

## CORE INCLUSION THEOREM FOR DOUBLE SEQUENCES

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*Abstract.* This article characterizes when four-dimensional matrices preserve generalized core inclusions for double sequences. Using the disk characterization of Pringsheim cores, this work establishes that a four-dimensional matrix  $A$  transforming bounded double sequences into bounded double sequences satisfies  $P-C\{A(x)\} \subseteq P-C^{(\alpha)}\{x\}$  for all bounded double sequences if and only if  $A$  is RH-regular and  $P\text{-}\limsup_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| \leq \alpha$ , which is a multidimensional extension of Natarajan's core inclusion theorem, with the special case  $\alpha = 1$  providing a double sequence analog of Knopp's core theorem.

### 1. Introduction

The core of a complex sequence, first introduced by Knopp [5], is defined as the intersection of closed convex hulls of sequence tails. For a bounded sequence  $x = (x_k)$ , if  $K_n(x)$  denotes the closed convex hull of  $\{x_k : k > n\}$ , then the core is  $\mathcal{K}(x) = \bigcap_{n=0}^{\infty} K_n(x)$ . Shcherbakov [14] provided an alternative characterization for bounded complex sequences, showing that  $\mathcal{K}(x) = \bigcap_{z \in \mathbb{C}} C_{\limsup_k |x_k - z|}(z)$ , where  $C_r(z)$  denotes the closed disk centered at  $z$  with radius  $r$ .

For double sequences, Patterson [9] introduced the Pringsheim core by extending Knopp's construction to the double sequence setting, defining it as the intersection of closed convex hulls of double sequence tails. The theory of double sequences and their summability has been comprehensively treated in Başar [1] and Mursaleen and Mohiuddine [7]. These monographs develop the framework of four-dimensional matrix transformations and RH-regularity, which are fundamental to studying core properties in the double sequence context.

Sever and Altay [13] extended Shcherbakov's disk characterization from single sequences to double sequences, showing that the Pringsheim core can be expressed as

$$P-C\{x\} = \bigcap_{z \in \mathbb{C}} B_x(z),$$

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where  $B_x(z)$  is the closed disk centered at  $z$  with radius  $P\text{-}\limsup_{k,l} |x_{k,l} - z|$ . In [2], this equivalence was revisited by means of the Hahn–Banach Separation Theorem and extended to the statistical setting.

Natarajan [8] characterized when matrix transformations of single sequences preserve core inclusions. For infinite matrices  $A = (a_{n,k})$  acting on single sequences, he showed that  $\mathcal{K}(A(x)) \subseteq \mathcal{K}^{(\alpha)}(x)$  for all bounded sequences if and only if the matrix is regular and satisfies  $\limsup_n \sum_{k=0}^{\infty} |a_{n,k}| \leq \alpha$ .

Here  $\mathcal{K}^{(\alpha)}(x) = \bigcap_{z \in \mathbb{C}} C_{\alpha\text{-}\limsup_k |x_k - z|}(z)$  is the generalized  $\alpha$ -core introduced by Shcherbakov [14]. This characterization, for  $\alpha = 1$ , yields an elegant proof of Knopp's core theorem.

The present work extends Natarajan's theorem to four-dimensional matrices acting on double sequences. Using the disk characterization from [2, 13], this article establishes necessary and sufficient conditions for a four-dimensional matrix to satisfy  $P\text{-}C\{A(x)\} \subseteq P\text{-}C^{(\alpha)}\{x\}$  for all bounded double sequences. The main result shows that, for matrices transforming bounded double sequences into bounded double sequences, this inclusion holds if and only if the matrix is RH-regular and satisfies

$$P\text{-}\limsup_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| \leq \alpha.$$

The extension from single to double sequences requires handling the six RH-regularity conditions and the more intricate convergence behavior in the Pringsheim sense. The disk characterization proves essential in managing these technical complexities.

The paper is organized as follows. Section 2 presents the necessary preliminaries, including Pringsheim convergence concepts, four-dimensional matrix transformations and RH-regularity, and reviews the disk characterization of the Pringsheim core from [2, 13]. Section 3 introduces the generalized  $\alpha$ -Pringsheim core and establishes the main theorem extending Natarajan's result to the double sequence setting. Section 4 provides concluding remarks.

## 2. Preliminaries

This section establishes the mathematical framework necessary for extending Natarajan's theorem on core inclusions to four-dimensional matrices, which begins with fundamental concepts of double sequence convergence, introduce four-dimensional matrix transformations and their regularity conditions, and then present the Pringsheim core along with its disk characterization from [2, 13]. This disk characterization serves as the crucial analytical tool that impacts the matrix transformations of double sequences core, ultimately allowing the generalization of Natarajan's single dimensional results to multidimensional setting.

## 2.1. Basic concepts for double sequences

The fundamental notion of convergence for double sequences was introduced by Pringsheim, and it forms the foundation for all subsequent developments.

DEFINITION 1. ([11]) A double sequence  $x = (x_{k,l})$  has Pringsheim limit  $L$  (denoted by  $P\text{-}\lim_{k,l} x_{k,l} = L$ ) provided that given  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $|x_{k,l} - L| < \varepsilon$  whenever  $k, l > N$ . We shall describe such a sequence more briefly as “P-convergent”.

For sequences that do not converge, we require the notion of limit superior in the Pringsheim sense, which plays a central role in defining cores and analyzing matrix transformations.

DEFINITION 2. ([9]) For a double sequence  $x = (x_{k,l})$  of real numbers, let  $\alpha_n = \sup\{x_{k,l} : k, l \geq n\}$ . The Pringsheim limit superior of  $x$  is defined as

$$P\text{-}\limsup_{k,l} x_{k,l} = \begin{cases} +\infty, & \text{if } \alpha_n = +\infty \text{ for each } n; \\ \inf_n \{\alpha_n\}, & \text{if } \alpha_n < \infty \text{ for some } n. \end{cases}$$

This notion is used exclusively with bounded double sequences, as the Pringsheim core is defined only for such sequences.

