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FOURTH HANKEL AND TOEPLITZ DETERMINANTS OF SYMMETRIC STARLIKE FUNCTIONS CONNECTED WITH GREGORY COEFFICIENTS

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Abstract. This paper introduces and investigates a subclass of analytic functions consisting of symmetric starlike functions associated with Gregory coefficients. Sharp coefficient estimates are established, along with results on the Fekete-Szegö functional and the Generalized Zalcman conjecture. Additionally, we derive estimates for the third and fourth Hankel determinants, as well as bounds for Toeplitz determinants of a certain order. The study further extends to the inverse functions, providing estimates for initial coefficients, the third Hankel determinant, and bounds for Toeplitz determinants. Moreover, Krushkal's inequality is examined for both functions and its inverse in the defined class. Our findings coincide with certain results in the existing literature while also providing notable improvements over previous works.

1. Introduction

Let $\mathscr{A}$ be the class of analytic functions $\mathfrak{f}$ defined in the unit disk $\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, expressed in the form:

$$\mathfrak{f}(\zeta) = \zeta + \sum_{j=2}^{\infty} b_j \zeta^j, \tag{1}$$

where the function satisfies the conditions $\mathfrak{f}(0) = 0$ and $\mathfrak{f}'(0) = 1$. Furthermore, let $\mathscr{S}$ represent a subclass of $\mathscr{A}$ consisting of functions that are univalent in $\mathbb{U}$. It is well established that if $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are two analytic functions defined in $\mathbb{U}$, then $\mathfrak{f}_1$ is said to be subordinate to $\mathfrak{f}_2$, denoted as $\mathfrak{f}_1 \prec \mathfrak{f}_2$, if there exists an Schwarz function ϕ such that $\phi(0) = 0$ and $|\phi(\zeta)| < 1$ for all $\zeta \in \mathbb{U}$, satisfying the relation $\mathfrak{f}_1(\zeta) = \mathfrak{f}_2(\phi(\zeta))$. In the particular case where $\mathfrak{f}_2$ is univalent in $\mathbb{U}$, this subordination condition is equivalent to the following:

$$\mathfrak{f}_1 \prec \mathfrak{f}_2, \qquad (\zeta \in \mathbb{U}) \qquad \Longleftrightarrow \qquad \mathfrak{f}_1(0) = \mathfrak{f}_2(0) \quad \text{and} \quad \mathfrak{f}_1(\mathbb{U}) \subset \mathfrak{f}_2(\mathbb{U}).$$

The notable subclasses of $\mathscr{S}$ include the starlike functions, denoted by $\mathscr{S}^*$, convex functions, $\mathscr{C}$, and close-to-convex functions, $\mathscr{K}$. The analytical definitions of

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these classes are as follows:

$$\begin{aligned}\mathscr{S}^* &= \left\{\mathfrak{f} \in \mathscr{S} : \operatorname{Re}\left(\frac{\zeta\mathfrak{f}'(\zeta)}{\mathfrak{f}(\zeta)}\right) > 0, \quad \zeta \in \mathbb{U}\right\}, \\ \mathscr{C} &= \left\{\mathfrak{f} \in \mathscr{S} : \operatorname{Re}\left(\frac{(\zeta\mathfrak{f}'(\zeta))'}{\mathfrak{f}'(\zeta)}\right) > 0, \quad \zeta \in \mathbb{U}\right\} \quad \text{and} \\ \mathscr{K} &= \left\{\mathfrak{f} \in \mathscr{S} : \operatorname{Re}\left(\frac{\zeta\mathfrak{f}'(\zeta)}{\mathfrak{g}(\zeta)}\right) > 0, \text{ where } \mathfrak{g} \in \mathscr{S}^*, \quad \zeta \in \mathbb{U}\right\}. \end{aligned} \tag{2}$$

We note that, taking $\mathfrak{g}(\zeta) = \zeta$ in (2) we get the class of functions of bounded turning denoted by $\mathscr{R}$.

Another important subclass of $\mathscr{S}$, denoted by $\mathscr{S}_s^*$ was introduced by Sakaguchi [53] consists of starlike functions about symmetric points and is defined by

$$\mathscr{S}_s^* = \left\{\mathfrak{f} \in \mathscr{S} : \left(\frac{2\zeta\mathfrak{f}'(\zeta)}{\mathfrak{f}(\zeta) - \mathfrak{f}(-\zeta)}\right) > 0, \quad \zeta \in \mathbb{U}\right\}. \tag{3}$$

DEFINITION 1. [51] A function $\mathfrak{f} \in \mathscr{A}$, is in the class $\mathscr{S}^*_{\text{sym}}(\phi)$ consisting of symmetric starlike functions if

$$\frac{2\zeta\mathfrak{f}'(\zeta)}{\mathfrak{f}(\zeta) - \mathfrak{f}(-\zeta)} \prec \phi(\zeta), \quad \zeta \in \mathbb{U}. \tag{4}$$

Replacing $\phi(\zeta)$ with e^ζ leads to the class $\mathscr{S}^*_{\text{sym}}(e^\zeta)$, which consists of starlike functions with respect to symmetric points associated with the exponential function. This class was introduced by Ganesh et al. [18]. Later, Zaprawa [70] refined some of the results from [18], which were further improved by Li et al. [35]. Recently, Tang et al. [61] and Abbas et al. [1] investigated the class of symmetric starlike functions associated with the petal-shaped domain. Several authors have examined the class defined in (4) from various viewpoints (see, for example, [3, 16, 19, 31, 37, 51, 54, 67]).

The problem of determining coefficient bounds provides valuable insights into the geometry of complex-valued functions and is essential in studying growth and distortion theorems for functions in the class S. Similarly, Hankel determinants are particularly useful in analyzing singularities and power series with integral coefficients. In the theory of univalent functions, significant attention has been given to estimating the bounds of Hankel and Toeplitz matrices, which play a crucial role in various branches of mathematics and have numerous applications (see [45, 68] for more details). While Toeplitz matrices have constant entries along the main diagonal, Hankel matrices are characterized by constant entries along the reverse diagonal. The Hankel determinant of $\mathfrak{f} \in \mathscr{A}$ for $q \geqslant 1$ and $j \geqslant 1$ was first defined by Pommerenke [45] as follows:

$$\mathbb{H}_{q,j}(\mathfrak{f}) = \begin{vmatrix} b_j & b_{j+1} & \cdots & b_{j+q-1} \\ b_{j+1} & b_{j+2} & \cdots & b_{j+q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{j+q-1} & b_{j+q} & \cdots & b_{j+2q-2} \end{vmatrix} \quad (b_1 = 1).$$

For certain specific values of j and q, we obtain the following selections.

1. For $q=2,\ j=1$:

$$\mathbb{H}_{2,1}(\mathfrak{f}) = \begin{vmatrix} 1 & b_2 \\ b_2 & b_3 \end{vmatrix} = b_3 - b_2^2. \tag{5}$$

It is evident that the particular case of Fekete-Szegö functional corresponds to $\mathbb{H}_{2,1}(\mathfrak{f})$. Fekete and Szegö [17] extended this estimate to $\left|b_3 - \mu b_2^2\right|$ for real μ and $\mathfrak{f} \in \mathscr{S}$.

2. For $q=2,\ j=2$:

$$\mathbb{H}_{2,2}(\mathfrak{f}) = \begin{vmatrix} b_2 & b_3 \\ b_3 & b_4 \end{vmatrix} = b_2 b_4 - b_3^2, \tag{6}$$

is the second Hankel determinant. Hayman [22] examined the second Hankel determinant for really mean univalent functions, and many other authors have also examined the second Hankel determinant for a variety of subclasses of $\mathscr{S}$, (see e.g., [24, 25, 32]).

3. For $q=3,\ j=1$:

$$\mathbb{H}_{3,1}(\mathfrak{f}) = \begin{vmatrix} 1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \\ b_3 & b_4 & b_5 \end{vmatrix} = b_3\left(b_2 b_4 - b_3^2\right) - b_4\left(b_4 - b_2 b_3\right) + b_5\left(b_3 - b_2^2\right)$$

is the third Hankel determinant. Babalola [10] established the upper bound for $|\mathbb{H}_{3,1}(\mathfrak{f})|$ in the classes $\mathscr{S}^*$, $\mathscr{C}$, and $\mathscr{R}$. Zaprawa [69] improved Babalola's findings and demonstrated that:

$$|\mathbb{H}_{3,1}(\mathfrak{f})| \leqslant \begin{cases} 1, & \mathfrak{f} \in \mathscr{S}^*, \\ \dfrac{49}{540}, & \mathfrak{f} \in \mathscr{C}, \\ \dfrac{41}{60}, & \mathfrak{f} \in \mathscr{R}. \end{cases}$$

Shi et al. [55] derived third Hankel determinant estimates for functions of certain subclasses of $\mathscr{S}$ which maps to cardioid domain. Subsequently, other researchers extended these results to various subclasses of $\mathscr{S}$, particularly in determining sharp bounds (see, e.g., [8, 12, 13, 18, 33, 34, 40, 57, 64]).

