

ON THE GENERAL MATRIX SUMMABILITY METHOD

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Abstract. This paper presents a theorem on $\varphi - |A; \delta|_k$ summability of the series $\sum a_n \lambda_n$ which generalizes a known result on the absolute Riesz summability factors of the same series.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n) . Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^n a_{nv} s_v, \quad n = 0, 1, \dots$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |A; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [8])

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty$$

where

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v,$$

and (p_n) is a sequence of positive numbers such that

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

If we take $\delta = 0$, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then $\varphi - |A; \delta|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability.

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2. Known result

In [2], the following theorem dealing with $\left|\bar{N}, p_n\right|_k$ summability factors of an infinite series has already been proved.

THEOREM 1. *Let (X_n) be a given sequence of positive numbers and let*

$$|s_n| = O(X_n) \quad \text{as } n \rightarrow \infty. \quad (1)$$

If (λ_n) is a sequence of complex numbers such that

$$\sum_{n=1}^{\infty} \left(\frac{p_n}{P_n} \right) (|\lambda_n| X_n)^k < \infty, \quad (2)$$

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty, \quad (3)$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3. Main result

For some papers on absolute matrix summability factors of infinite series, we can refer to [3, 4, 5, 6, 7, 9, 10, 11, 12, 13]. The aim of this paper is to generalize Theorem 1 for $\varphi - |A; \delta|_k$ summability. Before giving the main result, we must first introduce some further notations. Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^n a_{ni}, \quad n, v = 0, 1, \dots \quad (4)$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \dots \quad (5)$$

and

$$A_n(s) = \sum_{v=0}^n a_{nv} s_v = \sum_{v=0}^n \bar{a}_{nv} a_v$$

$$\bar{\Delta} A_n(s) = \sum_{v=0}^n \hat{a}_{nv} a_v. \quad (6)$$

THEOREM 2. *Let $\varphi_n p_n = O(P_n)$ and (X_n) be a given sequence of positive numbers. Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, \dots, \quad (7)$$

$$a_{n-1,v} \geq a_{nv} \quad \text{for } n \geq v+1, \quad (8)$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \quad (9)$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\hat{a}_{n,v+1}| = O\left(\varphi_v^{\delta k}\right) \quad \text{as } m \rightarrow \infty, \quad (10)$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\Delta_v(\hat{a}_{nv})| = O\left(\varphi_v^{\delta k-1}\right) \quad \text{as } m \rightarrow \infty, \quad (11)$$

where $\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1}$. If conditions (1), (3) of Theorem 1 and

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k-1} (|\lambda_n| X_n)^k < \infty, \quad (12)$$

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k} X_n |\Delta \lambda_n| < \infty, \quad (13)$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |A; \delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

4. Proof of Theorem 2

Let (θ_n) denotes A -transform of the series $\sum a_n \lambda_n$. Then by (6)

$$\begin{aligned} \bar{\Delta} \theta_n &= \sum_{v=1}^n \hat{a}_{nv} a_v \lambda_v \\ &= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv} \lambda_v) \sum_{r=1}^v a_r + \hat{a}_{nn} \lambda_n \sum_{v=1}^n a_v \\ &= \sum_{v=1}^{n-1} (\hat{a}_{nv} \lambda_v - \hat{a}_{n,v+1} \lambda_{v+1} - \hat{a}_{n,v+1} \lambda_v + \hat{a}_{n,v+1} \lambda_v) s_v + a_{nn} \lambda_n s_n \\ &= a_{nn} \lambda_n s_n + \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}) \lambda_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \Delta \lambda_v s_v \\ &= \theta_{n,1} + \theta_{n,2} + \theta_{n,3}. \end{aligned}$$

To prove Theorem 2, we will show that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |\theta_{n,r}|^k < \infty \quad \text{for } r = 1, 2, 3.$$

Case I:

$$\begin{aligned} \sum_{n=1}^m \varphi_n^{\delta k+k-1} |\theta_{n,1}|^k &= \sum_{n=1}^m \varphi_n^{\delta k+k-1} a_{nm}^k |\lambda_n|^k |s_n|^k \\ &= O(1) \sum_{n=1}^m \varphi_n^{\delta k-1} (|\lambda_n| X_n)^k = O(1), \quad m \rightarrow \infty \end{aligned}$$

by (12).

Case II: By Hölder's inequality

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |\theta_{n,2}|^k &\leq \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v| |s_v| \right)^k \\ &\leq \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |s_v|^k \right) \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right)^{k-1}. \end{aligned}$$

We easily show that

$$\begin{aligned} \Delta_v(\hat{a}_{nv}) &= \hat{a}_{nv} - \hat{a}_{n,v+1} \\ &= \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} \\ &= a_{nv} - a_{n-1,v} \end{aligned}$$

by (4) and (5).

Thus

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \leq a_{nn}$$

by (4), (7) and (8).

Then, by virtue of (9), (11), (12),

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |\theta_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| |\lambda_v|^k |s_v|^k \right) \\ &= O(1) \sum_{v=1}^m |\lambda_v|^k X_v^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\Delta_v(\hat{a}_{nv})| \\ &= O(1) \sum_{v=1}^m \varphi_v^{\delta k-1} (|\lambda_v| X_v)^k = O(1), \quad m \rightarrow \infty. \end{aligned}$$

Case III:

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |\theta_{n,3}|^k &\leq \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta \lambda_v| |s_v| \right)^k \\ &\leq \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}|^{k-1} |\hat{a}_{n,v+1}| |\Delta \lambda_v| X_v \right) \left(\sum_{v=1}^{n-1} |\Delta \lambda_v| X_v \right)^{k-1}. \end{aligned}$$

Here by (4), (5), (8),

$$|\hat{a}_{n,v+1}| = a_{nn} + \sum_{i=v+1}^{n-1} (a_{ni} - a_{n-1,i}) \leq a_{nn}.$$

Then, by using (9), (10) and (13), we have

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |\theta_{n,3}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta \lambda_v| X_v \right) \\ &= O(1) \sum_{v=1}^m |\Delta \lambda_v| X_v \sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\hat{a}_{n,v+1}| \\ &= O(1) \sum_{v=1}^m \varphi_v^{\delta k} |\Delta \lambda_v| X_v = O(1), \quad m \rightarrow \infty. \end{aligned}$$

Hence, the proof of Theorem 2 is completed. If we take $\delta = 0$, $\varphi_n = \frac{p_n}{p_n}$ and $a_{nv} = \frac{p_v}{p_n}$ in Theorem 2, then we get Theorem 1. In this case, the conditions (12) and (13) reduce to the conditions (2) and (3).

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