## 2.2. Four-dimensional matrix transformations

To study the behavior of double sequences under transformations, we recall the notion of four-dimensional matrices and their action on double sequences. These transformations generalize the notion of infinite matrix transformations from single sequences to the double sequence setting.

DEFINITION 3. A four-dimensional matrix  $A = (a_{m,n,k,l})$  maps a double sequence  $x = (x_{k,l})$  to the double sequence  $(A(x))_{m,n}$  where

$$(A(x))_{m,n} = \sum_{k,l=1,\infty}^{\infty,\infty} a_{m,n,k,l} x_{k,l},$$

assuming the series converges for each  $m, n \in \mathbb{N}$ .

For the analysis of core preservation, this article focuses on matrices that preserve the convergence properties of double sequences. This leads to the fundamental notion of RH-regularity.

DEFINITION 4. A four-dimensional matrix  $A$  is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

The following characterization, due independently to Hamilton and Robison, provides necessary and sufficient conditions for RH-regularity that are essential for the main results.

**THEOREM 1.** ([3, 12]) *A four-dimensional matrix  $A = (a_{m,n,k,l})$  is RH-regular if and only if the following conditions hold:*

$$\begin{aligned}
 RH_1 : \quad & P\text{-}\lim_{m,n} a_{m,n,k,l} = 0 \text{ for each } k \text{ and } l; \\
 RH_2 : \quad & P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1; \\
 RH_3 : \quad & P\text{-}\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0 \text{ for each } l; \\
 RH_4 : \quad & P\text{-}\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0 \text{ for each } k; \\
 RH_5 : \quad & \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| \text{ is } P\text{-convergent; and} \\
 RH_6 : \quad & \text{there exist positive numbers } A_0 \text{ and } B_0 \text{ such that} \\
 & \sum_{k,l > B_0} |a_{m,n,k,l}| < A_0 \text{ for all } m, n.
 \end{aligned}$$

### 2.3. Pringsheim core and its disk characterization

The Pringsheim core, introduced by Patterson for double sequences [9], generalizes the classical Knopp core to the double sequence setting. The standard definition using closed convex hulls and the disk characterization that was established in [13] and further studied in [2], which serves as the key analytical tool for studying how matrix transformations affect cores is as follows.

**DEFINITION 5.** ([9]) Let  $x = (x_{k,l})$  be a bounded double sequence in  $\mathbb{C}$ . For each  $n \in \mathbb{N}$ , let  $P\text{-}C_n\{x\}$  be the least closed convex set containing all points  $x_{k,l}$  for  $k, l > n$ . The Pringsheim core of  $x$  is defined as

$$P\text{-}C\{x\} = \bigcap_{n=1}^{\infty} P\text{-}C_n\{x\}.$$

The Pringsheim core captures the asymptotic behavior of a double sequence by identifying the set of all possible limit points and their convex combinations. To analyze how matrix transformations affect this core structure, the analysis utilizes the following disk characterization.

**DEFINITION 6.** ([13]) Let  $x = (x_{k,l})$  be a bounded double sequence in  $\mathbb{C}$ . For

each  $z \in \mathbb{C}$ , define the closed disk

$$B_x(z) = \left\{ w \in \mathbb{C} : |w - z| \leq P\text{-}\limsup_{k,l} |x_{k,l} - z| \right\},$$

where  $P\text{-}\limsup_{k,l} |x_{k,l} - z| = \inf_{n \in \mathbb{N}} \sup_{k,l > n} |x_{k,l} - z|$ .

For a bounded double sequence  $x$ , the quantity  $P\text{-}\limsup_{k,l} |x_{k,l} - z|$  above coincides with  $\inf_{m,n \in \mathbb{N}} \sup_{k > m, l > n} |x_{k,l} - z|$ , since every tail of the form  $\{(k,l) : k > m, l > n\}$  contains a tail of the form  $\{(k,l) : k, l > n_0\}$  and conversely; both expressions will be used interchangeably in what follows.

The following result, established in [13] and further studied in [2], provides a critical connection between the two characterizations, thus reinforcing the theoretical framework for this work.

**THEOREM 2.** ([2, 13]) *For any bounded double sequence  $x = (x_{k,l})$  in  $\mathbb{C}$ ,*

$$P\text{-}C\{x\} = \bigcap_{z \in \mathbb{C}} B_x(z).$$

This disk characterization, established in [13] and obtained in [2] by means of the Hahn–Banach Separation Theorem, expresses the Pringsheim core as the intersection of all closed disks  $B_x(z)$  as  $z$  ranges over the complex plane. This representation facilitates the analysis of matrix transformations by reducing the problem to examining how matrices affect the radii of these disks. This geometric approach leads directly to the generalized  $\alpha$ -cores introduced in Section 3.

## 2.4. Connection to Natarajan’s work

In 1990, Natarajan [8] characterized when matrix transformations of single sequences preserve core inclusions. In that work, the matrices are infinite matrices  $A = (a_{n,k})$  where  $n, k \in \mathbb{N}$ , acting on single sequences  $x = (x_k)$  to produce transformed sequences  $(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k$ . Natarajan established when such a matrix satisfies  $\mathcal{K}(A(x)) \subseteq \mathcal{K}^{(\alpha)}(x)$  for all bounded sequences  $x$ , where  $\mathcal{K}^{(\alpha)}(x) = \bigcap_{z \in \mathbb{C}} C_{\alpha\text{-}\limsup_n |x_n - z|}(z)$  is the generalized  $\alpha$ -core. The main result states that

this inclusion holds if and only if  $A$  is regular and satisfies  $\limsup_{n \rightarrow \infty} \sum_{k=0}^{\infty} |a_{nk}| \leq \alpha$ .