4. For $q=4,\ j=1$:

$$\mathbb{H}_{4,1}(\mathfrak{f}) = \begin{vmatrix} 1 & b_2 & b_3 & b_4 \\ b_2 & b_3 & b_4 & b_5 \\ b_3 & b_4 & b_5 & b_6 \\ b_4 & b_5 & b_6 & b_7 \end{vmatrix}, \tag{7}$$

is the fourth Hankel determinant. Arif et al. [7] investigated the fourth Hankel determinant problem for the class of bounded turning functions $\mathscr{R}$ for the first time and successfully established a bound for $\mathbb{H}_{4,1}(\mathfrak{f})$. Srivastava et al. [57] investigated the fourth Hankel determinant for a subclass of analytic functions with bounded turning functions associated with cardioid domains and provided sharp coefficient estimates and extended the understanding of function behavior in complex domains. Khan et al. [29] examined the same class of functions associated with sine functions and determined upper bounds for the third- and fourth-order Hankel determinants. Recently, Rahman et al. [48] introduced a new approach to finding bounds on the fourth Hankel determinant for a subclass of analytic functions. Wang et al. [65] established sharp estimates for Hankel determinants corresponding to the logarithmic and inverse functions of bounded turning functions, particularly those associated with the hyperbolic tangent function.

On the other Hand, Thomas and Halim [14] introduced the symmetric Toeplitz determinant $\mathscr{T}_{q,j}$ for $f \in \mathscr{A}$, defined as follows:

$$\mathscr{T}_{q,j}(\mathfrak{f}) = \begin{vmatrix} b_j & b_{j+1} & \cdots & b_{j+q-1} \\ b_{j+1} & b_j & \cdots & b_{j+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{j+q-1} & b_{j+q-2} & \cdots & b_j \end{vmatrix}, \tag{8}$$

where $j \geqslant 1$, $q \geqslant 1$, and $b_1 = 1$. Specifically,

$$\mathscr{T}_{2,2}(\mathfrak{f}) = \begin{vmatrix} b_2 & b_3 \\ b_3 & b_2 \end{vmatrix} \quad \text{and} \quad \mathscr{T}_{3,1}(\mathfrak{f}) = \begin{vmatrix} 1 & b_2 & b_3 \\ b_2 & 1 & b_2 \\ b_3 & b_2 & 1 \end{vmatrix}. \tag{9}$$

Recently, Allu et al. [5] investigated Toeplitz determinants for certain close-to-convex functions, while Altinkaya et al. [4] constructed Toeplitz matrices for a subclass of q-starlike functions. Wanas et al. [63] derived bounds for Toeplitz determinants for a certain family of analytic functions endowed with Borel distribution. Rahmatan [49] obtained bound for fifth Toeplitz determinant for the class of bounded turning functions. Zulfiqar et al. [72], estimated bound for Toeplitz determinant of fourth-order for the functions belong to the class $\mathscr{C}$ associated with sine function. Asih et al. [9], derived bound for Toeplitz determinants for the class of Bazilevic functions associated with Lemniscate of Bernoulli. Researchers have been actively deriving estimates for the Toeplitz determinant $|\mathscr{T}_{q,j}(\mathfrak{f})|$ for functions belonging to various subclasses of univalent functions (see, for example, [2,6,11,21,26,27,42,44,47,50,56,59,60,66,71]).

Now, we consider the function φ for which $\varphi(\mathbb{U})$ is starlike with respect to 1 and whose coefficients are the Gregory coefficients. These coefficients are decreasing rational numbers, such as $\frac{1}{2}, \frac{1}{12}, \frac{1}{24}, \frac{19}{720}, \cdots$, playing a role similar to Bernoulli numbers and appearing in numerous problems, particularly in numerical analysis and number theory. In 1671, James Gregory introduced these coefficients and have been rediscovered multiple times by renowned mathematicians, including Laplace, Hermite, Pearson, and others (see [28,41]). Due to their frequent rediscovery, they are known by various names in the literature, such as reciprocal logarithmic numbers, Bernoulli numbers of the second kind, and Cauchy numbers. The generating function of the Gregory coefficients, (see [28]) is given by (see Figure 1):

$$\psi(\zeta) := \frac{\zeta}{\log_e(1+\zeta)} = \sum_{j=0}^{\infty} \mathscr{G}_j \zeta^j, \quad \zeta \in \mathbb{U}. \tag{10}$$

Clearly, $\mathscr{G}_j$ for some values of j are $\mathscr{G}_0 = 1$, $\mathscr{G}_1 = \frac{1}{2}$, $\mathscr{G}_2 = -\frac{1}{12}$, $\mathscr{G}_3 = \frac{1}{24}$, $\mathscr{G}_4 = -\frac{19}{720}$, $\mathscr{G}_5 = \frac{3}{160}$, and so on. Srivastava et al. [58] investigated sharp inequalities for a novel subclass of convex functions associated with Gregory polynomials, which connect special functions and geometric function theory (GFT).

For further applications of this generating function in GFT, refer to [28,30,43,62] and the references therein. The generating function given in (10) is used to define a subclass of $\mathscr{S}$ as follows:

DEFINITION 2. A function $\mathfrak{f} \in \mathscr{A}$ is said to belong to the class $\mathscr{G}\mathscr{S}^*_{\text{sym}}$ if it satisfies the following condition:

$$\frac{2\zeta \mathfrak{f}'(\zeta)}{\mathfrak{f}(\zeta) - \mathfrak{f}(-\zeta)} \prec \psi(\zeta) := \frac{\zeta}{\log_e(1+\zeta)}, \quad \zeta \in \mathbb{U}.$$

EXAMPLE. Consider the function $\mathfrak{f}(\zeta) = \dfrac{\zeta}{1+0.125\zeta}$ in $\mathscr{S}$. A straightforward computation gives

$$\frac{2\zeta \mathfrak{f}'(\zeta)}{\mathfrak{f}(\zeta) - \mathfrak{f}(-\zeta)} = \frac{1-0.125\zeta}{1+0.125\zeta} = \mathfrak{g}(\zeta).$$

Clearly, $\mathfrak{g}(\zeta)$ belongs to $\mathscr{S}$. From Figure 2 (Right), it can be observed that $\mathfrak{g}(\mathbb{U})$ (pink region) is contained within $\psi(\mathbb{U})$, which confirms that $\mathfrak{f} \in \mathscr{G}\mathscr{S}^*_{\text{sym}}$.

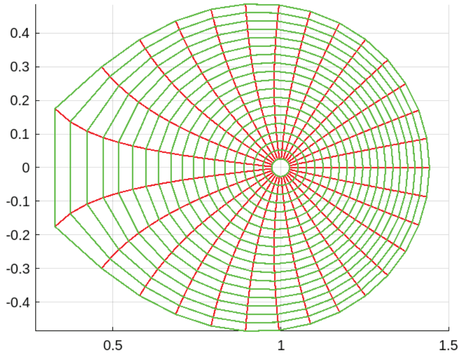

Figure 1: *(Left): Image of* $\psi(\mathbb{U})$.

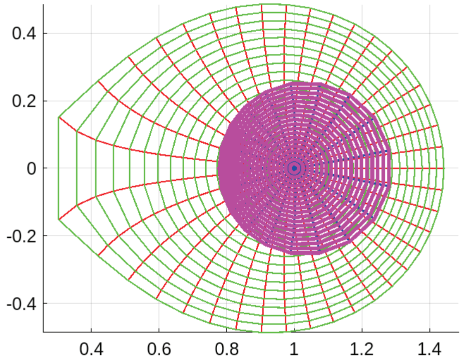

Figure 2: (Right): $\mathfrak{g}(\mathbb{U}) \subset \psi(\mathbb{U})$.

