

FEKETE–SZEGŐ AND TOEPLITZ TYPE RESULTS FOR q -CONVEX FUNCTION ASSOCIATED WITH MODIFIED q -SIGMOID FUNCTION

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Abstract. In this paper, we introduce a new class of q -convex functions that generalizes the classical convex functions by incorporating q -calculus and subordination to the modified q -sigmoid function. The primary aim is to estimate sharp bounds for the initial coefficients, the Fekete-Szegő functional, coefficient differences, and certain orders of the Toeplitz determinant associated with the defined class.

1. Introduction

Denote by $\mathcal{A}$ the class of all analytic functions f in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, i.e., functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \quad (1)$$

Let $\mathcal{S} \subset \mathcal{A}$ be the subclass consisting of functions that are univalent (i.e., injective) in $\mathbb{D}$. For two analytic functions f and g , we say that f is subordinate to g , denoted $f \prec g$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$.

In [18], Pommerenke defined the Hankel determinant of order r and index n for a function f of the form (1) as:

$$H_{r,n}(f) = \det \begin{bmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{bmatrix}, \quad (a_1 = 1). \quad (2)$$

In particular, the second-order Hankel determinant is defined as

$$|H_{2,1}(f)| := |a_3 - a_2^2|, \quad (3)$$

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which is closely related to the classical Fekete–Szegő functional. This concept originates from the work of Bieberbach (1916), and another important Hankel determinant is given by

$$|H_{2,2}(f)| := |a_2 a_4 - a_3^2|, \quad (4)$$

which has been extensively studied in various subclasses of $\mathcal{S}$ for estimating sharp bounds.

The Toeplitz determinant, a close relative of the Hankel determinant, has constant entries along its principal diagonals. Owing to its wide applications in pure and applied mathematics, it has gained considerable interest. The symmetric Toeplitz determinant $T_{q,s}(f)$ for f of the form (1) as:

$$T_{q,s}(f) = \begin{vmatrix} a_s & a_{s+1} & \cdots & a_{s+q-1} \\ a_{s+1} & a_s & \cdots & a_{s+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s+q-1} & a_{s+q-2} & \cdots & a_s \end{vmatrix}. \quad (5)$$

For instance,

$$T_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_1 \end{vmatrix}, \quad T_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_2 \\ a_3 & a_2 & a_1 \end{vmatrix}.$$

Many authors have investigated sharp bounds of Toeplitz determinants for different subclasses of analytic functions (see [1, 5, 12, 16, 17]).

Zalcman conjectured in the 1960s that for $f \in \mathcal{S}$,

$$|a_n^2 - a_{2n-1}|, \quad n \geq 2,$$

is maximized by the Koebe function. This conjecture implies the Bieberbach inequality $|a_n| \leq n$ for $n \geq 2$.

Once the Bieberbach conjecture was proved, further attention was given to differences in coefficients in $\mathcal{S}$ such as

$$-1 \leq |a_3| - |a_2| \leq \frac{3}{4} + e^{-\alpha_0}(2e^{-\alpha_0} - 1) \approx 1.029,$$

where α_0 is the unique root of $4\alpha = e^\alpha$ in $(0, 1)$. Recent works have extended this theme to functions with known inverses. For $f \in \mathcal{S}$, the inverse function f^{-1} has the form

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n, \quad |w| < r_0(f), \quad \text{where } r_0(f) \geq \frac{1}{4}. \quad (6)$$

The q -derivative of a complex-valued function f , defined on a domain $\mathcal{D}$, is given by:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad z \neq 0; \quad D_q f(0) = f'(0),$$

where $q \in \mathbb{C}$ and $q \neq 1$. As $q \rightarrow 1$, we recover the classical derivative:

$$\lim_{q \rightarrow 1} D_q f(z) = f'(z).$$

Based on above definition of q -analogue the Jackson q -exponential function $e_q(z)$ (see [9, 10]) is defined by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}.$$

where $[n]_q$ is the q -analogue of the integer n , defined as:

$$[n]_q = 1 + q + q^2 + \dots + q^{n-1} = \frac{1 - q^n}{1 - q}, \quad n \geq 1, \quad [0]_q = 0.$$

and the q -factorial, $[n]_q!$, is then given by:

$$[n]_q! = [1]_q [2]_q \cdots [n]_q, \quad [0]_q! = 1.$$

Building upon these contributions, Srivastava et al. [26] introduced and studied q -starlike function associated with q -exponential function, defined as:

A function $f \in \mathcal{S}$ is said to be in the function class $\mathcal{S}^*(\mathcal{L}, q)$ if it satisfies the condition:

$$\frac{z(D_q^n f)(z)}{f(z)} \prec e_q(z), \quad (z \in \mathbb{D})$$

and obtained bounds for the non-linear function related to initial coefficient.

Recently, Khan *et al.* [13] introduced the class of q -starlike function associated with q -lemniscate of Bernoulli, defined as

$$\mathcal{S}\mathcal{L}_q^* = \left\{ f \in \mathcal{A} : \frac{zD_q f(z)}{f(z)} \prec \sqrt{\frac{2(1+z)}{2+(1-q)z}}, \quad z \in \mathbb{D} \right\},$$

which is equivalent to:

$$\left| \left(\frac{zD_q f(z)}{f(z)} \right)^2 - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

Later, Banga *et al.* [6] obtained sharp bounds for the third Hankel determinant for such classes.