This paper extends Natarajan’s result to double sequences under four-dimensional matrices. This setting involves double sequences  $x = (x_{k,l})$  where  $k, l \in \mathbb{N}$ , and four-dimensional matrix transformations. The classical Silverman–Toeplitz regularity [4, 6, 10] is replaced by RH-regularity, which requires six conditions using Pringsheim notion of convergence.

The disk characterization from Theorem 2 provides the key tool for handling this more complex setting. It expresses the Pringsheim core through an intersection of disks, providing means to analyze how four-dimensional matrix transformations affect these geometric structures and establish the appropriate generalization of Natarajan's norm condition.

### 3. Generalized $\alpha$ -Pringsheim core

This study extends the generalized  $\alpha$ -core, introduced by Shcherbakov [14] for single sequences and employed by Natarajan [8] in his core inclusion theorem, to the double sequence setting. This generalization utilizes the disk characterization to examine the effect of four-dimensional matrix transformations on the cores of double sequences. Consequently, the following definition is introduced.

DEFINITION 7. For a bounded double sequence  $x = (x_{k,l})$  in  $\mathbb{C}$  and  $\alpha \geq 1$ , the generalized  $\alpha$ -Pringsheim core is defined as

$$P-C^{(\alpha)}\{x\} = \bigcap_{z \in \mathbb{C}} B_x^{(\alpha)}(z)$$

where  $B_x^{(\alpha)}(z) = \{w \in \mathbb{C} : |w - z| \leq \alpha \cdot P\text{-}\limsup_{k,l} |x_{k,l} - z|\}$ .

REMARK 1. When  $\alpha = 1$ , we recover  $P-C^{(1)}\{x\} = P-C\{x\}$  by Theorem 2. For  $\alpha > 1$ , the set  $P-C^{(\alpha)}\{x\}$  represents an expanded version of the Pringsheim core where each disk in the intersection has its radius scaled by the factor  $\alpha$ . This scaling mirrors Natarajan's generalization for single sequences, where  $\mathcal{H}^{(\alpha)}(x) = \bigcap_{z \in \mathbb{C}} C_{\alpha \cdot \limsup_n |x_n - z|}(z)$ . The disk characterization reveals the geometric principle at work: when a matrix transformation has norm bounded by  $\alpha$ , it maps the core of any sequence into the  $\alpha$ -expanded core of the original sequence.

#### 3.1. Main Theorem

Let us state and prove our main result, which characterizes when four-dimensional matrices preserve core inclusions. The theorem establishes that RH-regularity, together with a bounded norm condition on the matrix entries, completely characterizes this core preservation property.

Unlike in the two-dimensional setting, where the Silverman–Toeplitz conditions guarantee that every regular matrix transforms bounded sequences into bounded sequences, the RH-regularity conditions constrain only the behavior of the rows of  $A$  along Pringsheim tails. In order for the Pringsheim core of  $A(x)$  to be defined for every bounded double sequence  $x$ , the matrix  $A$  is required to transform bounded double sequences into bounded double sequences. Note that the definedness of  $A(x)$  for every bounded  $x$  already forces each row of  $A$  to be absolutely summable. For the bounded sequence  $x_{k,l} = \overline{a_{m,n,k,l}} / |a_{m,n,k,l}|$  (with  $x_{k,l} = 0$  when  $a_{m,n,k,l} = 0$ ), the rectangular

partial sums of the series defining  $(A(x))_{m,n}$  coincide with those of  $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ , which are monotone, so their Pringsheim convergence forces a finite supremum. Hence each functional  $x \mapsto (A(x))_{m,n}$  is bounded on the Banach space of bounded double sequences, with norm  $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ , and by the uniform boundedness principle the mapping property above is equivalent to the norm condition

$$\sup_{m,n \in \mathbb{N}} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| < \infty. \quad (1)$$

The sufficiency argument below is in the spirit of the norm estimate of [9, Lemma 3.1], adapted to the shifted sequences arising from the disk characterization, while the necessity part employs a block construction of the type introduced there.

**THEOREM 3.** *Let  $A = (a_{m,n,k,l})$  be a four-dimensional matrix satisfying (1) and let  $\alpha \geq 1$ . Then  $A$  satisfies*

$$P\text{-}C\{A(x)\} \subseteq P\text{-}C^{(\alpha)}\{x\}$$

for all bounded double sequences  $x = (x_{k,l})$  if and only if  $A$  is RH-regular and satisfies

$$P\text{-}\limsup_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| \leq \alpha. \quad (2)$$

*Proof. Sufficiency.* Suppose  $A$  is RH-regular and satisfies condition (2). Let  $x = (x_{k,l})$  be any bounded double sequence and let  $y \in P\text{-}C\{A(x)\}$ . Note that  $A(x)$  is a bounded double sequence by (1), so Theorem 2 applies to  $A(x)$  and, for any  $z \in \mathbb{C}$ , we are granted

$$\begin{aligned} |y - z| &\leq P\text{-}\limsup_{m,n} |z - (A(x))_{m,n}| \\ &= P\text{-}\limsup_{m,n} \left| z - \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l} \right|. \end{aligned}$$

By  $RH_2$ , setting  $\varepsilon_{m,n} = 1 - \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}$ , we have  $P\text{-}\lim_{m,n} \varepsilon_{m,n} = 0$ . It follows, using the

absolute convergence of the double series  $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$  granted by  $RH_5$ , that

$$\begin{aligned} |z - (A(x))_{m,n}| &= \left| z - \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l} \right| \\ &= \left| z \cdot \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} + z \cdot \varepsilon_{m,n} - \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l} \right| \end{aligned}$$

$$\begin{aligned}
&= \left| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} (z - x_{k,l}) + z \cdot \varepsilon_{m,n} \right| \\
&\leq \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| |z - x_{k,l}| + |z| |\varepsilon_{m,n}|.
\end{aligned}$$

