

ON APPROXIMATION OF GENERALIZED EULER MEANS AND RELATED COMPACT OPERATOR CHARACTERIZATIONS

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Abstract. Upon several investigations on difference sequence spaces, the current article aims to propose a new family of generalized Euler difference sequence spaces with the help of generalized Euler mean and fractional binomial difference operator. Also, this paper discusses certain theorems related to matrix characterizations, operator norms and Hausdorff measure of noncompactness of the new spaces. In addition, the compact operator characterization for the proposed spaces of the binomial difference via the Euler mean has been demonstrated. Moreover, towards the final section, we have proposed the Korovkin type approximation theorem for the proposed operator along with some convergence analysis via modulus of smoothness of order one.

1. Introduction, preliminaries and requisites

By the linear space ω , we mean the space containing all the complex and real valued sequences. However, any linear subspace of ω , is said to be a sequence space. The Banach space associated with continuous coordinates function p_t , i.e., $p_t : \mathfrak{Y} \rightarrow \mathbb{C}$, ($m \in \mathbb{N}_0$), where $p_m(\eta) = \eta_m$ for every $m \in \mathbb{N}_0$ and for all $\eta \in \mathfrak{Y}$, is coined to be an *BK*-space. Being specific some basic classical sequence spaces such as $\ell_\infty, c, c_0, \ell_1$, and ℓ_p ($1 < p < \infty$), *bs*, *cs*, represent for spaces of bounded, convergent, null sequences, absolutely summable sequences, p -summable sequences, and bounded series and convergent series respectively involves the norms $\|\eta\|_{\ell_\infty} = \|\eta\|_c = \|\eta\|_{c_0} = \sup_j \eta_j$, and $\|\eta\|_{bs} = \sup_m \left| \sum_{j=0}^m \eta_j \right|$.

However, both norms are isometrically isomorphic to ℓ_∞ , via the linear mapping

$$(\eta_m)_{m \in \mathbb{N}_0} \mapsto \left(\sum_{i=0}^m \eta_i \right)_{m \in \mathbb{N}}.$$

Suppose ϕ leverage the space of all finite sequences, $e = (1, 1, 1, \dots)$, and $e^{(k)}$ is the sequence which contains 1 at k th place for each $k \in \mathbb{N}_0$ and zero otherwise. In a

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BK subspace of ϕ , will satisfy AK property, whenever for each sequence $\eta = (\eta_k) \in \mathfrak{Y}$ posses a unique representation

$$\eta = \sum_{\mathfrak{k}=0}^{\infty} \eta_{\mathfrak{k}e(\mathfrak{k})}.$$

Let $A = (a_{mj})$ be an infinite matrix of real numbers a_{mj} , for $m, j \in \mathbb{N}_0$, and $\mathfrak{Y}_1, \mathfrak{Y}_2$ be BK -spaces. Then for every $\eta = (\eta_j) \in \mathfrak{Y}_1$, the matrix mapping $A : \mathfrak{Y}_1 \rightarrow \mathfrak{Y}_2$, defined by

$$(A\eta)_m = \sum_j a_{mj}\eta_j, \quad (m \in \mathbb{N}_0), \quad (1)$$

is called the A -transform of η provided the infinite series given in (1) converges for each $\eta \in \mathbb{N}_0$ and $A\eta \in \mathfrak{Y}_2$. The set $(\mathfrak{Y}_1, \mathfrak{Y}_2)$ signify the class containing all infinite matrices A from $\mathfrak{Y}_1$ to $\mathfrak{Y}_2$.

The set associated with an infinite matrix $\tilde{A}$ and the sequence space λ defined by

$$\mathfrak{S}_{\tilde{A}} = \{y = (y_j) \in w : \tilde{A}y \in \mathfrak{S}\},$$

represent the *matrix domain* $\mathfrak{S}_A$ of the matrix $\tilde{A}$ in the space $\mathfrak{S}$.

Let $\tilde{X}$ be a linear metric space. The sequence $(b_m)_{m=0}^{\infty}$ in $\tilde{X}$ with the criteria for every $y \in \mathfrak{Y}$, we can have a unique sequence $(a_m)_{m=0}^{\infty}$ of scalars such that

$$y = \sum_{m=0}^{\infty} a_m b_m.$$

The matrix domain $\mathfrak{S}_{\tilde{A}}$ of a normed linear space λ posses a basis if and only if the space λ has the basis.

The α - and β -dual of a sequence space X are, respectively defined by

$$\mathfrak{Y}^{\alpha} = \{a = (a_i) : ay \in \ell_1 \text{ for all } y \in \mathfrak{Y}\},$$

and

$$\mathfrak{Y}^{\beta} = \{a = (a_i) : ay \in cs \text{ for all } y \in \mathfrak{Y}\}.$$

We also write $\tilde{A} \in (\mathfrak{Y}_1, \mathfrak{Y}_2)$ if and only if $\tilde{A}_m \in \mathfrak{Y}_1^{\beta}$ for all $m \in \mathbb{N}_0$ and $\tilde{A}\eta \in \mathfrak{Y}$ for all $\eta \in \mathfrak{Y}$. Note that $\ell_{\infty}^{\beta} = c^{\beta} = c_0^{\beta} = \ell_1$, $\ell_1^{\beta} = \ell_{\infty}$ and $\ell_p^{\beta} = \ell_q$, where $1 < p < \infty$ and $q = p/(p-1)$.

Let $X \supset \phi$ be a BK space and $\mathfrak{a} = (a_k)$. Then, we have

$$\|\mathfrak{a}\|_{\mathfrak{Y}}^* = \sup_{\eta \in S_{\mathfrak{Y}}} \left| \sum_{i=0}^{\infty} a_i \eta_i \right|, \quad (2)$$

where the right side expression (2) exists and finite, indeed, this the case where $\mathfrak{a} \in \mathfrak{Y}^{\beta}$.

Due to numerous applications, the summability theory and the theory of linear bounded operators and sequence spaces have been used in many other achieved research field of pure and applied mathematics. The applications are more visible in matrices

and operator theory, approximation theory, calculus, etc. (see eg., [4, 6, 10, 13, 14, 21]). Recently, the sequence spaces generated by difference operators via different means are very much useful and effective for establishing relationships between pure and applied fields. For instance, the theory of Hausdorff measures of noncompactness related to compact operators are efficiently experimented in solving the functional integral equations, approximation theory, fixed point theory, etc. Motivated by the idea, the primary goal of the current investigations is to introduce a class of new generalized fractional binomial difference sequence spaces via Euler mean operator and find their algebraic and topological structures. We also discuss some results on the related matrix operator norms, compact operators via the Hausdorff measure of noncompactness over newly defined spaces.