Ramachandran et al [20] in 2017 examined the Fekete-Szegő problem for subclasses of analytic functions related to sigmoid functions, establishing coefficient inequalities and Hankel determinant results that demonstrated the fundamental role of sigmoid activation functions in geometric function theory. Building upon this foundation, Ramachandran et al. [21] further explored q -starlike and q -convex error functions associated with conic domains, utilizing both subordination and quasi-subordination techniques to derive coefficient estimates and Fekete-Szegő functional bounds. Recently, Kamali et al. [11] investigated initial coefficient bounds for generalized subclasses of

analytic functions related to modified sigmoid functions through the Hadamard product, proving sharp Fekete-Szegő inequalities and establishing Hankel determinant estimates that extended classical results in univalent function theory. Most recently, Akgül et al [2] introduced new subclasses of q -starlike and q -convex analytic error functions by applying modified sigmoid functions with q -derivative operators, establishing coefficient bounds and Fekete-Szegő inequalities for these novel function classes, thereby bridging q -calculus with error function theory in the most current developments of this research area, for similar works see [22, 23, 24].

Motivated by the aforementioned developments, we define and investigate q -convex functions associated with the modified q -sigmoid function, defined as:

$$\mathcal{K}^*(SG, q) = \left\{ f \in \mathcal{A} : \frac{D_q(zD_q f(z))}{D_q f(z)} \prec \frac{2}{1 + e_q(-z)}, \quad z \in \mathbb{D} \right\}.$$

This study aims to investigate the analytic and geometric properties of functions in the class $\mathcal{K}^*(SG, q)$ by deriving sharp estimates for the initial coefficients, Fekete-Szegő functionals, Hankel determinants, and symmetric Toeplitz determinants of various orders. Additionally, corresponding bounds are obtained for the inverse functions associated with this class.

Let $\mathcal{P}$ be the class of all functions $p(z)$ with a positive real part in $\mathbb{D}$, expressed as:

$$p(z) = 1 + \sum_{t=1}^{\infty} c_t z^t. \quad (7)$$

Such functions are known as Carathéodory functions [7].

For the proof of our main results we need the following lemmas.

LEMMA 1. ([8, 19]) For $p \in \mathcal{P}$, $|c_t| \leq 2$ for $t \in \mathbb{N}$, with equality for $p_0(z) = \frac{1+z}{1-z}$, $z \in \mathbb{D}$.

LEMMA 2. If $p \in \mathcal{P}$, then:

$$\begin{aligned} 2c_2 &= [c_1^2 + (4 - c_1^2) \zeta], \\ 4c_3 &= [c_1^3 + 2c_1(4 - c_1^2) \zeta - c_1(4 - c_1^2) \zeta^2 + 2(4 - c_1^2)(1 - |\zeta|^2)\eta], \end{aligned}$$

where $|\zeta| \leq 1$ and $|\eta| \leq 1$. Results for c_2 [19, p. 166] and c_3 [14] are referenced.

LEMMA 3. [15] If $p \in \mathcal{P}$, then:

$$|c_2 - \mu c_1^2| \leq \begin{cases} -4\mu + 2, & \mu < 0, \\ 2, & 0 \leq \mu \leq 1, \\ 4\mu - 2, & \mu > 1, \end{cases}$$

with the condition for $\mu < 0$ and $\mu > 1$, the equality holds for $p_0(z) = \frac{1+z}{1-z}$ and for $0 < \mu < 1$, the equality holds for $p_1(z) = \frac{1+z^2}{1-z^2}$.

LEMMA 4. ([19]) If $p \in \mathcal{P}$, then we have the estimate $|c_1^3 - 2c_1c_2 + c_3| \leq 2$.

Note that Arif et al. [4] proved the generalized result as

$$|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2(|J| + |K - 2J| + |J - K + L|). \quad (8)$$

LEMMA 5. [3] If $p \in \mathcal{P}$, with $0 \leq B \leq 1$ and $B(2B - 1) \leq D \leq B$, then:

$$|c_3 - 2Bc_1c_2 + Dc_1^3| \leq 2.$$

LEMMA 6. [25] If $p \in \mathcal{P}$, with B_1, B_2 and B_3 defined as $B_1 \geq 0, B_2 \in \mathbb{C}$ and $B_3 \in \mathcal{R}$, then:

$$\Psi_+(c_1, c_2) = |B_2c_1^2 + B_3c_2| - |B_1c_1|$$

and $\Psi_-(c_1, c_2) = -\Psi_+(c_1, c_2)$, then

$$\Psi_+(c_1, c_2) = \begin{cases} |4B_2 + 2b_3| - 2B_1, & \text{for } |2B_2 + B_3| \geq |B_3| + B_1, \\ 2|B_3|, & \text{otherwise,} \end{cases}$$

and

$$\Psi_-(c_1, c_2) = \begin{cases} 2B_1 - B_4, & \text{for } B_1 \geq 2|B_3| + B_4, \\ 2B_1 \sqrt{\frac{2|B_3|}{2|B_3| + B_4}}, & \text{for } B_1^2 \leq 2|B_3|(2|B_3| + B_4), \\ 2|B_3| + \frac{B_1^2}{2|B_3| + B_4}, & \text{otherwise,} \end{cases}$$

where, $B_4 = |4B_2 + 2B_3|$.

2. Results on $f \in \mathcal{K}^*(SG, q)$

THEOREM 1. If $f \in \mathcal{K}^*(SG, q)$, $0 < q < 1$, then the following bound holds true for the coefficient functional

$$|a_2| \leq \frac{1}{2q(q+1)}, \quad |a_3| \leq \begin{cases} \frac{q^2+1}{4q^2(q+1)^2(q^2+q+1)}, & \text{for } 0 < q < \sqrt{2}-1, \\ \frac{1}{2q^4+4q^3+4q^2+2q}, & \text{for } \sqrt{2}-1 \leq q < 1, \end{cases}$$

in which the inequality is sharp.