Since  $P\text{-}\lim_{m,n} \varepsilon_{m,n} = 0$ , we have  $|z| |\varepsilon_{m,n}| = o(1)$  in the Pringsheim sense. Let  $L = P\text{-}\limsup_{k,l} |z - x_{k,l}| = \inf_{m,n} \sup_{k>m, l>n} |z - x_{k,l}|$ . Since  $x$  is bounded, for any  $\varepsilon > 0$ , there exist  $M, N \in \mathbb{N}$  such that

$$\sup_{k>M, l>N} |z - x_{k,l}| < L + \varepsilon.$$

Thus the partition of the sum grants us the following,

$$\begin{aligned}
\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| |z - x_{k,l}| &= \sum_{k=1}^M \sum_{l=1}^N |a_{m,n,k,l}| |z - x_{k,l}| \\
&\quad + \sum_{k=1}^M \sum_{l=N+1}^{\infty} |a_{m,n,k,l}| |z - x_{k,l}| \\
&\quad + \sum_{k=M+1}^{\infty} \sum_{l=1}^N |a_{m,n,k,l}| |z - x_{k,l}| \\
&\quad + \sum_{k=M+1}^{\infty} \sum_{l=N+1}^{\infty} |a_{m,n,k,l}| |z - x_{k,l}|.
\end{aligned}$$

Let  $C = 1 + \sup\{|z - x_{k,l}| : (k,l) \in \mathbb{N} \times \mathbb{N}\}$ , which is finite since  $x$  is bounded. By  $RH_1$  and the finiteness of  $\{(k,l) : 1 \leq k \leq M, 1 \leq l \leq N\}$ , for each  $(k,l)$  in this set, there exists  $N(k,l)$  such that for all  $m, n > N(k,l)$ ,

$$|a_{m,n,k,l}| < \frac{\varepsilon}{3CMN}.$$

Let  $N_0 = \max\{N(k,l) : 1 \leq k \leq M, 1 \leq l \leq N\}$ . Then for all  $m, n > N_0$ ,

$$\sum_{k=1}^M \sum_{l=1}^N |a_{m,n,k,l}| < MN \cdot \frac{\varepsilon}{3CMN} = \frac{\varepsilon}{3C}.$$

For the second region, by  $RH_4$ , for each  $k = 1, \dots, M$ , there exists  $N_1(k)$  such that for all  $m, n > N_1(k)$ ,

$$\sum_{l=1}^{\infty} |a_{m,n,k,l}| < \frac{\varepsilon}{3CM}.$$

Let  $N_1 = \max_{k=1, \dots, M} \{N_1(k)\}$ . Then for all  $m, n > N_1$ ,

$$\sum_{k=1}^M \sum_{l=1}^{\infty} |a_{m,n,k,l}| < M \cdot \frac{\varepsilon}{3CM} = \frac{\varepsilon}{3C}.$$

Since  $\sum_{k \leq M, l > N} |a_{m,n,k,l}| \leq \sum_{k=1}^M \sum_{l=1}^{\infty} |a_{m,n,k,l}|$ , we have the desired bound for sufficiently large  $m$  and  $n$ . Similarly, for the third region, by  $RH_3$ , for each  $l = 1, \dots, N$ , there exists  $N_2(l)$  such that for all  $m, n > N_2(l)$ ,

$$\sum_{k=1}^{\infty} |a_{m,n,k,l}| < \frac{\varepsilon}{3CN}.$$

Let  $N_2 = \max_{l=1, \dots, N} \{N_2(l)\}$ . Then for all  $m, n > N_2$ ,

$$\sum_{l=1}^N \sum_{k=1}^{\infty} |a_{m,n,k,l}| < N \cdot \frac{\varepsilon}{3CN} = \frac{\varepsilon}{3C}.$$

For the first three regions, for sufficiently large  $m$  and  $n$ ,

$$\begin{aligned} \sum_{\{(k,l): k \leq M \text{ and } l \leq N\}} |a_{m,n,k,l}| |z - x_{k,l}| &< C \cdot \frac{\varepsilon}{3C} = \frac{\varepsilon}{3}, \\ \sum_{\{(k,l): k \leq M \text{ and } l > N\}} |a_{m,n,k,l}| |z - x_{k,l}| &< C \cdot \frac{\varepsilon}{3C} = \frac{\varepsilon}{3}, \quad \text{and} \\ \sum_{\{(k,l): k > M \text{ and } l \leq N\}} |a_{m,n,k,l}| |z - x_{k,l}| &< C \cdot \frac{\varepsilon}{3C} = \frac{\varepsilon}{3}. \end{aligned}$$

For the fourth region, the bound  $\sup_{k > M, l > N} |z - x_{k,l}|$  grants us

$$\begin{aligned} \sum_{\{(k,l): k > M \text{ and } l > N\}} |a_{m,n,k,l}| |z - x_{k,l}| &\leq (L + \varepsilon) \sum_{\{(k,l): k > M \text{ and } l > N\}} |a_{m,n,k,l}| \\ &\leq (L + \varepsilon) \sum_{k,l=1,1}^{\infty, \infty} |a_{m,n,k,l}|. \end{aligned}$$