Before moving to our main results, let us recall the following useful definitions:

The *Euler mean matrix* $E^t = (e_{mj}^t)$ of order t , ($0 < t < 1$) is defined by

$$e_{mj}^t = \begin{cases} \binom{m}{j} (1-t)^{m-j} t^j, & (j \leq m), \\ 0, & (j > m). \end{cases}$$

The *difference matrix* $B(r, s) = (b_{mj})$, $r, s \in \mathbb{R} \setminus \{0\}$ is defined by

$$b_{mj} = \begin{cases} r, & (j = m), \\ s, & (j = m - 1), \\ 0, & (\text{otherwise}). \end{cases}$$

The *fractional difference operator* $\Delta^{(\xi)} = (\delta_{mj}^{\xi})$ of order ξ (see [8]) is defined by the triangle i.e.,

$$\delta_{mj}^{\xi} = \begin{cases} \frac{(-\xi)_{m-j}}{\Gamma(m-j+1)}, & (0 \leq j \leq m), \\ 0, & (j > m), \end{cases}$$

where $\Gamma(\cdot)$ denotes the Gamma function and $(\xi)_k$ conceptualize the Pochhammer symbol for any real number ξ which is given by

$$(\xi)_k = \begin{cases} 1, & (\xi = 0, j \in \mathbb{N}), \\ \xi(\xi+1) \dots (\xi+j-1), & (j \in \mathbb{N}_0), \end{cases}$$

it is being understood conventionally that $(0)_0 = 1$.

The *fractional binomial difference operator* $B_{r,s}^{(\xi)} = (\delta_{mk}^{\xi}(r, s))$ of order ξ is defined by the triangle

$$\delta_{mk}^{\xi}(r, s) = \begin{cases} \frac{(-\xi)_{m-k} r^{\xi+k-m} s^{m-k}}{\Gamma(m-k+1)}, & (0 \leq k \leq m), \\ 0, & (k > m). \end{cases}$$

On combining operator E^t and $B_{r,s}^{(\xi)}$, we define the product operator $E_v^t B_{r,s}^{(\xi)}$ with the help of any sequence (fixed) $v = (v_\ell)$ of non-zero real numbers as

$$(E_v^t B_{r,s}^{(\xi)} x)_\ell = \sum_{j=0}^{\ell} \sum_{l=j}^{\ell} (-1)^j \frac{(-\xi)_{l-j} (-\ell)_l}{\Gamma(l-j+1) \Gamma(l+1)} t^l (1-t)^{\ell-l} r^{\xi+j-l} s^{l-j} v_j x_j. \quad (3)$$

Equivalently, in matrix notation, one may write

$$(E_{\nu}^t B_{r,s}^{(\xi)})_{m\ell} = \begin{cases} \sum_{l=\ell}^m (-1)^{\ell} \frac{(-\xi)_{l-\ell} (-m)_{\ell}}{(l-\ell)! \Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+\ell-l} s^{l-\ell} v_{\ell}, & (0 \leq \ell \leq m), \\ 0, & (\ell > m). \end{cases}$$

It is noted that $(E_{\nu}^t B_{r,s}^{(\xi)})^{-1}$ is given by a triangle $S = (s_{m\ell})$, where

$$s_{m\ell} = \begin{cases} \sum_{l=\ell}^m (-1)^{m-\ell} \frac{(\xi)_{m-l} (-l)_{\ell} t^l (t-1)^{l-\ell}}{\Gamma(m-l+1) \Gamma(\ell+1) v_m} r^{l-\xi-m} s^{m-l}, & (0 \leq \ell \leq m), \\ 0, & (\ell > m). \end{cases} \quad (4)$$

2. New sequence spaces

Let ξ be a positive fraction, $0 < t < 1$. Then, we introduce the matrix domain of $E_{\nu}^t B_{r,s}^{(\xi)}$, on $\mathfrak{Y}$, i.e., $\mathfrak{Y}(E_{\nu}^t B_{r,s}^{(\xi)})$, i.e.,

$$\mathfrak{Y}(E_{\nu}^t B_{r,s}^{(\xi)}) = \{x \in w : y \in \mathfrak{Y}\},$$

where, the sequence $y = (y_{\ell})$ can be computed as the $E_{\nu}^t B_{r,s}^{(\xi)}$ -transform of the sequence (x_{ℓ}) , i.e.,

$$y_{\ell} = \sum_{j=0}^{\ell} \sum_{l=j}^{\ell} (-1)^j \frac{(-\xi)_{l-j} (-\ell)_{\ell}}{\Gamma(l-j+1) \Gamma(l+1)} t^l (1-t)^{k-l} r^{\xi+j-l} s^{l-j} v_j x_j, \quad (\ell \in \mathbb{N}_0),$$

and $\mathfrak{Y} \in \{\ell_{\infty}, \ell_p, c, c_0\}$. It is remarked that the above spaces can be reduced to following classes under different special cases of ξ, r, s and ν . For instances,

1. The spaces e_0^t and e_c^t (see, [3]) under the case that $\xi = 0$, $r = s = 1$ and e_{∞}^t (see [4]) under the case that $\xi = 0$, $r = s = 1$ and $\nu = e$.
2. The spaces $b_0^{u,\nu}(\Delta^{\xi})$, $b_c^{u,\nu}(\Delta^{\xi})$ and $b_{\infty}^{u,\nu}(\Delta^{\xi})$ (see, [46]) under the case that $t = u$, $1 - t = \nu$, $r = s = 1$ and $\nu = e$.
3. The spaces $e_0^t(\Delta)$, $e_c^t(\Delta)$ and $e_{\infty}^t(\Delta)$ (see, [5]) under the case that $\xi = 1$, $r = 1$, $s = -1$ and $\nu = e$.
4. The spaces $\ell_{\infty}(E^t \Delta^{\xi})$, $c_0(E^t \Delta^{\xi})$, $c(E^t \Delta^{\xi})$ and $\ell_p(E^t \Delta^{\xi})$ (see, [24]) under the case that $r = 1$, $s = -1$ and $\nu = e$.
5. The spaces $e_0^t(\Delta^{(m)})$, $e_c^t(\Delta^{(m)})$ and $e_{\infty}^t(\Delta^{(m)})$ (see, [42]) under the case that $\xi = m$, $r = 1$, $s = -1$ and $\nu = e$.
6. The classes $e_0^t(B^{(m)})$ and $e_c^t(B^{(m)})$ (see, [26]) for the case that $\xi = m$ and $\nu = e$.
7. The classes $e_0^t(B_{\nu}^{(m)})$ and $e_c^t(B_{\nu}^{(m)})$ (see, [34]) for the case that $\xi = m$.

8. The spaces $b_0^{u,v}$, $b_c^{u,v}$ and $b_\infty^{u,v}$ (see, [16]) under the case that $\xi = 0$, $t = u$, $1 - t = v$, $r = s = 1$ and $v = e$.
9. The spaces $b_0^{u,v}(\nabla^{(m)})$, $b_c^{u,v}(\nabla^{(m)})$ and $b_\infty^{u,v}(\nabla^{(m)})$ (see, [35]) under the case that $\xi = m$, $t = u$, $1 - t = v$, $r = s = 1$ and $v = e$.

