each function $f \in \mathscr{F}$ is strongly separately continuous.

The following lemma follows directly from Definition 3.

LEMMA 2. *If $\mathscr{F}$ is equicontinuous at $\boldsymbol{x^0} = (x_j^0) \in \ell^2$, then $\mathscr{F}$ is strongly separately equicontinuous at $\boldsymbol{x^0}$.*

The converse implication does not hold in general. We show that there exists a family of functions which is strongly separately equicontinuous at every point but not equicontinuous at any point.

EXAMPLE 1. Let $S \subseteq \ell^2$ be a set of type $(P1)$ such that both S and $\ell^2 \setminus S$ are dense in ℓ^2. Such a set S exists according to [3, Example 1.3]. Define $g_n : \ell^2 \to \mathbb{R}$ by

$$g_n(\boldsymbol{x}) = \begin{cases} n, & \boldsymbol{x} \in S, \\ 0, & \boldsymbol{x} \in \ell^2 \setminus S. \end{cases}$$

Let $\mathscr{G} = \{g_n : \ell^2 \to \mathbb{R} \,|\, n \in \mathbb{N}\}$. Then $\mathscr{G}$ is strongly separately equicontinuous but not equicontinuous on ℓ^2. The first step is to show that $\mathscr{G}$ is strongly separately equicontinuous. Let $\boldsymbol{x^0} = (x_j^0) \in \ell^2$, $k \in \mathbb{N}$, and $\varepsilon > 0$ be fixed. Set $\delta = \varepsilon$. Let $\boldsymbol{x} = (x_j) \in B_{\ell^2}(\boldsymbol{x^0}, \delta)$ and $n \in \mathbb{N}$ be arbitrary. Since $\boldsymbol{x}_k(x_k^0)$ differs from $\boldsymbol{x}$ in at most one coordinate and S is of type $(P1)$, it follows that $\boldsymbol{x} \in S$ if and only if $\boldsymbol{x}_k(x_k^0) \in S$. Hence,

$$|g_n(\boldsymbol{x}) - g_n(\boldsymbol{x}_k(x_k^0))| = 0 < \varepsilon.$$

To show $\mathscr{G}$ is not equicontinuous at $\boldsymbol{x^0} \in \ell^2$, let $\varepsilon = \frac{1}{2}$ and $\delta > 0$ be arbitrary. The following two cases are considered.

1. $\boldsymbol{x^0} \in S$.
   Since $\ell^2 \setminus S$ is *dense* in ℓ^2, it follows that $B_{\ell^2}(\boldsymbol{x^0}, \delta) \cap (\ell^2 \setminus S) \neq \varnothing$, and therefore there exists $\boldsymbol{y} \in \ell^2 \setminus S$ such that $\boldsymbol{y} \in B_{\ell^2}(\boldsymbol{x^0}, \delta)$. Hence,

$$|g_1(\boldsymbol{x^0}) - g_1(\boldsymbol{y})| = |1 - 0| = 1 > \varepsilon.$$

2. $\boldsymbol{x^0} \notin S$.
   Because S is *dense* in ℓ^2, it follows that $B_{\ell^2}(\boldsymbol{x^0}, \delta) \cap S \neq \varnothing$, and therefore there exists $z \in S$ such that $z \in B_{\ell^2}(\boldsymbol{x^0}, \delta)$. Hence,

$$|g_1(\boldsymbol{x^0}) - g_1(z)| = |0 - 1| = 1 > \varepsilon. \quad \square$$

Recall that any finite family of continuous functions on a metric space is equicontinuous. A similar result holds in the context of strong separate continuity.

THEOREM 1. *If $\mathscr{F}$ is a finite collection of strongly separately continuous functions on ℓ^2, then $\mathscr{F}$ is strongly separately equicontinuous.*

Proof. Let $k \in \mathbb{N}$ and $\boldsymbol{x^0} = (x_j^0) \in \ell^2$. Since $\mathscr{F}$ is finite, it can be written as $\mathscr{F} = \{f_1, f_2, \ldots, f_j\}$ for some $j \in \mathbb{N}$. Let $\varepsilon > 0$. For each $m \in \{1, 2, \ldots, j\}$, the strong separate continuity of f_m at $\boldsymbol{x^0}$ implies that there exists $\delta_m > 0$ such that

$$|f_m(\boldsymbol{x}) - f_m(\boldsymbol{x}_k(x_k^0))| < \varepsilon$$

for all $\boldsymbol{x} \in B_{\ell^2}(\boldsymbol{x^0}, \delta)$. Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_j\}$. This yields

$$|f_m(\boldsymbol{x}) - f_m(\boldsymbol{x}_k(x_k^0))| < \varepsilon$$

for all $\boldsymbol{x} \in B_{\ell^2}(\boldsymbol{x^0}, \delta)$ and all $m = 1, 2, \ldots, j$. $\square$

The notion of weak local uniform convergence, considered in [3] as a weaker form of uniform convergence, is recalled in the following definition.