Combining these bounds with the fact that  $|z| |\varepsilon_{m,n}| < \varepsilon$  for sufficiently large  $m$  and  $n$ ; thus obtaining the following

$$\begin{aligned} |z - (A(x))_{m,n}| &\leq \sum_{k,l=1,1}^{\infty, \infty} |a_{m,n,k,l}| |z - x_{k,l}| + |z| |\varepsilon_{m,n}| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + (L + \varepsilon) \sum_{k,l=1,1}^{\infty, \infty} |a_{m,n,k,l}| + \varepsilon \\ &= 2\varepsilon + (L + \varepsilon) \sum_{k,l=1,1}^{\infty, \infty} |a_{m,n,k,l}|. \end{aligned}$$

Taking the  $P$ -lim sup on both sides and using condition (2), we obtain the following

$$\begin{aligned} P\text{-}\limsup_{m,n} |z - (A(x))_{m,n}| &\leq P\text{-}\limsup_{m,n} \left[ 2\varepsilon + (L + \varepsilon) \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| \right] \\ &= 2\varepsilon + (L + \varepsilon) \cdot P\text{-}\limsup_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| \\ &\leq 2\varepsilon + (L + \varepsilon)\alpha. \end{aligned}$$

Since this inequality holds for arbitrary  $\varepsilon > 0$ , it follows that

$$P\text{-}\limsup_{m,n} |z - (A(x))_{m,n}| \leq L\alpha = \alpha \cdot P\text{-}\limsup_{k,l} |z - x_{k,l}|.$$

Therefore,

$$|y - z| \leq \alpha \cdot P\text{-}\limsup_{k,l} |z - x_{k,l}|.$$

Thus,  $y \in B_x^{(\alpha)}(z)$  for all  $z \in \mathbb{C}$ , which means  $y \in P\text{-}C^{(\alpha)}\{x\}$  by Definition 7.

*Necessity.* Suppose that  $P\text{-}C\{A(x)\} \subseteq P\text{-}C^{(\alpha)}\{x\}$  for all bounded double sequences  $x$ . The analysis begins by showing that  $A$  must be RH-regular. Let  $x = (x_{k,l})$  be a bounded double sequence that is  $P$ -convergent to some limit  $L$ . Then  $P\text{-}\limsup_{k,l} |x_{k,l} - z| = |L - z|$  for every  $z \in \mathbb{C}$ , so that  $B_x^{(\alpha)}(L) = \{L\}$ , while

$L \in B_x^{(\alpha)}(z)$  for every  $z \in \mathbb{C}$  because  $|L - z| \leq \alpha |L - z|$ . Hence  $P\text{-}C^{(\alpha)}\{x\} = \{L\}$ , and the hypothesis yields  $P\text{-}C\{A(x)\} \subseteq \{L\}$ .

By (1),  $A(x)$  is a bounded double sequence. Suppose that  $A(x)$  is not  $P$ -convergent to  $L$ . Then there exist  $\varepsilon_0 > 0$  and indices  $(k_i, l_i)$  with  $k_i, l_i > i$  such that  $|(A(x))_{k_i, l_i} - L| \geq \varepsilon_0$  for each  $i \in \mathbb{N}$ . By boundedness, the sequence  $\{(A(x))_{k_i, l_i}\}_{i=1}^{\infty}$  has a convergent subsequence with limit  $y$  satisfying  $|y - L| \geq \varepsilon_0$ . For each fixed  $n_0 \in \mathbb{N}$ , all but finitely many of the points  $(A(x))_{k_i, l_i}$  belong to the set  $\{(A(x))_{k,l} : k, l > n_0\}$ ; since  $P\text{-}C_{n_0}\{A(x)\}$  is closed, it follows that  $y \in P\text{-}C_{n_0}\{A(x)\}$  for every  $n_0 \in \mathbb{N}$ , hence  $y \in P\text{-}C\{A(x)\} \subseteq \{L\}$ , a contradiction. Therefore  $A(x)$  is  $P$ -convergent to  $L$ , and  $A$  is RH-regular. Let us now establish condition (2) by contradiction. Suppose

$$P\text{-}\limsup_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| > \alpha.$$

Since  $P\text{-}\limsup_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| \leq \sup_{m,n \in \mathbb{N}} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| < \infty$  by (1), there exists  $h > 0$  such that

$$P\text{-}\limsup_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| = \alpha + h.$$

By the definition of  $P$ -lim sup, there exists a strictly increasing sequence of indices  $\{(m(i), n(i))\}_{i=1}^{\infty}$  such that

$$\lim_{i \rightarrow \infty} \sum_{k,l=1,1}^{\infty,\infty} |a_{m(i), n(i), k, l}| = \alpha + h.$$

We will now construct, by induction on  $i$ , strictly increasing sequences of nonnegative integers  $\{K(i)\}_{i=0}^{\infty}$  and  $\{L(i)\}_{i=0}^{\infty}$  with  $K(0) = L(0) = 0$ , and a subsequence of  $\{(m(i), n(i))\}$  (which is still denoted by  $\{(m(i), n(i))\}$  for simplicity). For each  $i \geq 1$ , given the pair  $(m(i), n(i))$  and the numbers  $K(i-1)$ ,  $L(i-1)$ , let us define the following regions:

$$\begin{aligned} R_1(i) &= \{(k, l) : 1 \leq k \leq K(i-1), 1 \leq l \leq L(i-1)\}, \\ R_2(i) &= \{(k, l) : 1 \leq k \leq K(i-1), L(i-1) < l \leq L(i)\}, \\ R_3(i) &= \{(k, l) : K(i-1) < k \leq K(i), 1 \leq l \leq L(i-1)\}, \\ R_4(i) &= \{(k, l) : K(i-1) < k \leq K(i), L(i-1) < l \leq L(i)\}, \\ R_5(i) &= \{(k, l) : k > K(i) \text{ or } l > L(i)\}. \end{aligned}$$