2.1. Topological and algebraic properties

Now, we state the following basic theorems and results related to inclusion relations, topological properties and certain operator norms of the above constructed spaces.

THEOREM 1. *The following inclusion relation holds strictly*

$$c_0(E_v^t B_{r,s}^{(\xi)}) \subset c(E_v^t B_{r,s}^{(\xi)}) \subset \ell_\infty(E_v^t B_{r,s}^{(\xi)}).$$

Proof. The proof is followed by the well known fact that $c_0 \subset c \subset \ell_\infty$ (see, Remark 3.1 of [26]). $\square$

THEOREM 2. *For $X \in \{\ell_\infty, c, c_0\}$, $X(E_v^t B_{r,s}^{(\xi)})$ are BK spaces with the following norm*

$$\|x\|_{\ell_\infty(E_v^t B_{r,s}^{(\xi)})} = \sup_{\mathfrak{k}} \left| \sum_{j=0}^{\mathfrak{k}} \sum_{l=j}^{\mathfrak{k}} (-1)^j \frac{(-\xi)_{l-j} (-\mathfrak{k})_l t^l (1-t)^{\mathfrak{k}-l}}{\Gamma(l-j+1) \Gamma(l+1)} r^{\xi+j-l} s^{l-j} v_j x_j \right|.$$

Similarly, the space $\ell_p(E_v^t B_{r,s}^{(\xi)})$ ($1 \leq p < \infty$) is a BK space with the following norm

$$\|x\|_{\ell_p(E_v^t B_{r,s}^{(\xi)})} = \left(\sum_{\mathfrak{k}} \left| \sum_{j=0}^{\mathfrak{k}} \sum_{l=j}^{\mathfrak{k}} (-1)^j \frac{(-\xi)_{l-j} (-\mathfrak{k})_l t^l (1-t)^{\mathfrak{k}-l}}{\Gamma(l-j+1) \Gamma(l+1)} r^{\xi+j-l} s^{l-j} v_j x_j \right|^p \right)^{1/p}.$$

Proof. The proof of this theorem is a routine verification (see, Theorem 3.1 of [26]). $\square$

Let the infinite matrix $A = (a_{nl})$ with $a_{nl} \geq 0$ for all $n, l \in \mathbb{N}_0$. Then we reach at the following desired results on operator norms by using A :

THEOREM 3. *Let $X \in \{\ell_\infty, c, c_0\}$. Then the infinite matrix A is a bounded operator from X to $X(E_v^t B_{r,s}^{(\xi)})$ if*

$$M = \sup_m \left\{ \sum_{\mathfrak{k}=0}^{\infty} \left| \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j} (-m)_l}{\Gamma(l-j+1) \Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{m\mathfrak{k}} \right| \right\} < \infty$$

and

$$\|A\|_{(E_v^t B_{r,s}^{(\xi)})_\infty} = M.$$

Proof. Let us assume the case of $X = \ell_\infty$. Suppose $x \in \ell_\infty$ and we have

$$\begin{aligned}
 & \|Ax\|_{E_V^t B_{r,s}^{(\xi)}, \infty} \\
 &= \sup_m \left| \sum_{\mathfrak{k}=0}^{\infty} \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j} (-m)_l}{\Gamma(l-j+1) \Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{m\mathfrak{k}l} x_{\mathfrak{k}} \right| \\
 &\leq \sup_m \left\{ \sum_{\mathfrak{k}=0}^{\infty} \left| \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j} (-n)_l}{\Gamma(l-j+1) \Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{m\mathfrak{k}l} x_{\mathfrak{k}} \right| \right\} \\
 &\leq \sup_m \left\{ \sum_{\mathfrak{k}=0}^{\infty} \left| \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j} (-m)_l}{\Gamma(l-j+1) \Gamma(l+1)} t^l (1-t)^{n-l} r^{\xi+j-l} s^{l-j} v_m a_{m\mathfrak{k}l} \right| \right\} \|x\|_\infty \\
 &\leq M \|x\|_\infty.
 \end{aligned}$$

Secondly, choose the sequence $x = e$, then we obtain

$$\begin{aligned}
 & \|Ae\|_{E_V^t B_{r,s}^{(\xi)}, \infty} \\
 &= \sup_m \left| \sum_{\mathfrak{k}=0}^{\infty} \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j} (-m)_l}{\Gamma(l-j+1) \Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{m\mathfrak{k}l} x_{\mathfrak{k}} \right| \\
 &= \sup_m \left\{ \left| \sum_{\mathfrak{k}=0}^{\infty} \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j} (-m)_l}{\Gamma(l-j+1) \Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{m\mathfrak{k}l} \right| \right\} \\
 &= \sup_m \left\{ \sum_{\mathfrak{k}=0}^{\infty} \left| \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j} (-n)_l}{\Gamma(l-j+1) \Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{m\mathfrak{k}l} \right| \right\} \\
 &= M.
 \end{aligned}$$

This proves the result. $\square$

THEOREM 4. Suppose that $A \in B\left(\ell_1, \ell_1(E_V^t B_{r,s}^{(\xi)})\right)$. If

$$\overline{M} = \sup_{\mathfrak{k}} \left\{ \sum_{m=0}^{\infty} \left| \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j} (-m)_l}{\Gamma(l-j+1) \Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{m\mathfrak{k}l} \right| \right\} < \infty,$$

and

$$\|A\|_{E_V^t B_{r,s}^{(\xi)}, 1} = \overline{M}.$$

Proof. Let us take $x \in \ell_1$. Then

$$\begin{aligned}
 & \|Ax\|_{E_V^t B_{r,s}^{(\xi)}, 1} \\
 &= \sum_{m=0}^{\infty} \left| \sum_{\mathfrak{k}=0}^{\infty} \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j} (-m)_l}{\Gamma(l-j+1) \Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{m\mathfrak{k}l} x_{\mathfrak{k}} \right|
 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} \left| \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j}(-m)_l}{\Gamma(l-j+1)\Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{m\ell} x_{\ell} \right| \\
&\leq \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \left| \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j}(-m)_l}{\Gamma(l-j+1)\Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{m\ell} \right| |x_{\ell}| \\
&\leq \overline{M} \|x\|_1.
\end{aligned}$$

Again consider $x = (e^{(m)})$, then we get

$$\begin{aligned}
&\|Ae_m\|_{E_V^t B_{r,s}^{(\xi)},1} \\
&= \sum_{m=0}^{\infty} \left| \sum_{\ell=0}^{\infty} \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j}(-m)_l}{\Gamma(l-j+1)\Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{m\ell} x_{\ell} \right| \\
&= \sum_{m=0}^{\infty} \left| \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j}(-m)_l}{\Gamma(l-j+1)\Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{mm} \right| \\
&= \overline{M}.
\end{aligned}$$

Hence, we reach at our required results. $\square$

THEOREM 5. Let $A \in \mathcal{B}(\ell_p, \ell_p(E_V^t B_{r,s}^{(\xi)}))$, $(1 \leq p < \infty)$ if

$$\overline{M}_p = \sup_{\ell} \left\{ \sum_{m=0}^{\infty} \left| \sum_{j=0}^m \sum_{l=j}^m (-1)^j \frac{(-\xi)_{l-j}(-m)_l}{\Gamma(l-j+1)\Gamma(l+1)} t^l (1-t)^{m-l} r^{\xi+j-l} s^{l-j} v_m a_{m\ell} \right|^p \right\} < \infty,$$

and

$$\|A\|_{E_V^t B_{r,s}^{(\xi)},p}^p = \overline{M}_p.$$

Proof. It follows from the similar arguments in Theorem 4. $\square$

THEOREM 6. The space $\mathfrak{Y}(E_V^t B_{r,s}^{(\xi)})$ is linearly isometric to the space $\mathfrak{Y}$ for $\mathfrak{Y} \in \{\ell_{\infty}, c, c_0\}$.