Proof. For $f \in \mathcal{K}^*(SG, q)$, from the definition (1), we have

$$\frac{D_q(zD_qf(z))}{D_qf(z)} \prec \frac{2}{1+e_q(-z)}, \quad z \in \mathbb{D}.$$

By the definition of subordination, one can find a Schwarz's map $w(z)$ aiming to

$$\frac{D_q(zD_qf(z))}{D_qf(z)} = \frac{2}{1+e_q(-w(z))}, \quad z \in \mathbb{D}. \quad (9)$$

Using (1) for $f(z)$ and (7) for $p(z)$ and $w(z) = \frac{p(z)-1}{p(z)+1}$ in (9), we have

$$\begin{aligned} & \frac{D_q(zD_q f(z))}{D_q f(z)} \\ &= 1 + z(a_2(q+1)^2 - a_2q - a_2) + z^2(a_2^2q^2 - a_3q^2 + a_3(q^2 + q + 1)^2 \\ & \quad + 2a_2^2q - a_3q + a_2(q+1)^2(a_2(-q) - a_2) + a_2^2 - a_3) + z^3(-a_2^3q^3 + 2a_2a_3q^3 \\ & \quad - a_4q^3 - 3a_2^3q^2 + 4a_2a_3q^2 - a_4q^2 + a_3(q^2 + q + 1)^2(a_2(-q) - a_2) \\ & \quad + a_2(q+1)^2(a_2^2q^2 - a_3q^2 + 2a_2^2q - a_3q + a_2^2 - a_3) + a_4(q^3 + q^2 + q + 1)^2 \\ & \quad - 3a_2^3q + 4a_2a_3q - a_4q - a_2^3 + 2a_2a_3 - a_4) + \dots \end{aligned} \quad (10)$$

and

$$\begin{aligned} & \frac{2}{1 + e_q(-w(z))} \\ &= 1 + \frac{c_1z}{4} + z^2\left(\frac{c_2}{4} - \frac{c_1^2(q+3)}{16(q+1)}\right) + \frac{1}{64(q+1)(q^2+q+1)}(z^3(c_1^3(q^3+6q^2 \\ & \quad + 6q+9) - 8c_2c_1(q^3+4q^2+4q+3) + 16c_3(q^3+2q^2+2q+1))) + \dots \end{aligned} \quad (11)$$

From above equations (10), (11) and (9), with simple computation, we have

$$a_2 = \frac{c_1}{4q(q+1)}, \quad a_3 = \frac{4c_2q(q+1) - c_1^2(q^2+2q-1)}{16q^2(q+1)^2(q^2+q+1)}. \quad (12)$$

By applying Lemma 1 to a_2 in (12), we have

$$|a_2| \leq \frac{1}{2q(q+1)},$$

which is sharp for $p_0(z) = \frac{1+z}{1-z}$.

Similarly, rearranging a_3 in (12) to apply Lemma 3, we obtain

$$a_3 = \frac{c_2 - \frac{c_1^2(q^2+2q-1)}{4q(q+1)}}{4q(q+1)(q^2+q+1)}.$$

Taking absolute value on both side with $0 < q < 1$ and applying Lemma 2, we get

$$|a_3| \leq \begin{cases} \frac{q^2+1}{4q^2(q+1)^2(q^2+q+1)}, & \text{for } 0 < q < \sqrt{2}-1, \\ \frac{1}{2q^4+4q^3+4q^2+2q}, & \text{for } \sqrt{2}-1 \leq q < 1, \end{cases}$$

the first inequality is sharp for $p_0(z) = \frac{1+z}{1-z}$ and the second one for $p_1(z) = \frac{1+z^2}{1-z^2}$. $\square$

THEOREM 2. *If $f \in \mathcal{H}^*(SG, q)$, $0 < q < 1$, then the following bound holds true for Fekete-Szegő functional*

$$|a_3 - \mu a_2^2| \leq \begin{cases} -\frac{\mu + (\mu - 1)q^2 + \mu q - 1}{4q^2(q+1)^2(q^2 + q + 1)}, & \text{for } \mu < \frac{-q^2 - 2q + 1}{q^2 + q + 1} \\ 1/(2q + 4q^2 + 4q^3 + 2q^4), & \text{for } \frac{-q^2 - 2q + 1}{q^2 + q + 1} \leq \mu \leq \frac{3q^2 + 2q + 1}{q^2 + q + 1}, \\ \frac{\mu + (\mu - 1)q^2 + \mu q - 1}{4q^2(q+1)^2(q^2 + q + 1)}, & \text{for } \mu > \frac{3q^2 + 2q + 1}{q^2 + q + 1}, \end{cases}$$

in which the inequality is sharp.

Proof. For $f \in \mathcal{H}^*(SG, q)$, using the values of a_2 and a_3 from the (12) in the Fekete-Szegő type functional, we get

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{4c_2q(q+1) - c_1^2(\mu + (\mu+1)q^2 + (\mu+2)q - 1)}{16q^2(q+1)^2(q^2 + q + 1)} \\ &= \frac{c_2 - \frac{c_1^2(\mu + (\mu+1)q^2 + (\mu+2)q - 1)}{4q(q+1)}}{4q(q+1)(q^2 + q + 1)}. \end{aligned} \quad (13)$$

Taking absolute value on both side of (13) and applying Lemma 3, we get

Case 1: for $\mu < \frac{-q^2 - 2q + 1}{q^2 + 2q + 1}$

$$|a_3 - \mu a_2^2| \leq -\frac{\mu + (\mu - 1)q^2 + \mu q - 1}{4q^2(q+1)^2(q^2 + q + 1)}.$$

Case 2: for $\frac{-q^2 - 2q + 1}{q^2 + 2q + 1} \leq \mu \leq \frac{3q^2 + 2q + 1}{q^2 + 2q + 1}$

$$|a_3 - \mu a_2^2| \leq \frac{1}{2q^4 + 4q^3 + 4q^2 + 2q}.$$

Case 3: for $\mu > \frac{3q^2 + 2q + 1}{q^2 + 2q + 1}$

$$|a_3 - \mu a_2^2| \leq \frac{\mu + (\mu - 1)q^2 + \mu q - 1}{4q^2(q+1)^2(q^2 + q + 1)}.$$

The bounds obtained in Case 1 and Case 3 are sharp for the function $p_0(z) = \frac{1+z}{1-z}$ and the bound in Case 2 is sharp for function $p_1(z) = \frac{1+z^2}{1-z^2}$. $\square$

THEOREM 3. *If $f \in \mathcal{H}^*(SG, q)$, $0 < q < 1$, then the following bound holds true for second Hankel determinant*

$$|H_{2,1}(f)| = |a_3 - a_2^2| \leq \frac{1}{2q(q+1)(q^2 + q + 1)},$$

in which the inequality is sharp.