These sequences are constructed so that, for each  $i \geq 1$ , the following conditions are satisfied:

$$\sum_{(k,l) \in R_1(i)} |a_{m(i),n(i),k,l}| < \frac{h}{16}, \quad (3)$$

$$\sum_{(k,l) \in R_2(i)} |a_{m(i),n(i),k,l}| < \frac{h}{16}, \quad (4)$$

$$\sum_{(k,l) \in R_3(i)} |a_{m(i),n(i),k,l}| < \frac{h}{16}, \quad (5)$$

$$\sum_{(k,l) \in R_4(i)} |a_{m(i),n(i),k,l}| > \alpha + \frac{h}{2}, \quad (6)$$

$$\sum_{(k,l) \in R_5(i)} |a_{m(i),n(i),k,l}| < \frac{h}{16}. \quad (7)$$

The inductive construction is as follows:

Set  $K(0) = 0$ ,  $L(0) = 0$ . Then  $R_1(1)$ ,  $R_2(1)$ ,  $R_3(1)$  are empty, so conditions (3)–(5) are trivially satisfied. Choose  $(m(1), n(1))$  from the subsequence such that

$$\sum_{k,l=1,1}^{\infty, \infty} |a_{m(1),n(1),k,l}| > \alpha + \frac{3h}{4}.$$

By  $RH_5$ , the double series  $\sum_{k,l=1,1}^{\infty, \infty} |a_{m(1),n(1),k,l}|$  converges, so its tails vanish, and we may choose  $K(1)$  and  $L(1)$  sufficiently large such that

$$\sum_{(k,l) \in R_5(1)} |a_{m(1),n(1),k,l}| < \frac{h}{16}.$$

Thus

$$\sum_{(k,l) \in R_4(1)} |a_{m(1),n(1),k,l}| > \left( \alpha + \frac{3h}{4} \right) - \frac{h}{16} > \alpha + \frac{h}{2}.$$

For  $i \geq 2$ , let us choose  $(m(i), n(i))$  such that:

1. For  $k = 1, \dots, K(i-1)$  and  $l = 1, \dots, L(i-1)$ ,  $RH_1$  grants us

$$\sum_{k=1}^{K(i-1)} \sum_{l=1}^{L(i-1)} |a_{m(i),n(i),k,l}| < \frac{h}{16}.$$

2. For  $k = 1, \dots, K(i-1)$ ,  $RH_4$  yields

$$\sum_{l=1}^{\infty} |a_{m(i),n(i),k,l}| < \frac{h}{16K(i-1)},$$

thus

$$\sum_{k=1}^{K(i-1)} \sum_{l=1}^{\infty} |a_{m(i),n(i),k,l}| < \frac{h}{16}.$$

3. For  $l = 1, \dots, L(i-1)$ ,  $RH_3$  ensures

$$\sum_{k=1}^{\infty} |a_{m(i),n(i),k,l}| < \frac{h}{16L(i-1)},$$

therefore,

$$\sum_{l=1}^{L(i-1)} \sum_{k=1}^{\infty} |a_{m(i),n(i),k,l}| < \frac{h}{16}.$$

4. Additionally,

$$\sum_{k,l=1,1}^{\infty,\infty} |a_{m(i),n(i),k,l}| > \alpha + \frac{3h}{4}.$$

Now, invoking  $RH_5$  once more, choose  $K(i) > K(i-1)$  and  $L(i) > L(i-1)$  sufficiently large such that

$$\sum_{(k,l) \in R_5(i)} |a_{m(i),n(i),k,l}| < \frac{h}{16},$$

which ensures condition (7) holds. Consequently,

$$\sum_{(k,l) \in R_4(i)} |a_{m(i),n(i),k,l}| > \left( \alpha + \frac{3h}{4} \right) - \left( \frac{h}{16} + \frac{h}{16} + \frac{h}{16} + \frac{h}{16} \right) = \alpha + \frac{h}{2},$$

therefore condition (6) holds. This completes the construction. Let us consider the double sequence  $x = (x_{kl})$ :

$$x_{kl} = \begin{cases} \operatorname{sgn}(a_{m(i),n(i),k,l}), & \text{if } (k,l) \in R_4(i) \text{ for some } i; \\ 0, & \text{otherwise,} \end{cases}$$

where  $\operatorname{sgn}(z) = \frac{\bar{z}}{|z|}$  if  $z \neq 0$  and  $\operatorname{sgn}(0) = 0$ .

The regions  $R_4(i)$  are pairwise disjoint since  $K(i) < K(i+1) < \dots$  and  $L(i) < L(i+1) < \dots$  by construction. This ensures the sequence  $x$  is well-defined. For each  $i$ , consider the transformed sequence value,

$$\begin{aligned} (A(x))_{m(i),n(i)} &= \sum_{k,l=1,1}^{\infty,\infty} a_{m(i),n(i),k,l} x_{k,l} \\ &= \sum_{(k,l) \in R_4(i)} a_{m(i),n(i),k,l} x_{k,l} + \sum_{(k,l) \notin R_4(i)} a_{m(i),n(i),k,l} x_{k,l}. \end{aligned} \quad (8)$$

For the first sum in (8), by our construction of  $x$ , thus  $x_{k,l} = \text{sgn}(a_{m(i),n(i),k,l})$  for  $(k,l) \in R_4(i)$ . Therefore

$$\begin{aligned} \sum_{(k,l) \in R_4(i)} a_{m(i),n(i),k,l} x_{k,l} &= \sum_{(k,l) \in R_4(i)} a_{m(i),n(i),k,l} \cdot \text{sgn}(a_{m(i),n(i),k,l}) \\ &= \sum_{(k,l) \in R_4(i)} |a_{m(i),n(i),k,l}| > \alpha + \frac{h}{2}, \end{aligned}$$

where the second equality holds since  $z \cdot \text{sgn}(z) = z\bar{z}/|z| = |z|$  for every  $z \neq 0$ . For the second sum in (8), let us first show that  $x_{kl} = 0$  for all  $(k,l) \in R_2(i) \cup R_3(i)$ . To see this, suppose that some  $(k,l) \in R_2(i)$ , where  $k \leq K(i-1)$  and  $L(i-1) < l \leq L(i)$ , belongs to  $R_4(j)$  for some  $j \in \mathbb{N}$ :