Proof. The proof of the theorem is provided only for $\ell_{\infty}(E_V^t B_{r,s}^{(\xi)})$ and those of other are being verified similarly. In this context, we look into a linear bijective map from $\ell_{\infty}(E_V^t B_{r,s}^{(\xi)})$ to ℓ_{∞} .

Consider the mapping $T : \ell_{\infty}(E_V^t B_{r,s}^{(\xi)}) \rightarrow \ell_{\infty}$, defined by $T(x) = (E_V^t B_{r,s}^{(\xi)})x$. Then it implies the mapping T to be injective, since, T is linear and $T(\theta) = 0$, where $\theta = (0, 0, 0, \dots)$.

Now, for the second part, we define a sequence $x = (x_{\ell})$ via the sequence $y = (y_{\ell}) \in \ell_{\infty}$ as

$$x_{\ell} = \sum_{j=0}^{\ell} \sum_{l=j}^{\ell} (-1)^{\ell-j} \frac{(\xi)_{\ell-l}(-l)_j t^l (t-1)^{l-j}}{\Gamma(\ell-l+1)\Gamma(j+1)} v_{\ell} r^{l-\xi-\ell} s^{\ell-l} y_j. \quad (5)$$

Then, we observe that

$$\begin{aligned} \sup_{\mathfrak{k}} [(E_{\mathfrak{v}}^t B_{r,s}^{(\alpha)})x]_{\mathfrak{k}} &= \sup_{\mathfrak{k}} \left[\sum_{j=0}^{\mathfrak{k}} \sum_{l=j}^{\mathfrak{k}} (-1)^j \frac{(-\xi)_{l-j}(-\mathfrak{k})_l}{\Gamma(l-j+1)\Gamma(l+1)} t^l (1-t)^{\mathfrak{k}-l} r^{\xi+j-l} s^{l-j} v_j x_j \right] \\ &= \sup_{\mathfrak{k}} [y_{\mathfrak{k}}]. \end{aligned}$$

As a result, we have $x \in \ell_{\infty}(E_{\mathfrak{v}}^t B_{r,s}^{(\xi)})$. Therefore, we conclude that T is surjective and preserve the norm. $\square$

Since the spaces c_0 and c have Schauder bases, next, we construct the Schauder bases of the spaces $c_0(E_{\mathfrak{v}}^t B_{r,s}^{(\xi)})$ and $c(E_{\mathfrak{v}}^t B_{r,s}^{(\xi)})$. For this purpose, we define the sequences $\lambda = (\lambda_n)$, as

$$\lambda_n = \sum_{j=0}^n \sum_{i=j}^n (-1)^j \frac{(-\xi)_{l-j}(-n)_l}{\Gamma(l-j+1)\Gamma(l+1)} t^l (1-t)^{n-l} r^{\xi+j-l} s^{l-j} v_j x_j, \quad (n \in \mathbb{N}_0),$$

such that $\lim_{n \rightarrow \infty} \lambda_n = \eta^*$ and the sequence $\beta^{(\mathfrak{k})} = \{\beta_n^{(\mathfrak{k})}\}_{n \in \mathbb{N}_0}$ for fixed $\mathfrak{k} \in \mathbb{N}_0$ as

$$\beta_m^{(\mathfrak{k})} = \begin{cases} \frac{1}{v_m} \left[\sum_{l=\mathfrak{k}}^m (-1)^{m-\mathfrak{k}} \frac{(\xi)_{m-l}(-l)_{\mathfrak{k}} t^l (t-1)^{l-\mathfrak{k}}}{\Gamma(m-l+1)\Gamma(\mathfrak{k}+1)} r^{l-\xi-m} s^{m-l} \right], & (\mathfrak{k} \leq m), \\ 0, & (\mathfrak{k} > m), \end{cases}$$

for all $m, \mathfrak{k} \in \mathbb{N}_0$.

Define the sequence $\gamma^* = (\gamma_m^*)$ as

$$\gamma_m^* = \sum_{\mathfrak{k}=0}^m \frac{1}{v_m} \left[\sum_{l=\mathfrak{k}}^m (-1)^{m-\mathfrak{k}} \frac{(\xi)_{m-l}(-i)_{\mathfrak{k}} t^l (t-1)^{l-\mathfrak{k}}}{\Gamma(m-l+1)\Gamma(\mathfrak{k}+1)} r^{l-\xi-m} s^{m-l} \right].$$

Then we have the following estimations:

THEOREM 7.

- (i) $(\beta^{(l)})_{l \in \mathbb{N}_0}$ forms a basis for $c_0(E_{\mathfrak{v}}^t B_{r,s}^{(\xi)})$ and for each $y \in c_0(E_{\mathfrak{v}}^t B_{r,s}^{(\xi)})$ has a unique representation as given below

$$y = \sum_i \lambda_i \beta^{(i)}.$$

- (ii) $\{\gamma^*, \beta^{(l)}\}$ forms basis for the space $c(E_{\mathfrak{v}}^t B_{r,s}^{(\xi)})$ and any $x \in c(E_{\mathfrak{v}}^t B_{r,s}^{(\xi)})$ is associated with a unique representation below

$$x = \eta^* \gamma^* + \sum_l (\lambda_l - \eta^*) \beta^{(l)}.$$

Proof. The proof follows from [30]. $\square$

Now, we determine the α - and β -duals for $\mathfrak{Y}(E_V^t B_{r,s}^{(\xi)})$ for $\mathfrak{Y} \in \{\ell_\infty, c, c_0\}$. For this we quote the following lemmas:

THEOREM 8. *Consider the following notations.*

$$c_{nk} = \begin{cases} \sum_{l=\mathfrak{k}}^m (-1)^{m-\mathfrak{k}} \frac{(\xi)_{m-l} (-l)_{\mathfrak{k}} t^l (t-1)^{l-\mathfrak{k}}}{\Gamma(m-l+1) \Gamma(\mathfrak{k}+1) v_m} r^{l-\xi-m} s^{m-l} a_m, & (0 \leq \mathfrak{k} \leq n), \\ 0, & (\mathfrak{k} > m), \end{cases}$$

$$R_{\mathfrak{k}}(a) = \sum_{l=0}^{\infty} \sum_{j=0}^l a_{k+l} (-1)^l \frac{(\xi)_{l-j} (-\mathfrak{k}-j)_{\mathfrak{k}} t^{\mathfrak{k}+j} (t-1)^j}{\Gamma(l-j+1) \Gamma(k+1) v_{\mathfrak{k}+l}} r^{j-l-\xi} s^{l-j},$$

$$w_{m\mathfrak{k}}(a) = \sum_{l=0}^{\infty} \sum_{j=0}^{l+m-\mathfrak{k}} a_{m+l} (-1)^{l+m-\mathfrak{k}} \frac{(\xi)_{l+m-j-\mathfrak{k}} (-\mathfrak{k}-j)_{\mathfrak{k}} t^{\mathfrak{k}+j} (t-1)^j}{\Gamma(m+l-j-\mathfrak{k}+1) \Gamma(k+1) v_{m+l}} r^{j+\mathfrak{k}-l-\xi-m} s^{l+m-j-k},$$

and define the sets $S_1(a)$, $S_2(a)$, $S_3(a)$, $S_4(a)$, $S_5(a)$, $S_6(a)$ and $S_7(a)$ is as follows:

$$\begin{aligned} S_1(a) &= \left\{ a \in w : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{\mathfrak{k} \in K} c_{n\mathfrak{k}} \right| < \infty \right\}, \\ S_2(a) &= \left\{ a \in w : \sum_{\mathfrak{k}=0}^{\infty} |R_{\mathfrak{k}}(a)| < \infty \right\}, \\ S_3(a) &= \left\{ a \in w : \lim_{m \rightarrow \infty} \left(\sum_{\mathfrak{k}=0}^{\infty} |w_{m\mathfrak{k}}(a)| \right) = 0 \right\}, \\ S_4(a) &= \left\{ a \in w : \lim_{m \rightarrow \infty} w_{m\mathfrak{k}}(a) = 0 \text{ for all } \mathfrak{k} \in \mathbb{N}_0 \right\}, \\ S_5(a) &= \left\{ a \in w : \sup_m \left(\sum_{\mathfrak{k}=0}^{\infty} |w_{m\mathfrak{k}}(a)| \right) < \infty \right\}, \\ S_6(a) &= \left\{ a \in w : \lim_{m \rightarrow \infty} w_{m\mathfrak{k}}(a) = \eta \text{ for all } \mathfrak{k} \in \mathbb{N}_0 \right\}, \\ S_7(a) &= \left\{ a \in w : \lim_{m \rightarrow \infty} \sum_{\mathfrak{k}=0}^{\infty} |w_{m\mathfrak{k}}(a)| = \gamma \right\}. \end{aligned}$$

Then

- (i) $[\ell_\infty(E_V^t B_{r,s}^{(\xi)})]^\alpha = [c_0(E_V^t B_{r,s}^{(\xi)})]^\alpha = [c(E_V^t B_{r,s}^{(\xi)})]^\alpha = S_1(a)$,
- (ii) $[\ell_\infty(E_V^t B_{r,s}^{(\xi)})]^\beta = S_2(a) \cap S_3(a)$,
- (iii) $[c_0(E_V^t B_{r,s}^{(\xi)})]^\beta = S_2(a) \cap S_4(a) \cap S_5(a)$,
- (iv) $[c(E_V^t B_{r,s}^{(\xi)})]^\beta = S_2(a) \cap S_5(a) \cap S_6(a) \cap S_7(a)$.

Proof. (i) Let $a = (a_m) \in w$ and $x = (x_m) \in \ell_\infty(E_v^t B_{r,s}^{(\xi)})$. Then

$$\begin{aligned} a_n x_n &= \sum_{j=0}^m \sum_{l=j}^m (-1)^{m-j} \frac{(\xi)_{n-l} (-l)_j t^l (t-1)^{l-j}}{\Gamma(m-l+1) \Gamma(j+1) v_m} r^{l-\xi-n} s^{m-l} a_m \\ &= \sum_{\ell=0}^m \sum_{l=\ell}^m (-1)^{m-\ell} \frac{(\xi)_{m-l} (-l)_\ell t^l (t-1)^{l-k}}{\Gamma(n-l+1) \Gamma(\ell+1) v_m} r^{l-\xi-m} s^{m-l} a_m y_k \\ &= (Cy)_m, \end{aligned}$$

while the matrix $C = (c_{m\ell})$ and

$$c_{n\ell} = \begin{cases} \sum_{l=\ell}^m (-1)^{m-\ell} \frac{(\xi)_{m-l} (-l)_\ell t^l (t-1)^{l-\ell}}{\Gamma(m-l+1) \Gamma(k+1) v_m} r^{l-\xi-m} s^{m-l} a_m, & (0 \leq \ell \leq m), \\ 0, & (\ell > m). \end{cases}$$

Now by a consequence of [23], we complete the proof of (i).

(ii) Suppose $S = (s_{m\ell})$ denotes the inverse of the matrix $E_v^t B_{r,s}^{(\xi)}$ and $R = S^T$, the transpose of the matrix S . By Theorem 3.2 of [31], we have

$$\begin{aligned} R_\ell(a) &= \sum_{j=\ell}^{\infty} a_j s_{j\ell} \\ &= \sum_{j=\ell}^{\infty} a_j \sum_{i=\ell}^j (-1)^{j-i} \frac{(\xi)_{j-l} (-l)_\ell t^l (t-1)^{l-k}}{\Gamma(j-l+1) \Gamma(k+1) v_j} r^{l-\xi-j} s^{j-l} \\ &= \frac{a_\ell}{r^\ell v_\ell} + \sum_{j=\ell+1}^{\infty} a_j \sum_{l=\ell}^j (-1)^{j-i} \frac{(\xi)_{j-i} (-l)_\ell t^l (t-1)^{l-\ell}}{\Gamma(j-l+1) \Gamma(\ell+1) v_j} r^{l-\xi-j} s^{j-l} \\ &= \frac{a_\ell}{r^\ell v_\ell} - a_{\ell+1} \sum_{l=\ell}^{k+1} \frac{(\xi)_{\ell+1-l} (-l)_k t^l (t-1)^{l-\ell}}{\Gamma(\ell-l+2) \Gamma(\ell+1) v_{\ell+1}} r^{l-\xi-\ell-1} s^{\ell+1-l} \\ &\quad + a_{\ell+2} \sum_{l=\ell}^{\ell+2} \frac{(\xi)_{\ell+2-l} (-l)_\ell t^l (t-1)^{l-\ell}}{\Gamma(\ell-l+3) \Gamma(\ell+1) v_{\ell+2}} r^{l-\xi-\ell-2} s^{\ell+2-l} - \dots \\ &= \sum_{l=0}^{\infty} \sum_{j=0}^l a_{\ell+l} (-1)^l \frac{(\xi)_{l-j} (-\ell-j)_\ell t^{\ell+j} (t-1)^j}{\Gamma(l-j+1) \Gamma(k+1) v_{k+l}} r^{j-l-\xi} s^{l-j}. \end{aligned}$$