Sakaguchi [53] proved that functions in the class defined by (3) are close-to-convex, ensuring their univalence. In this context, for each univalent function $\mathfrak{f} \in \mathscr{G}\mathscr{S}^*_{\text{sym}}$, Koebe's one-quarter theorem guarantees the existence of an inverse function $\mathfrak{f}^{-1}$ given by

$$\mathfrak{f}^{-1}(\omega) = \omega + \sum_{j=2}^{\infty} d_j \omega^j, \tag{11}$$

where $|\omega| < r_0$, with r_0 being greater than the radius of the Koebe domain for functions in the class $\mathscr{G}\mathscr{S}^*_{\text{sym}}$.

Motivated by the works of Srivastava et. al [58] and the studies in [23, 28, 39, 61], we aim to establish sharp estimates for coefficient inequalities, Hankel and Toeplitz determinants, as well as Krushkal, Zalcman, and Fekete-Szegö inequalities for the class $\mathscr{G}\mathscr{S}^*_{\text{sym}}$ of symmetric starlike functions associated with Gregory coefficients. Additionally, we derive coefficient estimates for the inverse functions in this class, along with bounds for their Hankel and Toeplitz determinants.

To proceed with the main results, we first recall the following lemmas.

2. A set of lemmas

Let $\mathscr{P}$ denote the class of analytic functions within the unit disk $\mathbb{U}$ of the form

$$p(\zeta) = 1 + \sum_{j=1}^{\infty} c_j \zeta^j \quad (\zeta \in \mathbb{U}) \tag{12}$$

that satisfy the condition $\operatorname{Re}(p(\zeta)) > 0$. It follows from the results of [20] that for any $p \in \mathscr{P}$, there exists a Schwarz function $w(\zeta)$ such that

$$p(\zeta) = \frac{1 + w(\zeta)}{1 - w(\zeta)}. \tag{13}$$

LEMMA 1. [46] *If $p \in \mathscr{P}$ and is given by the series representation* (12)*, then*

$$|p_n| \leqslant 2.$$

LEMMA 2. [52] *For any $p \in \mathscr{P}$, the following inequality holds:*

$$|p_{m+n} - \mu p_m p_n| \leqslant \begin{cases} 2, & 0 \leqslant \mu \leqslant 1, \\ 2|2\mu - 1|, & \textit{otherwise}. \end{cases}$$

LEMMA 3. [8] *Let α, β, and γ be real numbers. Then, for any function $p \in \mathscr{P}$, the following inequality holds:*

$$\left|\alpha p_1^3 - \beta p_1 p_2 + \gamma p_3\right| \leqslant 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \gamma|.$$

LEMMA 4. [52] *Let $p \in \mathcal{P}$ and let α, β, γ, and κ be real numbers satisfying $0 < \beta < 1$, $0 < \gamma < 1$ and*

$$8\beta(1-\beta)\left[(\gamma\kappa-2\alpha)^2+(\gamma(\beta+\gamma)-\kappa)^2\right]+\gamma(1-\gamma)(\kappa-2\beta\gamma)^2 \leqslant 4\gamma^2(1-\gamma)^2\beta(1-\beta),$$

then the following inequality holds:

$$\left|\alpha p_1^4+\beta p_2^2+2\gamma p_1 p_3-\frac{3}{2}\kappa p_1^2 p_2-p_4\right| \leqslant 2.$$

LEMMA 5. [36] *If $p \in \mathcal{P}$ is of the form (12) with $p_1 \geqslant 0$, then*

$$2p_2 = p_1^2+(4-p_1^2)x,$$
$$4p_3 = p_1^3+2p_1(4-p_1^2)x-p_1(4-p_1^2)x^2+2(4-p_1^2)(1-|x|^2)\sigma,$$

for some $x, \sigma \in \mathbb{C}$ satisfying $|x| \leqslant 1$ and $|\sigma| \leqslant 1$.

REMARK 1. The class $\mathcal{GS}^*_{sym}$ and the functional $|b_2b_4-b_3^2|$ are invariant under rotations, we may assume without loss of generality that $p_1 \geqslant 0$ throughout the paper.

3. Main results

Now, we derive bounds for the initial coefficients of functions belonging to the class $\mathfrak{f} \in \mathcal{GS}^*_{\mathrm{sym}}$.

THEOREM 1. *If $\mathfrak{f} \in \mathcal{GS}^*_{\mathrm{sym}}$ of the form (1), then*

$$|b_2| \leqslant \frac{1}{4}, \quad |b_3| \leqslant \frac{1}{4}, \quad |b_4| \leqslant \frac{1}{8}, \quad |b_5| \leqslant \frac{1}{8}, \quad |b_6| \leqslant \frac{7891}{7680} \quad \text{and} \quad |b_7| \leqslant \frac{845293}{322560}.$$

The first four bounds are sharp.

Proof. Consider $p \in \mathcal{P}$, then using (13), we can have

$$p(\zeta) = \frac{1+w(\zeta)}{1-w(\zeta)} = 1+p_1\zeta+p_2\zeta^2+p_3\zeta^3+\cdots.$$

Writing $w(\zeta)$ in terms of $p(\zeta)$, we get

$$w(\zeta) = \frac{p(\zeta)-1}{p(\zeta)+1} = \frac{p_1}{2}\zeta+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)\zeta^2+\left(\frac{p_1^2}{8}-\frac{p_1p_2}{2}+\frac{p_3}{2}\right)\zeta^3+\cdots.$$

Using the above relation in the representation of $\psi(w(\zeta))$ stated in (10), we obtain

$$\psi(w(\zeta)) = 1+\frac{p_1\zeta}{4}+\frac{(12p_2-7p_1^2)\zeta^2}{48}+\frac{(17p_1^3-56p_1p_2+48p_3)\zeta^3}{192} + \frac{(-649p_1^4+3060p_1^2p_2-1680p_2^2-3360p_1p_3+2880p_4)\zeta^4}{11520}+\cdots. \tag{14}$$

Since, $\mathfrak{f}$ is of the form (1), we can compute

$$\frac{2\zeta\mathfrak{f}'(\zeta)}{\mathfrak{f}(\zeta)-\mathfrak{f}(-\zeta)} = 1+2b_2\zeta+2b_3\zeta^2+(4b_4-2b_2b_3)\zeta^3+(4b_5-2b_3^2)\zeta^4 + (2b_2b_3^2-4b_3b_4-2b_2b_5+6b_6)\zeta^5+(2b_3^3-6b_3b_5+6b_7)\zeta^6+\cdots. \tag{15}$$

By Definition 2, we have

$$\frac{2\zeta\mathfrak{f}'(\zeta)}{\mathfrak{f}(\zeta)-\mathfrak{f}(-\zeta)} = \psi(w(\zeta)) := \frac{w(\zeta)}{\log_e(1+w(\zeta))}.$$

Thus, equating the corresponding coefficients of (14) and (15), we obtain

$$b_2 = \frac{p_1}{8}, \tag{16}$$

$$b_3 = \frac{p_2}{8} - \frac{7p_1^2}{96}, \tag{17}$$

$$b_4 = \frac{p_3}{16} - \frac{25p_1p_2}{384} + \frac{9p_1^3}{512}, \tag{18}$$

$$b_5 = \frac{p_4}{16} - \frac{7p_1p_3}{96} - \frac{11p_2^2}{384} + \frac{11p_1^2p_2}{192} - \frac{117p_1^4}{10240}, \tag{19}$$

$$b_6 = \frac{p_5}{24} - \frac{53p_1p_4}{1152} - \frac{25p_2p_3}{576} + \frac{11p_1^2p_3}{288} + \frac{341p_1p_2^2}{9216} - \frac{4117p_1^3p_2}{138240} + \frac{3493p_1^5}{737280}, \tag{20}$$

$$b_7 = \frac{p_6}{24} - \frac{7p_1p_5}{144} - \frac{47p_2p_4}{1152} + \frac{61p_1^2p_4}{1536} - \frac{7p_3^2}{288} + \frac{61p_1p_2p_3}{768} - \frac{4457p_1^3p_3}{138240} + \frac{97p_2^3}{9216} - \frac{941p_1^2p_2^2}{20480} + \frac{55691p_1^4p_2}{2211840} - \frac{71807p_1^6}{20643840}. \tag{21}$$

By directly applying Lemma 1 to (16), it follows that

$$|b_2| \leqslant \frac{1}{4}. \tag{22}$$

Rewriting (17) and applying Lemma 2, we get

$$|b_3| = \frac{1}{8}\left|p_2 - \frac{7}{12}p_1^2\right| \leqslant \frac{1}{4}. \tag{23}$$