Proof. For $f \in \mathcal{H}^*(SG, q)$, using the values of a_2 and a_3 from the (12) in the second Hankel determinant $H_{2,1}(f)$ in (3), we get

$$H_{2,1}(f) = \frac{4c_2(q+1) - c_1^2(2q+3)}{16q(q+1)^2(q^2+q+1)} = \frac{c_2 - \frac{c_1^2(2q+3)}{4(q+1)}}{4q(q+1)(q^2+q+1)}. \quad (14)$$

Taking absolute value on both side of (14), keeping a note of the fact the for $0 < q < 1$ the following $0 \leq \frac{2q+3}{4(q+1)} \leq 1$ is true and applying Lemma 3, we get

$$|H_{2,1}(f)| \leq \frac{1}{2q(q+1)(q^2+q+1)},$$

the bound is achieved by the function $p_1(z) = \frac{1+z^2}{1-z^2}$. $\square$

THEOREM 4. *If $f \in \mathcal{H}^*(SG, q)$, $0 < q < 1$, then the following bound holds true for initial coefficient*

$$\begin{aligned} & \left. \begin{aligned} & -\frac{\sqrt{\frac{q(q+1)}{6q^2+4q+2}}}{q(q+1)}, \quad \text{for } 0 < q < q_2 \\ & -\frac{q^6+3q^5+5q^4+5q^3+9q^2+5q+2}{4q(q+1)(q^2+q+1)(3q^2+2q+1)}, \quad \text{for } q_2 < q < 1 \end{aligned} \right\} \\ & \leq |a_3| - |a_2| \leq \begin{cases} -\frac{2q^4+4q^3+3q^2+2q-1}{4q^2(q+1)^2(q^2+q+1)}, & \text{for } 0 < q \leq q_1, \\ \frac{1}{2q(q+1)(q^2+q+1)}, & \text{for } q_1 < q < 1 \end{cases} \end{aligned}$$

in which the inequalities are sharp.

Proof. For $f \in \mathcal{H}^*(SG, q)$, using the values of a_2 and a_3 from the (12), we see

$$\begin{aligned} |a_3| - |a_2| &= \left| \frac{4c_2q(q+1) - c_1^2(q^2+2q-1)}{16q^2(q+1)^2(q^2+q+1)} \right| - \left| \frac{c_1}{4q(q+1)} \right| \\ &= |B_2c_1^2 + B_3c_2| - |B_1c_1| \end{aligned} \quad (15)$$

where $B_1 = \frac{1}{4q(q+1)}$, $B_2 = \frac{-q^2-2q+1}{16q^2(q+1)^2(q^2+q+1)}$ and $B_3 = \frac{1}{4q(q+1)(q^2+q+1)}$.

Now from (15) with $q_1 \approx 0.194586$, we observe that

Case 1: for $0 < q \leq q_1$, the following holds $|2B_2 + B_3| \geq |B_3| + B_1$, hence applying Lemma 6, we have

$$|a_3| - |a_2| \leq |4B_2 + 2B_3| - 2B_1 = -\frac{2q^4+4q^3+3q^2+2q-1}{4q^2(q+1)^2(q^2+q+1)},$$

the inequality is achieved by the function $p_0(z) = \frac{1+z}{1-z}$.

Case 2: for $q_1 < q < 1$, following $|2B_2 + B_3| < |B_3| + B_1$ holds, hence applying Lemma 6, we have

$$|a_3| - |a_2| \leq 2|B_3| = \frac{1}{2q(q+1)(q^2+q+1)},$$

the inequality is achieved by the function $p_1(z) = \frac{1+z^2}{1-z^2}$.

Similarly, from (12), by observing the values of a_2 and a_3 , we see

$$\begin{aligned} |a_2| - |a_3| &= \left| \frac{c_1}{4q(q+1)} \right| - \left| \frac{4c_2q(q+1) - c_1^2(q^2+2q-1)}{16q^2(q+1)^2(q^2+q+1)} \right| \\ &= |B_1c_1| - |B_2c_1^2 + B_3c_2| \end{aligned} \quad (16)$$

where B_1 , B_2 and B_3 are same as above in (15), consider

$$B_4 = |4B_2 + B_3| = \frac{q^2 + 1}{4q^2(q+1)^2(q^2+q+1)},$$

with $q_2 \approx 0.821952$, we observe that

Case 3: for $0 < q \leq q_2$, the following is true $B_1^2 \leq 2|B_3|(2|B_3| + B_4)$, hence making use of Lemma 6 in (16), we get

$$|a_2| - |a_3| \leq 2B_1 \sqrt{\frac{2|B_3|}{2|B_3| + B_4}} = \frac{\sqrt{\frac{q(q+1)}{6q^2+4q+2}}}{q(q+1)},$$

the sharpness of the inequality can be seen with the function

$$p_3(z) = \frac{1 - z^2}{-2\sqrt{2}\sqrt{\frac{q(q+1)}{3q^2+2q+1}}z + z^2 + 1}.$$