- If  $j = i$ : Then  $R_4(i)$  requires  $k > K(i-1)$ , contradicting  $k \leq K(i-1)$ .
- If  $j < i$ : Then  $R_4(j)$  requires  $l \leq L(j) \leq L(i-1)$ , contradicting  $l > L(i-1)$ .
- If  $j > i$ : Then  $R_4(j)$  requires  $k > K(j-1) \geq K(i)$ , contradicting  $k \leq K(i-1)$ .

Similarly, no  $(k,l) \in R_3(i)$ , where  $K(i-1) < k \leq K(i)$  and  $l \leq L(i-1)$ , can belong to any  $R_4(j)$ :

- If  $j = i$ : Then  $R_4(i)$  requires  $l > L(i-1)$ , contradicting  $l \leq L(i-1)$ .
- If  $j < i$ : Then  $R_4(j)$  requires  $k \leq K(j) \leq K(i-1)$ , contradicting  $k > K(i-1)$ .
- If  $j > i$ : Then  $R_4(j)$  requires  $l > L(j-1) \geq L(i)$ , contradicting  $l \leq L(i-1)$ .

Therefore  $x_{k,l} = 0$  on  $R_2(i) \cup R_3(i)$ , and thus,

$$\begin{aligned} \left| \sum_{(k,l) \notin R_4(i)} a_{m(i),n(i),k,l} x_{k,l} \right| &= \left| \sum_{(k,l) \in R_1(i)} a_{m(i),n(i),k,l} x_{k,l} + \sum_{(k,l) \in R_5(i)} a_{m(i),n(i),k,l} x_{k,l} \right| \\ &\leq \sum_{(k,l) \in R_1(i)} |a_{m(i),n(i),k,l}| |x_{k,l}| + \sum_{(k,l) \in R_5(i)} |a_{m(i),n(i),k,l}| |x_{k,l}| \\ &\leq \sum_{(k,l) \in R_1(i)} |a_{m(i),n(i),k,l}| + \sum_{(k,l) \in R_5(i)} |a_{m(i),n(i),k,l}| \\ &< \frac{h}{16} + \frac{h}{16} = \frac{h}{8}. \end{aligned}$$

By the reverse triangle inequality,

$$\begin{aligned} |(A(x))_{m(i),n(i)}| &\geq \left| \sum_{(k,l) \in R_4(i)} a_{m(i),n(i),k,l} x_{kl} \right| - \left| \sum_{(k,l) \notin R_4(i)} a_{m(i),n(i),k,l} x_{kl} \right| \\ &> \left( \alpha + \frac{h}{2} \right) - \frac{h}{8} = \alpha + \frac{3h}{8}. \end{aligned}$$

Moreover, since  $|x_{k,l}| \leq 1$  for all  $k$  and  $l$ , condition (1) grants, for every  $i$ ,

$$|(A(x))_{m(i),n(i)}| \leq \sum_{k,l=1,1}^{\infty,\infty} |a_{m(i),n(i),k,l}| \leq \sup_{m,n \in \mathbb{N}} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| < \infty,$$

so the sequence  $\{(A(x))_{m(i),n(i)}\}_{i=1}^{\infty}$  is bounded.

By the Bolzano-Weierstrass theorem, the bounded sequence  $\{(A(x))_{m(i),n(i)}\}_{i=1}^{\infty}$  has a convergent subsequence. Without loss of generality, we may assume that the sequence itself converges to some  $y \in \mathbb{C}$ . Since  $|(A(x))_{m(i),n(i)}| > \alpha + \frac{3h}{8}$  for all  $i$ , it follows that  $|y| \geq \alpha + \frac{3h}{8} > \alpha$ .

To verify that  $y \in P\text{-}C\{A(x)\}$ : Since  $y$  is the limit of the subsequence  $\{(A(x))_{m(i),n(i)}\}_{i=1}^{\infty}$  where  $m(i), n(i) \rightarrow \infty$ , for any fixed  $n_0 \in \mathbb{N}$ , there exists  $i_0$  such that  $m(i), n(i) > n_0$  for all  $i \geq i_0$ . This means that for each  $i \geq i_0$ , the point  $(A(x))_{m(i),n(i)}$  belongs to the set  $\{(A(x))_{k,l} : k, l > n_0\}$ . Since  $y = \lim_{i \rightarrow \infty} (A(x))_{m(i),n(i)}$  and  $P\text{-}C_{n_0}\{A(x)\}$  is the closed convex hull of  $\{(A(x))_{k,l} : k, l > n_0\}$ , and closed sets contain all their limit points, thus  $y \in P\text{-}C_{n_0}\{A(x)\}$ . Since this holds for every  $n_0 \in \mathbb{N}$ , it follows that  $y \in P\text{-}C\{A(x)\} = \bigcap_{n=1}^{\infty} P\text{-}C_n\{A(x)\}$ . Let us now analyze  $P\text{-}C^{(\alpha)}\{x\}$  for the constructed sequence. For  $z = 0$ ,

$$P\text{-}\limsup_{k,l} |0 - x_{k,l}| = P\text{-}\limsup_{k,l} |x_{k,l}| = 1,$$

since by (6) each  $R_4(i)$  contains at least one pair  $(k, l)$  with  $a_{m(i),n(i),k,l} \neq 0$ , hence with  $|x_{k,l}| = 1$ , and the regions  $R_4(i)$  eventually lie in every tail  $\{(k, l) : k, l > n\}$ . Therefore

$$B_x^{(\alpha)}(0) = \{w \in \mathbb{C} : |w| \leq \alpha \cdot 1 = \alpha\}.$$

Since  $P\text{-}C^{(\alpha)}\{x\} = \bigcap_{z \in \mathbb{C}} B_x^{(\alpha)}(z) \subseteq B_x^{(\alpha)}(0)$ , we have  $P\text{-}C^{(\alpha)}\{x\} \subseteq \{w \in \mathbb{C} : |w| \leq \alpha\}$ .