Applying Lemma 3 to (18) yields

$$|b_4| = \left|\frac{9p_1^3}{512} - \frac{25p_1p_2}{384} + \frac{p_3}{16}\right| \leqslant 2\left|\frac{9}{512}\right| + 2\left|\frac{25}{384} - \frac{18}{512}\right| + 2\left|\frac{9}{512} - \frac{25}{384} + \frac{1}{16}\right| = \frac{1}{8}. \tag{24}$$

Rewriting (19) and taking the modulus gives

$$|b_5| = \frac{1}{16}\left|\frac{117p_1^4}{640} + \frac{11p_2^2}{24} + \frac{7p_1p_3}{6} - \frac{11p_1^2p_2}{12} - p_4\right|. \tag{25}$$

Choosing $\alpha = \frac{117}{640}$, $\beta = \frac{11}{24}$, $\gamma = \frac{7}{12}$ and $\kappa = \frac{11}{18}$, we have

$$\begin{aligned}
&0 < \gamma < 1, \quad 0 < \beta < 1,\\
&8\beta(1-\beta)\left((\gamma\kappa - 2\alpha)^2 + (\gamma(\beta+\gamma)-\kappa)^2\right) + \gamma(1-\gamma)(\kappa - 2\beta\gamma)^2 = \frac{8644163}{5374771200}
\end{aligned}$$

and

$$4\gamma^2(1-\gamma)^2\beta(1-\beta) = \frac{175175}{2985984}.$$

Hence, the conditions of Lemma 4 are satisfied by (25), leading to

$$|b_5| \leqslant \frac{1}{8}. \tag{26}$$

Expressing (20) in an equivalent form results in

$$b_6 = \frac{p_5 - \frac{53p_1p_4}{24}}{48} + \frac{p_5 - \frac{25p_2p_3}{12}}{48} + \frac{11p_1^2p_3 - \frac{4117p_1p_2}{5280}}{288} + \frac{341p_1p_2^2}{9216} + \frac{3493p_1^5}{737280}.$$

Applying Lemmas 1 and 2, the following result is obtained:

$$|b_6| \leqslant \frac{2}{48}\left(\frac{53}{12} - 1\right) + \frac{2}{48}\left(\frac{50}{12} - 1\right) + \frac{88}{288} + \frac{341}{1152} + \frac{3493}{23040} = \frac{7891}{7680}.$$

Rewriting (21), we get

$$\begin{aligned}
b_7 = {} & \frac{p_6 - \frac{7p_3^2}{12}}{24} + \frac{7p_1p_5 - \frac{183p_1p_4}{224}}{144} + \frac{47p_2p_4 - \frac{97p_2^2}{376}}{1152} + \frac{61p_1p_2p_3 - \frac{2823p_1p_2}{4880}}{768}\\
&+ \frac{4457p_1^3p_3}{138240} + \frac{55691p_1^4p_2}{2211840} + \frac{71807p_1^6}{20643840}.
\end{aligned}$$

Using Lemma 1 and Lemma 2, we arrive at

$$|b_7| \leqslant \frac{845293}{322560}.$$

The first four results are sharp for the function $\mathfrak{f}_j : \mathbb{U} \to \mathbb{C}$ given by

$$\mathfrak{f}_1(\zeta) = \zeta \exp\left(\int_0^\zeta \frac{\psi(t/2)-1}{t}dt\right) = \zeta + \frac{1}{4}\zeta^2 + \cdots, \tag{27}$$

$$\mathfrak{f}_2(\zeta) = \zeta \exp\left(\int_0^\zeta \frac{\psi(t^2)-1}{t}dt\right) = \zeta + \frac{1}{4}\zeta^3 + \cdots, \tag{28}$$

$$\mathfrak{f}_3(\zeta) = \zeta \exp\left(\int_0^\zeta \frac{\psi(3t^3/4)-1}{t}dt\right) = \zeta + \frac{1}{8}\zeta^4 + \cdots, \tag{29}$$

$$\mathfrak{f}_4(\zeta) = \zeta \exp\left(\int_0^\zeta \frac{\psi(t^4)-1}{t}dt\right) = \zeta + \frac{1}{8}\zeta^5 + \cdots. \tag{30}$$

This completes the proof of Theorem 1. □

REMARK 2.

1. The estimates $|b_2|, |b_3|, |b_4|$, and $|b_5|$ align with the results in [16, Theorem 1, p. 4] and [23, Theorem 2.1, p. 6448] and provide an improvement over those in [1, Theorem 1, p. 5], [18, Theorem 6, p. 134] and [61, Theorem 1, p. 5].
2. The estimates $|b_4|$ and $|b_5|$ provide an improvement over the result obtained in [69, Theorem 1, p. 4].

Next, we determine the sharp bound for the second Hankel determinant of $\mathfrak{f} \in \mathcal{G}\mathcal{S}^*_{sym}$.

THEOREM 2. *If $\mathfrak{f} \in \mathcal{G}\mathcal{S}^*_{\mathrm{sym}}$ of the form (1), then*

$$|b_2b_4 - b_3^2| \leqslant \frac{1}{16}.$$

The obtained result is sharp for the function given in (28).

Proof. Using (16), (17) and (18), we can get

$$|b_2b_4 - b_3^2| = \left|\frac{p_1p_3}{128} - \frac{p_2^2}{64} + \frac{31p_1^2p_2}{3072} - \frac{115p_1^4}{36864}\right|.$$

Using Lemma 5, we get

$$|b_2b_4 - b_3^2| = \left|\frac{p_1(4-p_1^2)(1-|x|^2)\sigma}{256} - \frac{(4-p_1^2)^2x^2}{256} - \frac{p_1^2(4-p_1^2)x^2}{512} + \frac{7p_1^2(4-p_1^2)x}{6144} - \frac{p_1^4}{36864}\right|.$$

Let $|x| = t \in [0,1]$ and $p_1 = p \in [0,2]$ along with triangle inequality, we have

$$|b_2b_4 - b_3^2| \leqslant \frac{p(4-p^2)(1-t^2)}{256} + \frac{(4-p^2)^2t^2}{256} + \frac{p^2(4-p^2)t^2}{512} + \frac{7p^2(4-p^2)t}{6144} + \frac{p^4}{36864} = G(p,t).$$

Differentiating $G(p,t)$ partially with respect to t, we get

$$\frac{\partial G}{\partial t} = \frac{7p^2(4-p^2)+24(p+2)(p+4)(p-2)^2 t}{6144} \geqslant 0.$$

Hence $G(p,t)$ is an increasing function in the interval $t \in [0,1]$ and $p \in [0,2]$. So the maximum is attained at $t = 1$, that is,

$$\max G(p,t) = G(p,1) = \frac{(4-p^2)^2}{256} + \frac{19p^2(4-p^2)}{6144} + \frac{p^4}{36864} = F(p).$$

Now,

$$F'(p) = -\frac{7p^3}{1152} - \frac{29p(4-p^2)}{3072}.$$

If $F'(p) = 0$, then

$$p = 0 \quad \text{and} \quad p = \pm\frac{2\sqrt{87}}{\sqrt{31}} \notin [0,2].$$

Now, it is easy to check that

$$F''(0) = -\frac{29}{768} < 0,$$

Hence, the maximum is attained at $p = 0$; therefore, we have

$$|b_2 b_4 - b_3^2| \leqslant F(0) = \frac{1}{16}.$$

Hence, the proof of this Theorem is complete. □

REMARK 3. The bounds coincide with [16, Theorem 5, p. 7] and [23, Theorem 2.4, p.] while providing an improvement over the results in [1, Theorem 7, p. 10], [18, Theorem 9, p. 135], [40, Theorem 2.1, p. 4], [61, Theorem 4, p. 8] and [69, Theorem 4, p. 8].

The following are two additional estimates for the differences in the modulus of coefficients for functions in the class $\mathscr{G}\mathscr{S}^*_{\mathrm{sym}}$.