DEFINITION 4. [3, Definition 2.2] Let X be a metric space and (f_n) be a sequence of functions from X to $\mathbb{R}$. A sequence (f_n) is said to converge weakly locally uniformly to a function $f : X \to \mathbb{R}$ at $x^0 \in X$ if for every $\varepsilon > 0$, there exist $\delta > 0$ and $p \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon$$

holds for all $n \in \mathbb{N}$ with $n \geqslant p$ and all $x \in B_X(x^0, \delta)$. The sequence (f_n) is said to converge weakly locally uniformly to f on X if it converges weakly locally uniformly to f at every point $x^0 \in X$. Moreover, (f_n) is said to converge weakly locally uniformly if there exists a function $f : X \to \mathbb{R}$ such that (f_n) converges weakly locally uniformly to f on X.

The next theorem, originally proved by [3], states an important property of weakly locally uniformly convergent sequences.

THEOREM 2. [3, Theorem 2.3] *Let (f_n) be a sequence of strongly separately continuous functions on ℓ^2. If (f_n) converges weakly locally uniformly to a function $f : \ell^2 \to \mathbb{R}$ at $\boldsymbol{x^0} = (x_j^0) \in \ell^2$, then f is strongly separately continuous at $\boldsymbol{x^0}$.*

It is well known that a uniformly convergent sequence of continuous functions is equicontinuous. The following theorem extends this idea to the setting of strong separate equicontinuity, where convergence is considered in the weak local uniform sense.

THEOREM 3. *Let $n \in \mathbb{N}$ and $f_n : \ell^2 \to \mathbb{R}$ be a strongly separately continuous function on ℓ^2. If (f_n) converges weakly locally uniformly to a function $f : \ell^2 \to \mathbb{R}$ at $\boldsymbol{x^0} = (x_j^0) \in \ell^2$, then $\{f_n : n \in \mathbb{N}\}$ is strongly separately equicontinuous at $\boldsymbol{x^0}$.*

Proof. Let $k \in \mathbb{N}$ and $\varepsilon > 0$. Since (f_n) converges weakly locally uniformly to f at $\boldsymbol{x^0} = (x_j^0) \in \ell^2$, there exist $\delta_0 > 0$ and $p \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geqslant p$ and all $\boldsymbol{x} = (x_j) \in B_{\ell^2}(\boldsymbol{x^0}, \delta_0)$, the inequality

$$|f_n(\boldsymbol{x}) - f(\boldsymbol{x})| < \frac{\varepsilon}{3}$$

holds. Since f is strongly separately continuous at $\boldsymbol{x^0}$ by Theorem 2, there exists $\delta_1 > 0$ such that

$$|f(\boldsymbol{x}) - f(\boldsymbol{x}_k(x_k^0))| < \frac{\varepsilon}{3}$$

for all $\boldsymbol{x} \in B_{\ell^2}(\boldsymbol{x^0}, \delta_1)$. Let $\delta_2 = \min\{\delta_0, \delta_1\}$. Therefore, for all $n \in \mathbb{N}$ with $n \geqslant p$ and all $\boldsymbol{x} \in B_{\ell^2}(\boldsymbol{x^0}, \delta_2)$, it holds that

$$\begin{aligned} |f_n(\boldsymbol{x}) - f_n(\boldsymbol{x}_k(x_k^0))| &\leqslant |f_n(\boldsymbol{x}) - f(\boldsymbol{x})| + |f(\boldsymbol{x}) - f(\boldsymbol{x}_k(x_k^0))| \\ &\quad + |f(\boldsymbol{x}_k(x_k^0)) - f_n(\boldsymbol{x}_k(x_k^0))| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

According to Theorem 1, $\{f_n : 1 \leqslant n < p\}$ is strongly separately equicontinuous at $\boldsymbol{x^0}$, and therefore there exists $\delta_3 > 0$ such that

$$|f_n(\boldsymbol{x}) - f_n(\boldsymbol{x}_k(x_k^0))| < \varepsilon$$

for all $\boldsymbol{x} \in B_{\ell^2}(\boldsymbol{x^0}, \delta_3)$ and all $1 \leqslant n < p$. Let $\delta = \min\{\delta_2, \delta_3\}$. This leads to

$$|f_m(\boldsymbol{x}) - f_m(\boldsymbol{x}_k(x_k^0))| < \varepsilon$$

for all $\boldsymbol{x} \in B_{\ell^2}(\boldsymbol{x^0}, \delta)$ and all $m \in \mathbb{N}$. □

Motivated by the concept of weak local uniform convergence, as considered in [3], we introduce the notion of a weakly locally uniformly Cauchy sequence of functions, as defined below. Before doing so, we briefly recall the classical definition of a uniformly Cauchy sequence of functions. A sequence of functions (f_n) defined on a metric space X is called uniformly Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \varepsilon$$

for all $m, n \in \mathbb{N}$ with $m, n \geqslant N$ and all $x \in X$.