Case 4: for $q_2 < q < 1$ the following, $B_1 \geq 2|B_3| + B_4$ is not true and hence making use of Lemma 6 in (16), we get

$$|a_2| - |a_3| \leq 2|B_3| + \frac{B_1^2}{2|B_3| + B_4} = \frac{q^6 + 3q^5 + 5q^4 + 5q^3 + 9q^2 + 5q + 2}{4q(q+1)(q^2+q+1)(3q^2+2q+1)},$$

the sharpness of the inequality can be seen with the function

$$p_4(z) = \frac{1 - z^2}{-\frac{2q(q+1)(q^2+q+1)z}{3q^2+2q+1} + z^2 + 1}. \quad \square$$

THEOREM 5. *If $f \in \mathcal{H}^*(SG, q)$, then the following bound holds true for the second-order Toeplitz determinants*

$$|T_{2,1}(f)| \leq \frac{1}{4q^2(q+1)^2} + 1, \quad |T_{2,2}(f)| \leq \frac{4q^2(q+1)^2 + \frac{(q^2+1)^2}{(q^2+q+1)^2}}{16q^4(q+1)^4}$$

in which the inequalities are sharp.

Proof. For $f \in \mathcal{H}^*(SG, q)$, based on equation (1.4), we now revisit $T_{2,1}(f) := 1 - a_2^2$, imputing the value of a_2 from equation (12), by taking the absolute value and utilizing Lemma 1, we obtain

$$|T_{2,1}(f)| \leq 1 + \frac{|c_1|^2}{16q^2(q+1)^2} = \frac{1}{4q^2(q+1)^2} + 1.$$

Similarly, using the values of a_2 and a_3 from the (12) in the Toeplitz determinant $T_{2,2}(f) := a_2^2 - a_3^2$, we have

$$T_{2,2}(f) = \frac{c_1^2}{16q^2(q+1)^2} - \frac{(4c_2q(q+1) - c_1^2(q^2 + 2q - 1))^2}{256q^4(q+1)^4(q^2 + q + 1)^2} \quad (17)$$

Now substituting the value of c_2 from Lemma 2 in (17), we have

$$\begin{aligned} T_{2,2}(f) = & -\frac{(4 - c_1^2)^2 \zeta^2}{64q^2(q+1)^2(q^2 + q + 1)^2} + \frac{c_1^2}{16q^2(q+1)^2} \\ & - \frac{c_1^4(q^2 + 1)^2}{256q^4(q+1)^4(q^2 + q + 1)^2} - \frac{(4 - c_1^2)c_1^2(q^2 + 1)\zeta}{64q^3(q+1)^3(q^2 + q + 1)^2} \end{aligned} \quad (18)$$

Taking absolute value in (18) while considering $c_1 := c \in [0, 2]$ and $|\zeta| \leq 1$, we have

$$\begin{aligned} |T_{2,2}(f)| \leq & \frac{c^4(q^2 + 1)^2}{256q^4(q+1)^4(q^2 + q + 1)^2} + \frac{c^2}{16q^2(q+1)^2} \\ & + \frac{(4 - c^2)^2}{64q^2(q+1)^2(q^2 + q + 1)^2} + \frac{(4 - c^2)c^2(q^2 + 1)}{64q^3(q+1)^3(q^2 + q + 1)^2}. \end{aligned} \quad (19)$$

We observe the function in (19) is an increasing function of c , hence its maximum exists at $c = 2$, so

$$|T_{2,2}(f)| \leq \frac{4q^2(q+1)^2 + \frac{(q^2+1)^2}{(q^2+q+1)^2}}{16q^4(q+1)^4}.$$

The sharpness of both the bounds can be seen with the function

$$p_5(z) = \frac{1 + iz}{1 - iz}. \quad \square$$

THEOREM 6. *If $f \in \mathcal{H}^*(SG, q)$, then the following bound holds true for the third-order Toeplitz determinants*

$$|T_{3,1}(f)| \leq \frac{8(q^2 + q + 1)^2 q^2 + q^2 + 16(q^3 + 2q^2 + 2q + 1)^2 q^4 + 1}{16q^4(q+1)^2(q^2 + q + 1)^2}$$

in which the inequality is sharp.

Proof. For $f \in \mathcal{K}^*(SG, q)$, using the values of a_2 and a_3 from the (12) in $T_{3,1}(f) := 1 - 2a_2^2 - a_3(a_3 - 2a_2^2)$, we have

$$T_{3,1}(f) = \frac{1}{256q^4(q+1)^3(q^2+q+1)^2} \left(-8c_1^2q^2 \left(4(q+1)(q^2+q+1)^2 - c_2(2q+3) \right) + c_1^4(-3q^3 - 7q^2 + q + 1) + 16q^2(q+1) \left(16q^2(q^3 + 2q^2 + 2q + 1)^2 - c_2^2 \right) \right). \quad (20)$$

Imputting the values of c_2 from the Lemma 2 in the (20), we obtain

$$T_{3,1}(f) = -\frac{(4 - c_1^2)^2 \zeta^2}{64q^2(q+1)^2(q^2+q+1)^2} + \frac{(4 - c_1^2) c_1^2 \zeta}{64q^2(q+1)^3(q^2+q+1)^2} + \frac{-32c_1^2(q^2+q+1)^2 q^2 + c_1^4(q^2+1) + 256(q^3 + 2q^2 + 2q + 1)^2 q^4}{256q^4(q+1)^2(q^2+q+1)^2}. \quad (21)$$

Taking absolute value in (21) while considering $|\zeta| \leq 1$, $0 < q < 1$ and $c_1 := c \in [0, 2]$, we have

$$|T_{3,1}(f)| \leq \frac{(4 - c^2) c^2}{64q^2(q+1)^3(q^2+q+1)^2} + \frac{(4 - c^2)^2}{64q^2(q+1)^2(q^2+q+1)^2} + \frac{c^4(q^2+1) + 32c^2q^2(q^2+q+1)^2 + 256q^4(q^3 + 2q^2 + 2q + 1)^2}{256q^4(q+1)^2(q^2+q+1)^2}. \quad (22)$$