THEOREM 3. *If* $\mathfrak{f} \in \mathscr{G}\mathscr{S}^*_{\mathrm{sym}}$ *of the form* (1), *then*

$$|b_2 b_5 - b_3 b_4| \leqslant \frac{199}{1920}.$$

Proof. From (16)–(19), we have

$$|b_2 b_5 - b_3 b_4| = \left| \frac{p_1 p_4}{128} - \frac{p_2 p_3}{128} - \frac{7p_1^2 p_3}{1536} + \frac{7p_1 p_2^2}{1536} + \frac{p_1^3 p_2}{4608} - \frac{3p_1^5}{20480} \right|.$$

Rearranging the above terms, we get

$$|b_2b_5-b_3b_4| = \left|\frac{13p_1(p_4-p_1p_3)}{1536} - \frac{p_1(p_4-7p_2^2)}{1536} - \frac{p_3\left(p_2-\frac{p_1^2}{2}\right)}{128} + \frac{p_1^3\left(p_2-\frac{p_1^2}{2}\right)}{4608} - \frac{7p_1^5}{184320}\right|$$

$$\leqslant \frac{13|p_1|\,|p_4-p_1p_3|}{1536} + \frac{|p_1|\,|p_4-7p_2^2|}{1536} + \frac{|p_3|}{128}\left|p_2-\frac{p_1^2}{2}\right| + \frac{|p_1^3|}{4608}\left|p_2-\frac{p_1^2}{2}\right| + \frac{7|p_1^5|}{184320}.$$

Using Lemma 1 and Lemma 2, we can obtain

$$|b_2b_5-b_3b_4| \leqslant \frac{199}{1920}.$$

The proof of the Theorem is complete. □

THEOREM 4. *If $\mathfrak{f}$ of the form (1) belongs to $\mathcal{GS}^*_{\text{sym}}$, then*

$$|b_3b_5-b_4^2| \leqslant \frac{5187}{20480}.$$

Proof. Using (17), (18) and (19), we can compute

$$|b_3b_5-b_4^2| = \left|\frac{p_2p_4}{128} - \frac{7p_1^2p_4}{1536} - \frac{p_3^2}{256} - \frac{p_1p_2p_3}{1024} + \frac{115p_1^3p_3}{36864} - \frac{11p_2^3}{3072} + \frac{739p_1^2p_2^2}{147456} - \frac{4891p_1^4p_2}{1474560} + \frac{687p_1^6}{1310720}\right|.$$

Rewriting and using triangle inequality, we get

$$|b_3b_5-b_4^2| \leqslant \frac{|p_2|\left|p_4-\frac{3p_1p_3}{16}\right|}{128} + \left|\frac{7p_1^2p_4}{1536}\right| + \frac{|p_3|\left|p_3-\frac{p_1p_2}{8}\right|}{256} + \frac{115|p_1^3|\left|p_3-\frac{489p_1p_2}{460}\right|}{36864} + \frac{475|p_1^2p_2|\left|p_2-\frac{p_1^2}{4750}\right|}{147456} + \frac{11|p_2^2|\left|p_2-\frac{p_1^2}{2}\right|}{3072} + \frac{687|p_1^6|}{1310720}.$$

Using Lemma 1 and Lemma 2, we get

$$|b_3b_5 - b_4^2| \leqslant \frac{5187}{20480}.$$

This completes the proof of the Theorem 4. □

Next, we establish the bounds for the inverse functions as defined in (11).

THEOREM 5. *If* $\mathfrak{f} \in \mathcal{GS}^*_{\text{sym}}$ *and* $\mathfrak{f}^{-1}$ *is the inverse function defined by* (11), *then*

$$|d_2| \leqslant \frac{1}{4}, \quad |d_3| \leqslant \frac{1}{4}, \quad |d_4| \leqslant \frac{1}{6} \quad \text{and} \quad |d_5| \leqslant \frac{21}{32}.$$

Proof. Since $\mathfrak{f}^{-1}(\omega) = \omega + \sum_{j=2}^{\infty} d_n\omega^j$ is the inverse of $\mathfrak{f} \in \mathcal{GS}^*_{\text{sym}}$, it is clear that,

$$\mathfrak{f}^{-1}(\mathfrak{f}(\zeta)) = \mathfrak{f}(\mathfrak{f}^{-1}(\zeta)) = \zeta. \tag{31}$$

From (1) and (31), we can write

$$\mathfrak{f}^{-1}\left(\zeta + \sum_{j=2}^{\infty} b_j\zeta^j\right) = \zeta.$$

With simple calculations, we arrive at

$$\zeta + (b_2 + d_2)\zeta^2 + (b_3 + 2b_2d_2 + d_3)\zeta^3 + \cdots = \zeta. \tag{32}$$

Equating the coefficients in (32), we get

$$d_2 = -b_2 \tag{33}$$

$$d_3 = 2b_2^2 - b_3 \tag{34}$$

$$d_4 = -5b_2^3 + 5b_2b_3 - b_4 \tag{35}$$

$$d_5 = 14b_2^4 - 21b_2^2b_3 + 6b_2b_4 + 3b_3^2 - b_5 \tag{36}$$

Using (22) in (33), we get

$$|d_2| \leqslant |b_2| = \frac{1}{4}. \tag{37}$$

Substituting (16) and (17) in (34) and applying Lemma 2, we have

$$|d_3| = |2b_2^2 - b_3| = \frac{1}{8}\left|p_2 - \frac{5}{6}p_1^2\right| \leqslant \frac{1}{4}. \tag{38}$$

Substituting (16), (17) and (18) in (35) and application of Lemma 2, we obtain

$$|d_4| = \left|\frac{7p_1^3}{96} - \frac{55p_1p_2}{384} + \frac{p_3}{16}\right| \leqslant 2\left|\frac{7}{96}\right| + 2\left|\frac{55}{384} - 2\frac{7}{96}\right| + 2\left|\frac{7}{96} - \frac{55}{384} + \frac{1}{16}\right| \tag{39}$$
$$\leqslant \frac{1}{6}.$$

From (16) to (19), we have

$$|d_5| = \left| -\frac{p_4}{16} + \frac{23p_1p_3}{192} + \frac{29p_2^2}{384} - \frac{155p_1^2p_2}{768} + \frac{1043p_1^4}{15360} \right| \tag{40}$$

By applying triangle inequality, we get

$$|d_5| \leqslant \left| \frac{23p_1p_3}{192} - \frac{155p_1^2p_2}{768} + \frac{1043p_1^4}{15360} \right| + \left| \frac{29p_2^2}{384} - \frac{p_4}{16} \right| \tag{41}$$

Using Lemma 1 and 3, we have

$$\left| \frac{23p_1p_3}{192} - \frac{155p_1^2p_2}{768} + \frac{1043p_1^4}{15360} \right| \leqslant |p_1| \left| \frac{1043p_1^3}{15360} - \frac{155p_1p_2}{768} + \frac{23p_3}{192} \right| \leqslant \frac{23}{48}. \tag{42}$$

Also, using Lemma 2, we get

$$\left| \frac{29p_2^2}{384} - \frac{p_4}{16} \right| \leqslant \frac{1}{16} \left| p_4 - \frac{29p_2^2}{24} \right| \leqslant \frac{17}{96}. \tag{43}$$

Using (42) and (43) in (41), we estimate

$$|d_5| \leqslant \frac{23}{48} + \frac{17}{96} = \frac{21}{32}. \tag{44}$$

Hence, the proof. □

REMARK 4. The estimates $|d_2|$ and $|d_3|$ align with the existing results, while $|d_4|$ provides an improvement over the result obtained in [23, Theorem 4.1, p. 6458].

4. Fekete-Szegö functional

In this section, we provide Fekete-Szegö estimate for the functions of the form (1) and (11) in the class $\mathcal{GS}^*_{sym}$.

THEOREM 6. *If $\mathfrak{f} \in \mathcal{GS}^*_{\text{sym}}$ and μ is any complex number, then*

$$|b_3 - \mu b_2^2| \leqslant \frac{1}{4} \max\left\{1, \left|\frac{2+3\mu}{12}\right|\right\}.$$

The result attains sharpness for the function specified in (28).