Now, we have

$$\begin{aligned} w_{mk}(a) &= \sum_{j=m}^{\infty} a_j s_{jk} \\ &= \sum_{j=m}^{\infty} \sum_{l=k}^j \frac{(-1)^{j-k} (\xi)_{j-l} (-l)_k t^l (t-1)^{l-k}}{\Gamma(j-l+1) \Gamma(k+1) v_j} r^{l-\xi-j} s^{j-l} a_j \end{aligned}$$

$$\begin{aligned}
&= a_m \sum_{l=k}^m \frac{(-1)^{m-k} (\xi)_{m-l} (-l)_k t^l (t-1)^{l-k}}{\Gamma(m-l+1) \Gamma(k+1) v_m} r^{l-\xi-m} s^{m-l} \\
&\quad + a_{m+1} \sum_{l=k}^{m+1} \frac{(-1)^{m+1-k} (\xi)_{m+1-l} (-l)_k t^l (t-1)^{l-k}}{\Gamma(m-l+2) \Gamma(k+1) v_{m+1}} r^{l-\xi-m-1} s^{m+1-l} + \dots \\
&= \sum_{l=0}^{\infty} \sum_{j=0}^{l+m-k} a_{m+l} (-1)^{l+m-k} \frac{(\xi)_{l+m-j-k} (-k-j)_k t^{k+j} (t-1)^j}{\Gamma(m+l-j-k+1) \Gamma(k+1) v_{m+l}} r^{j+k-l-\xi-m} s^{l+m-j-k}.
\end{aligned}$$

For β -duals for the developed spaces, we usually prove that $R(a) = (R_k(a)) \in \ell_1$ and use the conditions for matrix characterizations of the classes $W \in (\ell_\infty, c_0)$, $W \in (c_0, c_0)$ and $W \in (c, c)$, where $W = (w_{mk}(a))$. Using Theorem 3.2 of [31] and [23], we conclude that $[\ell_\infty(E_V^t B_{r,s}^{(\xi)})]^\beta = S_1(a) \cap S_2(a)$, $[c_0(E_V^t B_{r,s}^{(\xi)})]^\beta = S_1(a) \cap S_3(a) \cap S_4(a)$ and $[c(E_V^t B_{r,s}^{(\xi)})]^\beta = S_1(a) \cap S_4(a) \cap S_5(a) \cap S_6(a)$. $\square$

Suppose $\mathcal{F}$ stand for the all nonempty and finite subsets of $\mathbb{N}_0$ and ξ is a positive proper fraction, and $\bar{A} = \bar{a}_{nk}$ be the associated matrix of $A = (a_{nk})$, defined as follows

$$\bar{a}_{nk} = \sum_{j=k}^{\infty} \sum_{l=k}^j (-1)^{j-k} \frac{(\xi)_{j-l} (-l)_k t^l (t-1)^{l-k}}{\Gamma(j-l+1) \Gamma(k+1) v_j} r^{l-\xi-j} s^{j-l} a_{nj}, \quad (n, k \in \mathbb{N}_0). \quad (6)$$

Then, we list the following conditions on the matrix characterization from the defined spaces to the basic sequence spaces such as ℓ_∞, c, c_0 and ℓ_1 .

$$\sup_{n \in \mathbb{N}} \left(\sum_k |\bar{a}_{nk}| \right) < \infty, \quad (7)$$

$$\lim_{n \rightarrow \infty} \left(\sum_k |\bar{a}_{nk} - \xi'_k| \right) = 0, \quad (\exists \xi'_k \in \mathbb{R}) \quad (8)$$

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} \bar{a}_{nk} \right| < \infty, \quad (9)$$

$$\sup_{n \in \mathbb{N}} \left(\sum_k |\bar{a}_{nk} - \alpha'_k| \right) < \infty, \quad (\exists \alpha'_k \in \mathbb{R}) \quad (10)$$

$$\sup_{n \in \mathbb{N}} \left| \sum_k \bar{a}_{nk} \right| < \infty, \quad (11)$$

$$\lim_{n \rightarrow \infty} |\bar{a}_{nk} - \beta'_k| = 0, \quad (\exists \beta'_k \in \mathbb{R}) \quad (12)$$

$$\sum_n \left| \sum_k \bar{a}_{nk} \right| < \infty, \quad (13)$$

$$\sup_{n,k} |\bar{a}_{nk}| < \infty, \quad (14)$$

$$\sup_{n,k} |\bar{a}_{nk} - \beta'_k| < \infty, \quad (\exists \beta'_k \in \mathbb{R}) \quad (15)$$

$$\lim_{n \rightarrow \infty} \left| \sum_k \bar{a}_{nk} \right| = 0. \quad (16)$$

THEOREM 9. *Using conditions (7)–(16), we have the following results:*

X	Y	Necessary and sufficient condition(s) for $A \in (X(E_V^t B_{r,s}^{(\xi)}), Y)$
ℓ_∞	ℓ_∞	(7)
ℓ_∞	ℓ_1	(9)
ℓ_∞	c	(7) and (8)
ℓ_∞	c_0	(8) with $\xi'_k = 0$ for every $k \in \mathbb{N}_0$
c	ℓ_∞	(7) and (11)
c	ℓ_1	(9) and (13)
c	c	(7), (8), (10) and (12)
c	c_0	(7), (16) and (12) with $\beta'_k = 0$ for every $k \in \mathbb{N}_0$
c_0	ℓ_∞	(7)
c_0	ℓ_1	(9)
c_0	c	(7), (10) and (12)
c_0	c_0	(7) and (12) with $\beta'_k = 0$ for every $k \in \mathbb{N}_0$
ℓ_1	ℓ_∞	(7) and (14)
ℓ_1	ℓ_1	(7)
ℓ_1	c	(7), (12), (14) and (15)
ℓ_1	c_0	(7), (14) and (12) with $\beta'_k = 0$ for every $k \in \mathbb{N}_0$

Table 1: Matrix characterizations on the space $X(E_V^t B_{r,s}^{(\xi)})$ for $X = \ell_\infty, c, c_0$ and ℓ_1 .

Proof. The proof is followed by Theorem 5.1 of [23]. $\square$

2.2. Results on related compact operators

The theory of noncompactness of a set was developed by Kuratowski [29] by defining the measure of non-compactness of a function and Darbo [18] used the idea in fixed point theory. Later on, the idea of Hausdorff measures of noncompactness (HMNC) was initiated by Goldenstein et al. [22]. This was extensively used in differential equations, optimizations, fixed point theory, etc (see eg., [1, 7, 11, 12, 21]).