We observe that the above function in (22) attain its maximum at $c = 2$ and hence

$$|T_{3,1}(f)| \leq \frac{8(q^2+q+1)^2 q^2 + q^2 + 16(q^3 + 2q^2 + 2q + 1)^2 q^4 + 1}{16q^4(q+1)^2(q^2+q+1)^2}.$$

The sharpness of the bound obtained can be seen with the function $p_5(z) = \frac{1+iz}{1-iz}$. $\square$

3. Results on f^{-1} when $f \in \mathcal{K}^*(SG, q)$

THEOREM 7. *If $f \in \mathcal{K}^*(SG, q)$, $0 < q < 1$, then the following bound holds true for the coefficient functional*

$$|t_2| \leq \frac{1}{2q(q+1)}, \quad |t_3| \leq \frac{1}{4q^2(q^2+q+1)}$$

in which the inequalities are sharp.

Proof. In view of (1) and (6), we obtain

$$w = f(f^{-1}) = f^{-1}(w) + \sum_{n=2}^{\infty} a_n(f^{-1}(w))^n,$$

further, we have

$$w = f(f^{-1}) = w + \sum_{n=2}^{\infty} t_n w^n + \sum_{n=2}^{\infty} a_n (w + \sum_{n=2}^{\infty} t_n w^n)^n,$$

and upon simplification, we obtain

$$(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4 \\ + (t_5 + 2a_2t_4 + 2a_2t_2t_3 + 3a_3t_3 + 3a_3t_2^2 + 4a_4t_2 + a_5)w^5 + \dots = 0. \quad (23)$$

Equating the coefficients of like power in (23), upon simplification, we obtain

$$t_2 = -a_2; t_3 = \{-a_3 + 2a_2^2\}. \quad (24)$$

For $f \in \mathcal{K}^*(SG, q)$, using the values of $a_n (n = 2, 3, 4, 5)$ from (12) in (24), upon simplification, we obtain

$$t_2 = -\frac{c_1}{4q(q+1)} \quad t_3 = \frac{c_1^2(3q+1) - 4c_2q}{16q^2(q+1)(q^2+q+1)}. \quad (25)$$

By applying Lemma 1 to t_2 in (25), we have

$$|t_2| \leq \frac{1}{2q(q+1)},$$

which is sharp for $p_0(z) = \frac{1+z}{1-z}$.

Similarly, rearranging t_3 in (25) to apply Lemma 3, we obtain

$$t_3 = -\frac{c_2 - \frac{c_1^2(3q+1)}{4q}}{4q(q+1)(q^2+q+1)}.$$

Taking absolute value on both side, keeping a note of the fact that $\frac{3q+1}{4q} > 1$ for $0 < q < 1$ and applying Lemma 2, we observe that for $0 < q < 1$ the following

$$|t_3| \leq \frac{1}{4q^2(q^2+q+1)},$$

where the inequality is sharp for $p_0(z) = \frac{1+z}{1-z}$. $\square$

THEOREM 8. *If $f \in \mathcal{K}^*(SG, q)$, $0 < q < 1$, then the following bound holds true for Fekete-Szegő functional*

$$|t_3 - \mu t_2^2| \leq \begin{cases} -\frac{\mu + (\mu-5)q^2 + (\mu-6)q-1}{4q^2(q+1)^2(q^2+q+1)}, & \mu < \frac{3q^2+4q+1}{q^2+q+1}, \\ \frac{1}{2q^4+4q^3+4q^2+2q}, & \frac{3q^2+4q+1}{q^2+q+1} \leq \mu \leq \frac{7q^2+8q+1}{q^2+q+1}, \\ \frac{\mu + (\mu-5)q^2 + (\mu-6)q-1}{4q^2(q+1)^2(q^2+q+1)}, & \mu > \frac{7q^2+8q+1}{q^2+q+1}, \end{cases}$$

in which the inequalities are sharp.

Proof. For $f \in \mathcal{H}^*(SG, q)$, using the values of t_2 and t_3 form the (25) in the Fekete-Szegő type functional, we get

$$\begin{aligned} t_3 - \mu t_2^2 &= -\frac{c_1^2(\mu + (\mu - 3)q^2 + (\mu - 4)q - 1) + 4c_2q(q + 1)}{16q^2(q + 1)^2(q^2 + q + 1)} \\ &= -\frac{\frac{c_1^2(\mu + (\mu - 3)q^2 + (\mu - 4)q - 1)}{4q(q + 1)} + c_2}{4q(q + 1)(q^2 + q + 1)}. \end{aligned} \quad (26)$$

Taking absolute value on both side of (26) and applying Lemma 3, we get

Case 1: for $\mu < \frac{3q^2 + 4q + 1}{q^2 + q + 1}$

$$|t_3 - \mu t_2^2| \leq -\frac{\mu + (\mu - 5)q^2 + (\mu - 6)q - 1}{4q^2(q + 1)^2(q^2 + q + 1)}.$$

Case 2: for $\frac{3q^2 + 4q + 1}{q^2 + q + 1} \leq \mu \leq \frac{7q^2 + 8q + 1}{q^2 + q + 1}$

$$|t_3 - \mu t_2^2| \leq \frac{1}{2q^4 + 4q^3 + 4q^2 + 2q}.$$

Case 3: for $\mu > \frac{7q^2 + 8q + 1}{q^2 + q + 1}$

$$|t_3 - \mu t_2^2| \leq \frac{\mu + (\mu - 5)q^2 + (\mu - 6)q - 1}{4q^2(q + 1)^2(q^2 + q + 1)}.$$

The bounds obtained in Case 1 and Case 3 are sharp for the function $p_0(z) = \frac{1+z}{1-z}$ and the bound in Case 2 is sharp for function $p_1(z) = \frac{1+z^2}{1-z^2}$. $\square$

THEOREM 9. If $f \in \mathcal{H}^*(SG, q)$, $0 < q < 1$, then the following bound holds true for second Hankel determinant

$$|H_{2,1}(f^{-1})| = |t_3 - t_2^2| \leq \frac{1}{2q^4 + 4q^3 + 4q^2 + 2q}$$

in which the inequality is sharp.