Proof. For any complex number μ and from (16) and (17), we have

$$|b_3 - \mu b_2^2| = \left| \frac{p_2}{8} - \frac{7p_1^2}{96} - \frac{\mu p_1^2}{64} \right| = \frac{1}{8} \left| p_2 - \left(\frac{14+3\mu}{24} \right) p_1^2 \right|. \tag{45}$$

Based on the result in [15], for any $p \in \mathcal{P}$ and $\nu \in \mathbb{C}$, the following holds:

$$|p_2 - \nu p_1^2| \leqslant 2\max\{1, |2\nu - 1|\}. \tag{46}$$

Applying (46) to (45), we get the desired result. □

THEOREM 7. *If $\mathfrak{f} \in \mathcal{GS}^*_{sym}$, with $\mathfrak{f}^{-1}$ as the inverse function defined by (11), and μ being any complex number, then*

$$|d_3 - \mu d_2^2| \leqslant \frac{1}{4} \max\left\{1, \left|\frac{3\mu - 8}{12}\right|\right\}.$$

Proof. For any complex number μ and from (33) and (34), we have

$$|d_3 - \mu d_2^2| = \left|-\frac{p_2}{8} - \frac{p_1^2\mu}{64} + \frac{5p_1^2}{84}\right| = \frac{1}{8}\left|p_2 - \left(\frac{20-3\mu}{24}\right)p_1^2\right|. \tag{47}$$

Applying (46) to (47), we get the desired estimate. □

5. Generalized Zalcman conjecture

Zalcman conjectured that if $\mathfrak{f} \in \mathcal{S}$, then $\left|b_j^2 - b_{2j-1}\right| \leqslant (j-1)^2$, for $j \geqslant 2$, with equality attained only for the Koebe function $k(\zeta) = \dfrac{\zeta}{(1-\zeta)^2}$ or its rotation. In 1999, Ma [38] proposed a generalized form of Zalcman's conjecture: If $\mathfrak{f} \in \mathcal{S}$, then

$$\left|b_j b_n - b_{j+n-1}\right| \leqslant (j-1)(n-1), \quad j \geqslant 2,\ n \geqslant 2.$$

For $j = n$, this reduces to the Zalcman conjecture. The following results are related to the Generalized Zalcman Conjecture for functions as well as inverse functions and their inverses in the class $\mathcal{GS}^*_{sym}$ for specific values of j and n.

THEOREM 8. *If $\mathfrak{f} \in \mathcal{GS}^*_{sym}$ of the form (1), then*

$$|b_3 - b_2^2| \leqslant \frac{1}{4} < (2-1)^2 = 1.$$

The sharpness of the result is achieved for the function given in (28).

Proof. Using (16), (17) and Lemma 2, we have

$$|b_3 - b_2^2| = \frac{1}{8}\left|p_2 - \frac{17p_1^2}{24}\right| \leqslant \frac{1}{4}.$$

Hence, the theorem is proved. □

THEOREM 9. *If $\mathfrak{f}$ of the form (1) belongs to $\mathcal{GS}^*_{sym}$, then*

$$|b_2 b_3 - b_4| \leqslant \frac{1}{8} < (2-1)(3-1) = 2.$$

The result is sharp for the function defined in (29).

Proof. From (16), (17) and (18), we can compute

$$|b_2b_3 - b_4| = \left| \frac{p_3}{16} - \frac{31p_1p_2}{384} + \frac{41p_1^3}{1536} \right|.$$

Application of Lemma 3, we obtain

$$\left| \frac{p_3}{16} - \frac{31p_1p_2}{384} + \frac{41p_1^3}{1536} \right| \leqslant 2\left|\frac{41}{1536}\right| + 2\left|\frac{31}{384} - 2\left(\frac{41}{1536}\right)\right| + 2\left|\frac{41}{1536} - \frac{31}{384} + \frac{1}{16}\right|$$
$$\leqslant \frac{1}{8}.$$

This concludes the proof. □

THEOREM 10. *If $\mathfrak{f} \in \mathcal{G}\mathcal{S}^*_{\mathrm{sym}}$ of the form* (1)*, then*

$$|b_5 - b_2b_4| \leqslant \frac{1}{8} < (2-1)(4-1) = 3.$$

The sharpness of the result is attained for the function defined in (30)*.*

Proof. From (16), (18) and (19), we have

$$|b_5 - b_2b_4| = \left| \frac{p_4}{16} - \frac{31p_1p_3}{384} - \frac{11p_2^2}{384} + \frac{67p_1^2p_2}{1024} - \frac{279p_1^4}{20480} \right|$$
$$= \frac{1}{16}\left| \frac{279p_1^4}{1280} + \frac{11p_2^2}{24} + \frac{31p_1p_3}{24} - \frac{67p_1^2p_2}{64} - p_4 \right|. \tag{48}$$

It is easy to verify that the above equation satisfies the conditions of Lemma 4. Therefore,

$$|b_5 - b_2b_4| \leqslant \frac{1}{8}.$$

The proof is now complete. □

THEOREM 11. *If $\mathfrak{f} \in \mathcal{G}\mathcal{S}^*_{\mathrm{sym}}$ of the form* (1)*, then*

$$|b_3^2 - b_5| \leqslant \frac{1}{8} < (3-1)^2 = 4.$$

The result is sharp for the function defined in (30)*.*

Proof. Using (17) and (19), we get

$$|b_3^2 - b_5| = \left| -\frac{p_4}{16} + \frac{7p_1p_3}{96} + \frac{17p_2^2}{384} - \frac{29p_1^2p_2}{384} + \frac{1543p_1^4}{92160} \right|$$
$$= \frac{1}{16}\left| \frac{1543}{5760}p_1^4 + \frac{17}{24}p_2^2 + \frac{7}{6}p_1p_3 - \frac{29}{24}p_1^2p_2 - p_4 \right|. \tag{49}$$

It can be verified that (49) meets the conditions of Lemma 4, which gives

$$|b_3^2 - b_5| \leqslant \frac{1}{8}.$$

Thus, the theorem has been proved. □

THEOREM 12. *If* $\mathfrak{f} \in \mathscr{G}\mathscr{S}^*_{\mathrm{sym}}$ *and* $\mathfrak{f}^{-1}$ *is the inverse function defined by* (11)*, then*

$$|d_2 d_3 - d_4| \leqslant \frac{1}{8} \quad \textit{and} \quad |d_3 - d_2^2| \leqslant \frac{1}{4}.$$

Proof. From (33) to (35), along with Lemma (3), we can estimate

$$|d_4 - d_2 d_3| = |-b_4 + 4b_2 b_3 - 3b_2^3| = \left|\frac{23p_1^3}{384} - \frac{49p_1 p_2}{384} + \frac{p_3}{16}\right| \leqslant \frac{1}{8}.$$

And, from (33) and (34), along with (16),(17) and Theorem 8, we have

$$|d_3 - d_2^2| = |b_2^2 - b_3| \leqslant \frac{1}{4}.$$

This concludes the proof. □

6. Estimates of the third and fourth Hankel determinants

We proceed to investigate the third and fourth Hankel determinants bounds for functions in $\mathscr{G}\mathscr{S}^*_{\mathrm{sym}}$, motivated by the work of Zaprawa [70]. The main result is stated as follows.

THEOREM 13. *If* $\mathfrak{f}$ *of the form* (1) *belongs to* $\mathscr{G}\mathscr{S}^*_{\mathrm{sym}}$*,*

$$|\mathbb{H}_{3,1}(\mathfrak{f})| \leqslant \frac{1}{16}.$$

Proof. From (7), we have

$$\mathbb{H}_{3,1}(\mathfrak{f}) = b_3\left(b_2 b_4 - b_3^2\right) - b_4\left(b_4 - b_2 b_3\right) + b_5\left(b_3 - b_2^2\right).$$

By using the triangle inequality, we obtain

$$|\mathbb{H}_{3,1}(\mathfrak{f})| \leqslant |b_3|\left|b_2 b_4 - b_3^2\right| + |b_4|\,|b_4 - b_2 b_3| + |b_5|\left|b_3 - b_2^2\right|.$$

Utilizing the results from (23), (24), (26), and Theorem 2, 8, 9, it follows that

$$|\mathbb{H}_{3,1}(\mathfrak{f})| \leqslant \frac{1}{4}\left(\frac{1}{16}\right) + \frac{1}{8}\left(\frac{1}{8}\right) + \frac{1}{8}\left(\frac{1}{4}\right) = \frac{1}{16}.$$

The proof is now complete. □

REMARK 5. The above result coincides with [23, Theorem 2.5, p. 6453] and improves upon [18, Theorem 10, p. 136], [35, Theorem 2, p. 4] [40, Theorem 2.2, p. 6], and [70, Theorem 6, p. 17].