DEFINITION 5. Let X be a metric space and (f_n) be a sequence of functions from X to $\mathbb{R}$. A sequence (f_n) is said to be weakly locally uniformly Cauchy at $x^0 \in X$ if for every $\varepsilon > 0$, there exist $\delta > 0$ and $p \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \varepsilon$$

holds for all $m,\ n \in \mathbb{N}$ with $m,\ n \geqslant p$ and all $x \in B_X(x^0, \delta)$. The sequence (f_n) is said to be weakly locally uniformly Cauchy on X if it is weakly locally uniformly Cauchy at every point $x^0 \in X$.

We now present a theorem that plays a central role in establishing our main result.

THEOREM 4. *Let $X \subset \ell^2$ be a compact set and (f_n) be a sequence of functions $f_n : X \to \mathbb{R}$. If (f_n) is strongly separately equicontinuous and weakly locally uniformly Cauchy on X, then (f_n) is uniformly Cauchy on X.*

Proof. Let $\varepsilon > 0$ and $k \in \mathbb{N}$. Let $\boldsymbol{x} = (x_j) \in X$. Since (f_n) is strongly separately equicontinuous at $\boldsymbol{x}$, there exists $\delta_{\boldsymbol{x}} > 0$ such that for all $n \in \mathbb{N}$ and all $\boldsymbol{y} = (y_j) \in \ell^2$, the condition $d(\boldsymbol{x},\boldsymbol{y}) < \delta_{\boldsymbol{x}}$ implies

$$|f_n(\boldsymbol{y}) - f_n(\boldsymbol{y}_k(x_k))| < \frac{\varepsilon}{3}.$$

Since (f_n) is weakly locally uniformly Cauchy at $\boldsymbol{x}$, there exist $\alpha_{\boldsymbol{x}} > 0$ and $N_{\boldsymbol{x}} \in \mathbb{N}$ such that for all $n,\ m \in \mathbb{N}$ with $n,\ m \geqslant N_{\boldsymbol{x}}$ and all $\boldsymbol{y} \in B_{\ell^2}(\boldsymbol{x}, \alpha_{\boldsymbol{x}})$, the inequality

$$|f_n(\boldsymbol{y}) - f_m(\boldsymbol{y})| < \frac{\varepsilon}{3}$$

holds. Let $\gamma_{\boldsymbol{x}} = \min\{\delta_{\boldsymbol{x}}, \alpha_{\boldsymbol{x}}\}$ and $\mathscr{B} = \{B_{\ell^2}(\boldsymbol{x}, \gamma_{\boldsymbol{x}}) \,|\, \boldsymbol{x} \in X\}$. Since X is compact and $\mathscr{B}$ is an open cover of X, there exist $\boldsymbol{x}^{\boldsymbol{1}}, \ldots, \boldsymbol{x}^{\boldsymbol{p}} \in X$ such that

$$X \subset \bigcup_{i=1}^{p} B_{\ell^2}(\boldsymbol{x}^{\boldsymbol{i}}, \gamma_{\boldsymbol{x}^i}).$$

Let $N = \max\{N_{\boldsymbol{x}^i} \,|\, i = 1, \ldots, p\}$, $n,\ m \in \mathbb{N}$ with $n,\ m \geqslant N$, and $\boldsymbol{y} \in X$. There exists $i^* \in \{1, \ldots, p\}$ such that $\boldsymbol{y} \in B_{\ell^2}(\boldsymbol{x}^{\boldsymbol{i}^*}, \gamma_{\boldsymbol{x}^{i^*}})$, which implies $\boldsymbol{y}_k(x_k^{i^*}) \in B_{\ell^2}(\boldsymbol{x}^{\boldsymbol{i}^*}, \gamma_{\boldsymbol{x}^{i^*}})$. Consequently,

$$\begin{aligned} |f_n(\boldsymbol{y}) - f_m(\boldsymbol{y})| &\leqslant |f_n(\boldsymbol{y}) - f_n(\boldsymbol{y}_k(x_k^{i^*}))| + |f_n(\boldsymbol{y}_k(x_k^{i^*})) - f_m(\boldsymbol{y}_k(x_k^{i^*}))| \\ &\quad + |f_m(\boldsymbol{y}_k(x_k^{i^*})) - f_m(\boldsymbol{y})| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \quad \square \end{aligned}$$

Note that weak local uniform convergence implies the weak local uniform Cauchy property. Combined with the fact that a uniformly Cauchy sequence of functions converges uniformly, this leads to the following corollary which is the main result of this paper.

COROLLARY 1. *Let $X \subset \ell^2$ be a compact set and (f_n) be a sequence of functions $f_n : X \to \mathbb{R}$. If (f_n) is strongly separately equicontinuous and converges weakly locally uniformly on X, then (f_n) converges uniformly on X.*

It is known that if an equicontinuous sequence of functions converges pointwise on a compact metric space, then it actually converges uniformly. The corollary above can be viewed as an analogue of this classical result in the setting of strong separate equicontinuity. Motivated by the classical Arzela-Ascoli theorem, we pose the following question.

QUESTION 1. *Could one find sufficient conditions that ensure a strongly separately equicontinuous sequence of functions admits a uniformly or pointwise convergent subsequence?*

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