Proof. For $f \in \mathcal{H}^*(SG, q)$, using the values of t_2 and t_3 form the (12) in the second Hankel determinant $H_{2,1}(f^{-1}) := t_3 - t_2^2$, we get

$$H_{2,1}(f^{-1}) = \frac{c_1^2(2q + 3) - 4c_2(q + 1)}{16q(q + 1)^2(q^2 + q + 1)} = -\frac{c_2 - \frac{c_1^2(2q + 3)}{4(q + 1)}}{4q(q + 1)(q^2 + q + 1)}. \quad (27)$$

Taking absolute value on both side of (27), with the note of the fact that for $0 < q < 1$ the following $0 \leq \frac{2q+3}{4(q+1)} \leq 1$ is true and applying Lemma 3, we get

$$|H_{2,1}(f^{-1})| \leq \frac{1}{2q^4 + 4q^3 + 4q^2 + 2q},$$

the bound is achieved by the function $p_1(z) = \frac{1+z^2}{1-z^2}$. $\square$

THEOREM 10. *If $f \in \mathcal{K}^*(SG, q)$, $0 < q < 1$, then the following bound holds true for initial coefficient*

$$\leq |t_3| - |t_2| \leq \left\{ \begin{array}{l} -\frac{\sqrt{\frac{q}{6q+2}}}{q(q+1)}, \quad \text{for } 0 < q < q_4, \\ -\frac{q^5 + 2q^4 + 3q^3 + 2q^2 + 7q + 2}{4q(q+1)(3q+1)(q^2+q+1)}, \quad \text{for } q_4 < q < 1 \end{array} \right\}$$

$$\leq |t_3| - |t_2| \leq \left\{ \begin{array}{l} -\frac{2q^3 + 2q^2 + q - 1}{4q^2(q+1)(q^2+q+1)}, \quad \text{for } 0 < q < q_3, \\ \frac{1}{2q^4 + 4q^3 + 4q^2 + 2q}, \quad \text{for } q_3 < q < 1, \end{array} \right.$$

in which the inequalities are sharp.

Proof. For $f \in \mathcal{K}^*(SG, q)$, using the values of t_2 and t_3 from the (12), we see

$$\begin{aligned} |t_3| - |t_2| &= \left| \frac{c_1^2(3q+1) - 4c_2q}{16q^2(q+1)(q^2+q+1)} \right| - \left| \frac{c_1}{4q(q+1)} \right| \\ &= |B_2c_1^2 + B_3c_2| - |B_1c_1|, \end{aligned} \quad (28)$$

where $B_1 = \frac{1}{4q(q+1)}$, $B_2 = \frac{3q+1}{16q^2(q+1)(q^2+q+1)}$ and $B_3 = -\frac{1}{4q(q+1)(q^2+q+1)}$. Now from (28) with $q_3 \approx 0.271069$, we observe that

Case 1: for $0 < q \leq q_3$ the following holds $|2B_2 + B_3| \geq |B_3| + B_1$, hence applying Lemma 6, we have

$$|t_3| - |t_2| \leq |4B_2 + 2B_3| - 2B_1 = -\frac{2q^3 + 2q^2 + q - 1}{4q^2(q+1)(q^2+q+1)},$$

the inequality is achieved by the function $p_0(z) = \frac{1+z}{1-z}$.

Case 2: for $q_3 < q < 1$ following $|2B_2 + B_3| < |B_3| + B_1$ holds, hence applying Lemma 6, we have

$$|t_3| - |t_2| \leq \frac{1}{2q^4 + 4q^3 + 4q^2 + 2q},$$

the inequality is achieved by the function $p_1(z) = \frac{1+z^2}{1-z^2}$.

Similarly, from (25), by observing the values of t_2 and t_3 , we see

$$\begin{aligned} |t_2| - |t_3| &= \left| \frac{c_1}{4q(q+1)} \right| - \left| \frac{c_1^2(3q+1) - 4c_2q}{16q^2(q+1)(q^2+q+1)} \right| \\ &= |B_1c_1| - |B_2c_1^2 + B_3c_2|, \end{aligned} \quad (29)$$

where B_1 , B_2 and B_3 are same as above in (28), consider $B_4 = \frac{1}{4q^2(q^2+q+1)}$, with $q_4 \approx 0.94832$, we observe that

Case 3: for $0 < q \leq q_4$, the following is true $B_1^2 \leq 2|B_3|(2|B_3| + B_4)$, hence making use of Lemma 6 in (29), we get

$$|a_2| - |a_3| \leq 2B_1 \sqrt{\frac{2|B_3|}{2|B_3| + B_4}} = \frac{\sqrt{\frac{q}{6q+2}}}{q(q+1)},$$

the sharpness of the inequality can be seen with the function $p_6(z) = \frac{2\sqrt{2}\sqrt{\frac{q}{3q+1}}z + z^2 + 1}{1-z^2}$.

Case 4: for $q_4 < q < 1$ the following, $B_1 \geq 2|B_3| + B_4$ is not true and hence making use of Lemma 6 in (16), we get

$$|a_2| - |a_3| \leq 2|B_3| + \frac{B_1^2}{2|B_3| + B_4} = \frac{q^5 + 2q^4 + 3q^3 + 2q^2 + 7q + 2}{4q(q+1)(3q+1)(q^2+q+1)},$$

the sharpness of the inequality can be seen with the function $p_7(z) = \frac{\frac{2q(q^2+q+1)z}{3q+1} + z^2 + 1}{1-z^2}$. □

THEOREM 11. *If $f \in \mathcal{K}^*(SG, q)$, then the following bound holds true for the second-order Toeplitz determinants*

$$|T_{2,1}(f^{-1})| \leq \frac{1}{4q^2(q+1)^2} + 1, \quad |T_{2,2}(f^{-1})| \leq \frac{\frac{4q^2}{(q+1)^2} + \frac{1}{(q^2+q+1)^2}}{16q^4}$$

in which the inequality is sharp.