THEOREM 14. *If $\mathfrak{f} \in \mathcal{GS}^*_{sym}$ of the form (1), then*

$$|\mathbb{H}_{4,1}(\mathfrak{f})| \leqslant \frac{855385879}{1651507200}.$$

Proof. Expanding the determinant stated in (7), we get

$$\mathbb{H}_{4,1}(\mathfrak{f}) = b_7\mathbb{H}_{3,1}(\mathfrak{f}) - b_6\delta_1 + b_5\delta_2 - b_4\delta_3, \tag{50}$$

where,

$$\delta_1 = b_3(b_2b_5 - b_3b_4) - b_4(b_5 - b_2b_4) + b_6\left(b_3 - b_2^2\right),$$
$$\delta_2 = b_3\left(b_3b_5 - b_4^2\right) - b_5(b_5 - b_2b_4) + b_6(b_4 - b_2b_3) \text{ and}$$
$$\delta_3 = b_4\left(b_3b_5 - b_4^2\right) - b_5(b_2b_5 - b_3b_4) + b_6(b_4 - b_2b_3).$$

Using the results from (23), (24), (27), and Theorems 3, 8, and 10, the following holds:

$$|\delta_1| \leqslant |b_3||b_2b_5 - b_3b_4| + |b_4||b_5 - b_2b_4| + |b_6|\left|b_3 - b_2^2\right| \leqslant \frac{27421}{92160}. \tag{51}$$

From (23), (26), (27), and Theorems 4, 9, and 10, it follows that,

$$|\delta_2| \leqslant |b_3|\left|b_3b_5 - b_4^2\right| + |b_5||b_5 - b_2b_4| + |b_6||b_4 - b_2b_3| \leqslant \frac{9167}{30720}. \tag{52}$$

Applying (24), (26), (27), and Theorems 3, 4, and 9, we get,

$$|\delta_3| \leqslant |b_4|\left|b_3b_5 - b_4^2\right| + |b_5||b_2b_5 - b_3b_4| + |b_6||b_4 - b_2b_3| \leqslant \frac{85057}{491520}. \tag{53}$$

Using (51), (52), (53), Theorem 1 and Theorem 13 in (50), we get

$$|\mathbb{H}_{4,1}(\mathfrak{f})| \leqslant \frac{855385879}{1651507200} \approxeq 0.5179425672500853.$$

Hence, the proof is complete. □

REMARK 6. The above bound is an improvement over the result presented in [23, Theorem 3.4, p. 6457].

THEOREM 15. *If $\mathfrak{f} \in \mathcal{GS}^*_{sym}$ and $\mathfrak{f}^{-1}$ is the inverse function defined by (11), then*

$$|\mathbb{H}_{3,1}(\mathfrak{f}^{-1})| \leqslant \frac{77}{984}.$$

Proof. From (33)–(35) and (16)–(18), we have

$$|d_2d_4-d_3^2| = |b_2b_4-b_3^2-b_2^2b_3+b_2^4| = \left|\frac{p_1p_3}{128}-\frac{p_2^2}{64}+\frac{25p_1^2p_2}{3072}-\frac{p_1^2}{576}\right|. \tag{54}$$

Using Lemma 5, we get

$$|d_2d_4-d_3^2| = \left|\frac{7p_1^4}{18432}-\frac{p_1^2(4-p_1^2)x^2}{512}+\frac{p_1^2(4-p_1^2)x}{6144}-\frac{(4-p_1^2)^2x^2}{256}\right.$$
$$\left.+\frac{p_1(4-p_1^2)(1-|x|^2)\sigma}{256}\right|.$$

We denote $p_1 = p \in [0,2]$, $|x| = t \in [0,1]$ and applying triangle inequality, we have

$$|d_2d_4-d_3^2| \leqslant \frac{7p^4}{18432}+\frac{p^2(4-p^2)t^2}{512}+\frac{p^2(4-p^2)t}{6144}+\frac{(4-p^2)^2t^2}{256}+\frac{p(4-p^2)(1-t^2)}{256}.$$

Suppose that,

$$F(p,t) = \frac{7p^4}{18432}+\frac{p^2(4-p^2)t^2}{512}+\frac{p^2(4-p^2)t}{6144}+\frac{(4-p^2)^2t^2}{256}+\frac{p(4-p^2)(1-t^2)}{256}.$$

Then,

$$\frac{\partial F}{\partial t} = \frac{p^2(4-p^2)+24(p-2)^2(p+2)(p+4)t}{96} \geqslant 0.$$

Therefore, $F(p,y)$ is an increasing function in the interval $t \in [0,1]$ and $p \in [0,2]$. So the maximum is attained at $t = 1$, that is,

$$\max F(p,t) = F(p,1) = \frac{(4-p^2)^2}{256}+\frac{7p^2}{18432}+\frac{13p^2(4-p^2)}{6144} = G(p). \tag{55}$$

We can compute,

$$G'(p) = -\frac{25p^3}{9216}-\frac{35p(4-p^2)}{3072} \quad \text{and} \quad G''(p) = \frac{15p^2}{1024}-\frac{35(4-p^2)}{3072}.$$

If $G'(p) = 0$, then it leads to,

$$p = 0 \quad \text{and} \quad p = \pm\frac{\sqrt{21}}{2} \notin [0,2].$$

Also,

$$G''(0) = -\frac{35}{768} < 0.$$

Therefore G, attains its maximum at $p = 0$, hence we have

$$G(p) \leqslant G(0) = \frac{1}{16}.$$

Hence

$$|d_2d_4 - d_3^2| \leqslant \frac{1}{16}. \tag{56}$$

From (33) to (35), along with Lemma (3), we can estimate

$$|d_4 - d_2d_3| = |-b_4 + 4b_2b_3 - 3b_2^3| = \left|\frac{23p_1^3}{384} - \frac{49p_1p_2}{384} + \frac{p_3}{16}\right| \leqslant \frac{1}{8}. \tag{57}$$

From (33) and (34), along with (16),(17) and Theorem 8, we have

$$|d_3 - d_2^2| = |b_2^2 - b_3| \leqslant \frac{1}{4}. \tag{58}$$

Using (56), (57), (58) and Theorem 5, we estimate

$$|\mathbb{H}_{3,1}(\mathfrak{f}^{-1})| \leqslant |d_3|\left|d_2d_4 - d_3^2\right| + |d_4|\,|d_4 - d_2d_3| + |d_5|\left|d_3 - d_2^2\right| \leqslant \frac{77}{984}.$$

This concludes the proof. □

7. Bounds for Toeplitz determinants

In this section, we establish bounds for Toeplitz determinants of a particular order for functions $\mathfrak{f}$ and their inverses $\mathfrak{f}^{-1}$ in $\mathcal{GS}^*_{sym}$.

THEOREM 16. *If $\mathfrak{f} \in \mathcal{GS}^*_{sym}$ of the form (1), then*

$$|\mathcal{T}_{2,1}(\mathfrak{f})| \leqslant \frac{17}{16}, \quad |\mathcal{T}_{2,2}(\mathfrak{f})| \leqslant \frac{5}{24}, \quad |\mathcal{T}_{2,3}(\mathfrak{f})| \leqslant \frac{13}{48}, \quad |\mathcal{T}_{3,1}(\mathfrak{f})| \leqslant \frac{19}{16} \quad \text{and}$$
$$|\mathcal{T}_{4,1}(\mathfrak{f})| \leqslant \frac{1425}{1024}.$$

Proof. From (16) and Lemma 1, we can obtain

$$|\mathcal{T}_{2,1}(\mathfrak{f})| = |1 - b_2^2| \leqslant 1 + \frac{|p_1|^2}{64} \leqslant \frac{17}{16}.$$

Using 16 and (17), we can have

$$|\mathcal{T}_{2,2}(\mathfrak{f})| = |b_2^2 - b_3^2| = \frac{1}{9216}\left|144p_1^2 - 144p_2^2 + 168p_1^2p_2 - 49p_1^4\right| \tag{59}$$

Rearranging the terms, we get

$$|b_2^2 - b_3^2| = \frac{1}{9216}\left|144p_1^2 - 144p_2\left(p_2 - \frac{1}{2}p_1^2\right) + 96p_1^2\left(p_2 - \frac{49}{96}p_1^2\right)\right|$$

Applying Lemmas 1 and 2, we can estimate

$$|b_2^2 - b_3^2| \leqslant \frac{5}{24}.$$

Using (16) and (17) and applying Lemma 2 along with triangular inequality, we obtain

$$|\mathscr{T}_{2,3}(\mathfrak{f})| \leqslant |b_3^2| + |b_2^2| \leqslant \frac{7}{384}|p_1|^2 \left| p_2 - \frac{7}{24}p_1^2 \right| + \frac{1}{64}|p_1|^2 \leqslant \frac{13}{48}. \tag{60}$$

Applying triangle inequality to (9), we get

$$|\mathscr{T}_{3,1}(\mathfrak{f})| = \left|1 - 2b_2^2 - b_3(b_3 - 2b_2^2)\right| \leqslant 1 + 2|b_2^2| + |b_3||b_3 - 2b_2^2| \tag{61}$$