Since  $|y| > \alpha$ , we have  $y \notin P\text{-}C^{(\alpha)}\{x\}$ . However,  $y \in P\text{-}C\{A(x)\}$  as shown above. This contradicts our assumption that  $P\text{-}C\{A(x)\} \subseteq P\text{-}C^{(\alpha)}\{x\}$  for all bounded double sequences  $x$ . Therefore,  $P\text{-}\limsup_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| \leq \alpha$ .  $\square$

REMARK 2. For the case  $\alpha = 1$ , condition (2) becomes  $P\text{-}\limsup_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$

$\leq 1$  which, for RH-regular matrices, is equivalent to  $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| = 1$ . This

gives us an extension of Knopp's core theorem to the double sequence case, providing a simple and elegant proof for this important result. This special case deserves explicit formulation as a corollary, as it represents a direct generalization of one of the foundational results in summability theory.

To see the stated equivalence, note that condition  $RH_2$  and the triangle inequality yield

$$P\text{-}\liminf_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| \geq P\text{-}\liminf_{m,n} \left| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \right| = 1,$$

which together with condition (2) for  $\alpha = 1$  forces the Pringsheim limit to exist and equal 1. For real-valued double sequences, the corollary below is closely related to [9, Theorem 3.2], in which RH-regularity together with the same limit condition characterizes the inequality  $P\text{-}\limsup A(x) \leq P\text{-}\limsup x$ ; the present formulation provides a complex, core-theoretic counterpart.

COROLLARY 4. Let  $A = (a_{m,n,k,l})$  be a four-dimensional matrix satisfying (1). Then

$$P\text{-}C\{A(x)\} \subseteq P\text{-}C\{x\}$$

for any bounded double sequence  $x = (x_{k,l})$  if and only if  $A$  is RH-regular and satisfies

$$P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| = 1.$$

#### 4. Conclusion

This paper establishes an extension of Natarajan's theorem on core inclusions to four-dimensional matrices acting on double sequences. The main result establishes that a four-dimensional matrix  $A$  transforming bounded double sequences into bounded double sequences satisfies  $P\text{-}C\{A(x)\} \subseteq P\text{-}C^{(\alpha)}\{x\}$  for all bounded double sequences  $x$  if and only if  $A$  is RH-regular and satisfies the natural norm condition

$$P\text{-}\limsup_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| \leq \alpha.$$

The disk characterization from [2, 13] proved essential in establishing this result, allowing this work to proceed with scalar inequalities involving Pringsheim limit superiors rather than more complex set-theoretic arguments. The proof technique, particularly the construction used in the necessity part, provides insight into how four-dimensional matrices interact with the geometric structure of cores.

The techniques developed here may extend to other convergence concepts for double sequences, such as statistical convergence or ideal convergence, though adapting the

disk characterization to these settings would require addressing the measure-theoretic aspects of these convergence notions.

## REFERENCES

- [1] F. BAŞAR, *Summability Theory and its Applications*, 2nd ed., CRC Press/Taylor & Francis Group, Boca Raton, 2022.
- [2] S. M. HAMID, R. F. PATTERSON, *Disk Characterizations of Pringsheim and Statistical Cores for Complex Double Sequences*, *Bull. Math. Anal. Appl.* **17** (2025), no. 3, 115–125.
- [3] H. J. HAMILTON, *Transformations of Multiple Sequences*, *Duke Math. J.* **2**, 29–60 (1936).
- [4] G. H. HARDY, *Divergent Series*, Oxford University Press, Oxford, 1949.
- [5] K. KNOPP, *Zur Theorie der Limitierungsverfahren*, (Erste Mitteilung), *Math. Z.* **31**, 97–127 (1930).
- [6] I. J. MADDOX, *Elements of Functional Analysis*, Cambridge University Press, Cambridge, 1970.
- [7] M. MURSALEEN, S. A. MOHIUDDINE, *Convergence methods for double sequences and applications*, Springer, New Delhi, 2014.
- [8] P. N. NATARAJAN, *On the core of a sequence over valued fields*, *J. Indian Math. Soc.* **55**, 189–198 (1990).
- [9] R. F. PATTERSON, *Double sequence core theorems*, *Internat. J. Math. Math. Sci.* **22** (4), 785–793 (1999).
- [10] A. PEYERIMHOFF, *Lectures on Summability*, *Lecture Notes in Mathematics*, vol. 107, Springer, Berlin, 1969.
- [11] A. PRINGSHEIM, *Zur Theorie der zweifach unendlichen Zahlenfolgen*, *Math. Ann.* **53**, 289–321 (1900).
- [12] G. M. ROBISON, *Divergent Double Sequences and Series*, *Trans. Amer. Math. Soc.* **28**, (1), 50–73 (1926).
- [13] Y. SEVER, B. ALTAY, *On the core of double sequences*, *Sarajevo J. Math.* **11** (24), no. 2, 273–282 (2015).
- [14] A. A. SHCHERBAKOV, *Kernels of sequences of complex numbers and their regular transformations*, *Math. Notes* **22**, 948–953 (1977).

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