Proof. For $f \in \mathcal{K}^*(SG, q)$, based on equation (1.4), we now revisit $T_{2,1}(f^{-1}) := 1 - t_2^2$. Substituting the value of t_2 from equation (25), taking the absolute value and utilizing Lemma 1, we obtain

$$|T_{2,1}(f^{-1})| \leq 1 + \frac{|c_1|^2}{16q^2(q+1)^2} = \frac{1}{4q^2(q+1)^2} + 1.$$

Similarly, using the values of t_2 and t_3 form the (25) in the Toeplitz determinant $T_{2,2}(f^{-1}) := t_2^2 - t_3^2$, we have

$$T_{2,2}(f^{-1}) = \frac{c_1^2}{16q^2(q+1)^2} - \frac{(c_1^2(3q+1) - 4c_2q)^2}{256q^4(q+1)^2(q^2+q+1)^2}. \quad (30)$$

Now imputting the value of c_2 form Lemma 2 in (30), we have

$$\begin{aligned} T_{2,2}(f^{-1}) = & -\frac{(4-c_1^2)^2 \zeta^2}{64q^2(q+1)^2(q^2+q+1)^2} + \frac{c_1^2}{16q^2(q+1)^2} - \frac{c_1^4}{256q^4(q^2+q+1)^2} \\ & + \frac{(4-c_1^2)c_1^2 \zeta}{64q^3(q+1)(q^2+q+1)^2}. \end{aligned} \quad (31)$$

Taking absolute value in (31) while considering $c_1 := c \in [0, 2]$, $0 < q < 1$ and $|\zeta| \leq 1$, we have

$$\begin{aligned} |T_{2,2}(f^{-1})| \leq & \frac{c^4}{256q^4(q^2+q+1)^2} + \frac{c^2}{16q^2(q+1)^2} + \frac{(4-c^2)^2}{64q^2(q+1)^2(q^2+q+1)^2} \\ & + \frac{(4-c^2)c^2}{64q^3(q+1)(q^2+q+1)^2}. \end{aligned} \quad (32)$$

We observe the function in (32) is a increasing function of c , hence maximum exist at $c = 2$, hence we have

$$|T_{2,2}(f^{-1})| \leq \frac{\frac{4q^2}{(q+1)^2} + \frac{1}{(q^2+q+1)^2}}{16q^4}.$$

The sharpness of both the bounds can be seen with the function

$$p_5(z) = \frac{1+iz}{1-iz}. \quad \square$$

THEOREM 12. *If $f \in \mathcal{K}^*(SG, q)$, then the following bound holds true for the third-order Toeplitz determinants*

$$|T_{3,1}(f^{-1})| \leq \frac{8(q^2+q+1)^2 q^2 + q^2 + 16(q^3 + 2q^2 + 2q + 1)^2 q^4 + 1}{16q^4(q+1)^2(q^2+q+1)^2}$$

in which the inequality is sharp.

Proof. For $f \in \mathcal{K}^*(SG, q)$, using the values of t_2 and t_3 form the (25) in $T_{3,1}(f^{-1}) := 1 - 2t_2^2 - t_3(t_3 - 2t_2^2)$, we have

$$\begin{aligned} T_{3,1}(f^{-1}) = & \frac{1}{256} \left(-\frac{32c_1^2}{q^2(q+1)^2} + \frac{c_1^4(6q+2) - 8c_1^2c_2q}{q^4(q+1)^3(q^2+q+1)} \right. \\ & \left. - \frac{(c_1^2(3q+1) - 4c_2q)^2}{q^4(q+1)^2(q^2+q+1)^2} + 256 \right). \end{aligned} \quad (33)$$

Imputting the values of c_2 from the Lemma 2 in the (33), we obtain

$$T_{3,1}(f^{-1}) = -\frac{(4-c_1^2)^2 \zeta^2}{64q^2(q+1)^2(q^2+q+1)^2} + \frac{(4-c_1^2)c_1^2 x}{64q^2(q+1)^3(q^2+q+1)^2} \\ + \frac{-32c_1^2(q^2+q+1)^2 q^2 + c_1^4(q^2+1) + 256(q^3+2q^2+2q+1)^2 q^4}{256q^4(q+1)^2(q^2+q+1)^2}. \quad (34)$$

Taking absolute value in (34) while considering $|\zeta| \leq 1$ $0 < q < 1$ and $c_1 := c \in [0, 2]$, we have

$$|T_{3,1}(f^{-1})| \leq \frac{(4-c^2)c^2}{64q^2(q+1)^3(q^2+q+1)^2} + \frac{(4-c^2)^2}{64q^2(q+1)^2(q^2+q+1)^2} \\ + \frac{c^4(q^2+1) + 32c^2q^2(q^2+q+1)^2 + 256q^4(q^3+2q^2+2q+1)^2}{256q^4(q+1)^2(q^2+q+1)^2}. \quad (35)$$

We observe that the above function in (35) attain its maximum at $c = 2$ and hence

$$|T_{3,1}(f^{-1})| \leq \frac{8(q^2+q+1)^2 q^2 + q^2 + 16(q^3+2q^2+2q+1)^2 q^4 + 1}{16q^4(q+1)^2(q^2+q+1)^2}.$$

The sharpness of the bound obtained can be seen with the function $p_5(z) = \frac{1+iz}{1-iz}$. $\square$

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