It can be computed that

$$|b_3 - 2b_2^2| = \frac{1}{8}\left| p_2 - \frac{5}{6}p_1^2 \right| \leqslant \frac{1}{4}. \tag{62}$$

Using (62) in (61), yields

$$|\mathscr{T}_{3,1}(\mathfrak{f})| \leqslant 1 + 2\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{4}\right) = \frac{19}{16}.$$

Using (8), we can compute

$$\mathscr{T}_{4,1}(\mathfrak{f}) = (1 - b_2^2)^2 + (b_2b_4 - b_3^2)^2 - (b_2 - b_2b_3)^2 - (b_2b_3 - b_4)^2 \\ +2(b_3 - b_2b_4)(b_2^2 - b_3).$$

Applying Triangle inequality,

$$|\mathscr{T}_{4,1}(\mathfrak{f})| \leqslant |1 - b_2^2|^2 + |b_2b_4 - b_3^2|^2 + |b_2|^2|1 - b_3|^2 + |b_2b_3 - b_4|^2 \\ +2|b_3 - b_2b_4||b_2^2 - b_3|. \tag{63}$$

Consider,

$$|b_3 - b_2b_4| = \left| -\frac{p_1p_3}{128} + \frac{25p_1^2p_2}{3072} + \frac{p_2}{8} - \frac{9p_1^4}{4096} - \frac{7p_1^2}{96} \right|$$

Rewriting and applying triangle inequality, we get

$$|b_3 - b_2b_4| \leqslant |p_1| \left| \frac{9p_1^3}{4096} - \frac{25p_1p_2}{3072} + \frac{p_3}{128} \right| + \frac{1}{8}\left| p_2 - \frac{7}{12}p_1^2 \right|$$

Using Lemmas 2 and 3, we can estimate

$$|b_3 - b_2b_4| \leqslant \frac{9}{32}. \tag{64}$$

Also, from (17) we have

$$|1-b_3| = \left|1-\frac{p_2}{8}+\frac{7p_1^2}{96}\right| \leqslant 1+\frac{1}{8}\left|p_2-\frac{7}{12}p_1^2\right| \leqslant \frac{5}{4}. \tag{65}$$

Substituting the results obtained in Theorem 2 and (64), (65) in (63), we can estimate

$$|\mathscr{T}_{4,1}(\mathfrak{f})| \leqslant \frac{1425}{1024}.$$

The theorem is thus proved. □

THEOREM 17. *If* $\mathfrak{f} \in \mathscr{G}\mathscr{S}^*_{sym}$ *and* $\mathfrak{f}^{-1}$ *is the inverse function defined by* (11)*, then*

$$|\mathscr{T}_{2,1}(\mathfrak{f}^{-1})| \leqslant \frac{17}{16}, \quad |\mathscr{T}_{2,2}(\mathfrak{f}^{-1})| \leqslant \frac{1}{8}, \quad |\mathscr{T}_{2,3}(\mathfrak{f}^{-1})| \leqslant \frac{13}{144} \quad \text{and} \quad |\mathscr{T}_{3,1}(\mathfrak{f}^{-1})| \leqslant \frac{19}{16}.$$

Proof. From (37), we have

$$|\mathscr{T}_{2,1}(\mathfrak{f}^{-1})| = |1-d_2^2| \leqslant 1+|d_2^2| \leqslant \frac{17}{16}.$$

Using (37) and (38), we get

$$|\mathscr{T}_{2,2}(\mathfrak{f}^{-1})| = |d_2^2-d_3^2| \leqslant |d_2|^2+|d_3|^2 \leqslant \frac{1}{8}.$$

From (38) and (39), we obtain

$$|\mathscr{T}_{2,3}(\mathfrak{f}^{-1})| \leqslant |d_4|^2+|d_3|^2 \leqslant \frac{13}{144}.$$

From (9), along with triangle inequality, we have

$$|\mathscr{T}_{3,1}(\mathfrak{f}^{-1})| = \left|1-2d_2^2-d_3(d_3-2d_2^2)\right| \leqslant 1+2|d_2^2|+|d_3||d_3-2d_2^2| \tag{66}$$

From (33) and (34), and applying Lemma 2, we get

$$|d_3-2d_2^2| = \left|\frac{7p_1^2}{96}-\frac{p_2}{8}\right| \leqslant \frac{1}{8}\left|p_2-\frac{7}{12}p_1^2\right| \leqslant \frac{1}{4}. \tag{67}$$

Substituting (37), (38) and (67) in (66), we get

$$|\mathscr{T}_{3,1}(\mathfrak{f}^{-1})| \leqslant \frac{19}{16}.$$

This completes the proof of the theorem. □

REMARK 7. The estimates $|\mathscr{T}_{2,2}(\mathfrak{f}^{-1})|$, $|\mathscr{T}_{2,3}(\mathfrak{f}^{-1})|$, and $|\mathscr{T}_{3,1}(\mathfrak{f}^{-1})|$ show an improvement over the results obtained in [21].

8. Krushkal's inequality

For the functions $\mathfrak{f} \in \mathscr{S}$ of the form (1), the well-known Krushkal's inequality is given by

$$\left| b_j^n - b_2^{n(j-1)} \right| \leqslant 2^{n(j-1)} - j^n,$$

where j and n are positive integers. We now estimate these bounds for a particular values of j and n.

THEOREM 18. *If $\mathfrak{f} \in \mathscr{G}\mathscr{S}^*_{\text{sym}}$ of the form (1), then*

$$|b_4 - b_2^3| \leqslant \frac{1}{8} < 2^3 - 4 = 4 \quad \textit{and} \quad |b_5 - b_2^4| \leqslant \frac{1}{8} < 2^4 - 5 = 11.$$

The sharpness of the first inequality is attained for the function defined in (29), while the sharpness of second inequality holds for the function given in (30).

Proof. From (16) and (18) and application of Lemma 3, we get

$$|b_4 - b_2^3| = \left| \frac{p_1^3}{64} - \frac{25 p_1 p_2}{384} + \frac{p_3}{16} \right| \leqslant \frac{1}{8}.$$

Using (16) and (19), we can write

$$\begin{aligned} |b_5 - b_2^4| &= \left| \frac{p_4}{16} - \frac{7 p_1 p_3}{96} - \frac{11 p_2^2}{384} + \frac{11 p_1^2 p_2}{192} - \frac{239 p_1^4}{20480} \right| \\ &= \frac{1}{16} \left| \frac{239}{1280} p_1^4 + \frac{11}{24} p_2^2 + \frac{7}{6} p_1 p_3 - \frac{11}{12} p_1^2 p_2 - p_4 \right|. \end{aligned}$$

The above expression satisfies the condition stated in Lemma 4, therefore

$$|b_5 - b_2^4| \leqslant \frac{1}{16}(2) = \frac{1}{8}.$$

The theorem is thus proved. □

THEOREM 19. *If $\mathfrak{f} \in \mathscr{G}\mathscr{S}^*_{\text{sym}}$ and $\mathfrak{f}^{-1}$ is the inverse function defined by (11), then*

$$|d_4 - d_2^3| \leqslant \frac{1}{8}.$$

Proof. Using (33) and (34), and applying Lemma 3, we get

$$|d_4 - d_2^3| = \left| \frac{p_3}{16} - \frac{55 p_1 p_2}{384} + \frac{109 p_1^3}{1536} \right| \leqslant \frac{1}{8}.$$

This concludes the proof. □

9. Conclusions

One of the fundamental challenges in GFT is obtaining sharp estimates for functionals involving coefficients in the Taylor-Maclaurin series of analytic or univalent functions. Among these, an important problem is determining the sharp bounds of Hankel determinants. In this study, we examined a subclass of $\mathscr{S}$ consisting of symmetric starlike functions associated with Gregory coefficients. We established sharp coefficient estimates and derived results for the Fekete-Szegö functional and the Generalized Zalcman conjecture. Additionally, bounds for the third and fourth Hankel determinants, as well as Toeplitz determinants of a specific order, were obtained. The analysis was further extended to the inverse functions, where initial coefficient estimates, third Hankel determinant bounds, and Toeplitz determinant estimates were derived. Furthermore, Krushkal's inequality was explored for both functions in this class and their inverses. These findings not only enrich the existing theory but also pave the way for further studies involving special function classes and inverse function behavior. Future extensions of this work may include deriving sharp bounds for third, fourth, and fifth-order Hankel determinants for the same class, as well as exploring these bounds for newly defined subclasses of analytic functions. All graphical representations in this work were generated using MATLAB®.

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