DEFINITION 1. Suppose $\mathcal{M}_{\mathfrak{Y}}$ is the collection of all subsets of a metric space $(\mathfrak{Y}, d)$, being bounded, and $B, B_1, B_2 \in \mathcal{M}_{\mathfrak{Y}}$ and $\mathfrak{Y}$ is complete. Then the mapping $\psi : \mathcal{M}_{\mathfrak{Y}} \rightarrow [0, \infty)$ is said to be a measure of noncompactness (MNC) on $\mathfrak{Y}$ if it satisfies

- (a) $\psi(B) = 0$ if and only if B is precompact,
- (b) $\psi(B) = \phi(\overline{B})$,
- (c) $\psi(B_1 \cap B_2) = \max\{\psi(B_1), \psi(B_2)\}$.

Suppose that $M_1, M_2 \subseteq (\mathfrak{Y}, d)$ and $\varepsilon > 0$. By ε -net M_1 of M_2 in $\mathfrak{Y}$, we mean for every $x \in M_2$ there exists $y \in M_1$ such that $d(x, y) < \varepsilon$. Moreover, the function $\mathcal{X} : \mathcal{M}_{\mathfrak{Y}} \rightarrow [0, \infty)$ defined by

$$\mathcal{X}(\tilde{Q}) = \inf\{\varepsilon > 0 : \tilde{Q} \text{ has a finite } \varepsilon\text{-net in } \mathfrak{Y}\}$$

is called the HMNC.

DEFINITION 2. Let ψ_1 and ψ_2 be MNC on the Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$. An operator $L : \mathfrak{X} \rightarrow \mathfrak{Y}$ is said to be (ψ_1, ψ_2) -bounded if

$$L(B) \in \mathcal{M}_{\mathfrak{Y}} \text{ for all } B \in \mathcal{M}_{\mathfrak{X}},$$

and there exists a nonnegative real number c such that

$$\psi_2(L(B)) \leq c\psi_1(B) \text{ for all } B \in \mathcal{M}_{\mathfrak{X}}.$$

If an operator L is (ψ_1, ψ_2) -bounded, then the number

$$\|L\|_{(\psi_1, \psi_2)} = \inf\{c > 0 : \psi_2(L(B)) \leq c\psi_1(B) \text{ for all } B \in \mathcal{M}_{\mathfrak{X}}\}$$

is called the (ψ_1, ψ_2) -MNC of L .

Let $\mathcal{B}(\mathfrak{Y}, \tilde{Y})$ represent the set of all linear from $\mathfrak{Y}$ to $\tilde{Y}$, being bounded. Then the HMNC of $L \in B(\mathfrak{Y}, \tilde{Y})$, denoted by $\|L\|_{\mathcal{X}}$, is given by

$$\|L\|_{\mathcal{X}} = \mathcal{X}(L(S_{\mathfrak{Y}})) \quad (17)$$

and also,

$$L \text{ is compact if and only if } \|L\|_{\mathcal{X}} = 0. \quad (18)$$

Now, we may quote some useful lemmas. follows:

LEMMA 1. ([43]) Let $B \in \mathcal{M}_{\ell_p}$, $1 \leq p < \infty$. If $P_t : \ell_p \rightarrow \ell_p$ ($t \in \mathbb{N}_0$) is the operator defined by $P_t(y) = (y_0, y_1, \dots, y_t, 0, 0, \dots)$ for all $y \in \ell_p$, then we have

$$\mathcal{X}(B) = \lim_{t \rightarrow \infty} \left(\sup_{y \in B} \|(I - P_t)(y)\|_{\ell_p} \right).$$

LEMMA 2. ([26, Lemma 6.3]) Let $\mathfrak{Y} \supset \psi$ and $\mathfrak{Y}, \tilde{Y}$ be BK-spaces. Then,

- (a) $(\mathfrak{Y}, \tilde{Y}) \subset \mathcal{B}(\mathfrak{Y}, \tilde{Y})$,

(b) If $\mathfrak{Y}$ has AK, then $\mathcal{B}(\mathfrak{Y}, \tilde{Y}) \subset (X, Y)$.

LEMMA 3. ([45, Theorem 7.2.9]) Let $X \subset \{c, c_0, \ell_\infty, \ell_p \ (1 \leq p < \infty)\}$. Then, $\|a\|_X^* = \|\bar{a}\|_{X^\beta}^*$, where $\sum_k a_k x_k = \sum_k \bar{a}_k y_k$ for all $\bar{a} \in X^\beta$ and $a \in [X(E_V^t B_{r,s}^{(\xi)})]^\beta$.

LEMMA 4. ([36, Theorem 1.6]) Suppose that $\mathfrak{Y}$ is a BK-space with Schauder basis $(b_n)_{n=0}^\infty$, $B \in \mathcal{M}_{\mathfrak{Y}}$ and $P_t : \mathfrak{Y} \rightarrow \mathfrak{Y} \ (t \in \mathbb{N}_0)$ be the projector onto the linear span of $\{b_0, b_1, \dots, b_t\}$. Then we have

$$\frac{1}{\tilde{a}} \cdot \limsup_{t \rightarrow \infty} \left(\sup_{y \in B} \|(I - P_t)(y)\| \right) \leq \mathcal{X}(B) \leq \limsup_{t \rightarrow \infty} \left(\sup_{y \in B} \|(I - P_t)(y)\| \right),$$

where $\tilde{a} = \limsup_{t \rightarrow \infty} \|(I - P_t)\|$ and I denotes the identity operator on $\mathfrak{Y}$.

Throughout the further sections, let us choose $\mathcal{F}_t \ (t \in \mathbb{N}_0)$ is noted for the sub-collection of $\mathcal{F}$ defined and following manner,

$$\mathcal{F}_t = \{N \in \mathcal{F} : m > t \text{ for all } m \in N\}, \ (t \in \mathbb{N}_0)$$

and

$$\bar{A}_n(x) = \sum_{k=0}^\infty \sum_{j=k}^\infty \sum_{l=k}^j (-1)^{j-k} \frac{(\xi)_{j-l} (-l)_k t^l (t-1)^{l-k}}{\Gamma(j-l+1) \Gamma(k+1) v_j} r^{j-\xi-j} s^{j-l} a_{nj} x_k, \ (n \in \mathbb{N}_0). \quad (19)$$

THEOREM 10. Let X be a BK space with AK. Then, the following assertions are true:

(i) If $A \in (\mathfrak{Y}(E_V^t B_{r,s}^{(\xi)}), \ell_\infty)$, then, L_A is compact if $\lim_{n \rightarrow \infty} \|\bar{A}_n\|_{\mathfrak{Y}}^* = 0$ and in fact,

$$0 \leq \|L_A\|_{\mathcal{X}} \leq \limsup_{n \rightarrow \infty} \|\bar{A}_n\|_{\mathfrak{Y}}^*.$$

(ii) If $A \in (\mathfrak{Y}(E_V^t B_{r,s}^{(\xi)}), c_0)$, then L_A is compact if and only if $\lim_{n \rightarrow \infty} \|\bar{A}_n\|_{\mathfrak{Y}}^* = 0$, and in fact,

$$\|L_A\|_{\mathcal{X}} = \limsup_{n \rightarrow \infty} \|\bar{A}_n\|_{\mathfrak{Y}}^*.$$

(iii) If $A \in (\mathfrak{Y}(E_V^t B_{r,s}^{(\xi)}), c)$, then L_A is compact if and only if $\lim_{n \rightarrow \infty} \|\bar{A}_n - \bar{\xi}\|_{\mathfrak{Y}}^* = 0$, and in fact,

$$\frac{1}{2} \cdot \limsup_{n \rightarrow \infty} \|\bar{A}_n - \bar{\alpha}\|_X^* \leq \|L_A\|_{\mathcal{X}} \leq \limsup_{n \rightarrow \infty} \|\bar{A}_n - \bar{\alpha}\|_X^*, \quad (20)$$

where $\bar{\alpha} = (\bar{\alpha}_k)$ and $\bar{\alpha}_k = \lim_{n \rightarrow \infty} \bar{a}_{nk}$ for all $k \in \mathbb{N}_0$.

Proof. For proof of (i) and (ii), one may use Eq. (18) in Lemmas 2.8 of [9] and 3, i.e.,

$$\|A_n\|_X^* = \|\bar{A}_n\|_{X\beta}^*.$$

Whereas, to prove part (iii), we use Lemma 2.8 of [9], which yields that

$$\frac{1}{2} \cdot \limsup_{n \rightarrow \infty} \|\bar{A}_n - \bar{\alpha}\|_X^* \leq \|L_{\bar{A}}\|_{\mathcal{X}} \leq \limsup_{n \rightarrow \infty} \|\bar{A}_n - \bar{\alpha}\|_X^*. \quad (21)$$

Note that $\bar{\alpha} = (\bar{\alpha}_k)$ with $\bar{\alpha}_k = \lim_{n \rightarrow \infty} \bar{a}_{nk}$ for all $k \in \mathbb{N}_0$.

Let $S = S_X$ and $\tilde{S} = S_{X(E_V^t B_{r,s}^{(\xi)})}$. Now, consider the inclusion (17) in (i) part of Lemma 2, then one obtain that

$$\|L_A\|_{\mathcal{X}} = \mathcal{X}(L_A(\tilde{S})) = \mathcal{X}(\bar{A}\tilde{S}) \quad (22)$$

and

$$\|L_{\bar{A}}\|_{\mathcal{X}} = \mathcal{X}(L_{\bar{A}}(S)) = \mathcal{X}(\bar{A}S). \quad (23)$$

Combining (22) and (23), we have $\tilde{A}S = \bar{A}S$, as a consequence $\|L_A\|_{\mathcal{X}} = \|L_{\bar{A}}\|_{\mathcal{X}}$. Therefore, we derive (20) from (21). $\square$

COROLLARY 1. *If either $A \in (\ell_{\infty}(E_V^t B_{r,s}^{(\xi)}), c_0)$ or $A \in (\ell_{\infty}(E_V^t B_{r,s}^{(\xi)}), c)$, then L_A is compact.*

Proof. Consider $A \in (\ell_{\infty}(E_V^t B_{r,s}^{(\xi)}), c_0)$, then, it is derived that $\bar{A} \in (\ell_{\infty}, c_0)$ i.e. $\lim_n (\sum_k |\bar{a}_{nk}|) = 0$. Thus, by using Lemma 3 we have $\lim_n \|\bar{A}_n\|_{\ell_{\infty}}^* = 0$. Therefore, the operator L_A is compact which follows from Theorem 10(b) and similarly we can show that $A \in (\ell_{\infty}(E_V^t B_{r,s}^{(\xi)}), c)$ and hence L_A is compact. $\square$

THEOREM 11. *Let $\mathfrak{Y}$ be a BK space with AK and $A \in (\mathfrak{Y}(E_V^t B_{r,s}^{(\xi)}), \ell_1)$. Then if L_A is compact if and only if*

$$\lim_{r \rightarrow \infty} \left(\sup_{N \in \mathcal{F}_r} \left\| \sum_{n \in N} \bar{A}_n \right\|_X^* \right) = 0$$

and

$$\begin{aligned} \lim_{r \rightarrow \infty} \left(\sup_{N \in \mathcal{F}_r} \left\| \sum_{n \in N} \bar{A}_n \right\|_X^* \right) &\leq \|L_A\|_{\mathcal{X}} \\ &\leq 4 \cdot \lim_{r \rightarrow \infty} \left(\sup_{N \in \mathcal{F}_r} \left\| \sum_{n \in N} \bar{A}_n \right\|_X^* \right). \end{aligned}$$

Proof. To prove this we may use (18) and (20) with Lemma 2.9 of [9]. $\square$

LEMMA 5. (a) Let $1 \leq p < \infty$ and $A \in (\ell_1(E_V^t B_{r,s}^{(\xi)}), \ell_p)$. Then

$$\|L_A\| = \sup_k \left(\sum_{n=0}^{\infty} |\bar{a}_k|^p \right)^{1/p} < \infty.$$

(b) Let $q = p/(p-1)$ for $1 < p < \infty$ and $A \in (\ell_p(E_V^t B_{r,s}^{(\xi)}), \ell_1)$. Then

$$\sup_{N \in \mathcal{F}_r} \left(\sum_k \left| \sum_{n \in N} \bar{a}_{nk} \right|^q \right)^{1/q} \leq \|L_A\| \leq 4 \cdot \sup_{N \in \mathcal{F}_r} \left(\sum_k \left| \sum_{n \in N} \bar{a}_{nk} \right|^q \right)^{1/q},$$

where $\sup_{N \in \mathcal{F}_r} (\sum_k |\sum_{n \in N} \bar{a}_{nk}|^q)^{1/q} < \infty$.

THEOREM 12. Let $1 \leq p < \infty$ and $A \in (\ell_1(E_V^t B_{r,s}^{(\xi)}), \ell_p)$. Then L_A is compact if and only if

$$\lim_{r \rightarrow \infty} \left(\sup_k \left(\sum_{n=r}^{\infty} |\bar{a}_{nk}|^p \right)^{1/p} \right) = 0$$

and

$$\|L_A\|_{\mathcal{X}} = \lim_{r \rightarrow \infty} \left(\sup_k \left(\sum_{n=r}^{\infty} |\bar{a}_{nk}|^p \right)^{1/p} \right).$$

Proof. Suppose that $B = S_{\ell_1(E_V^t B_{r,s}^{(\xi)})}$. Then, from Lemma 2, it is obtained that $L_A(B) = AB \in \ell_p$. From Lemma 1 and (17) we derive that

$$\|L_A\|_{\mathcal{X}} = \mathcal{X}(AB) = \lim_{r \rightarrow \infty} \left(\sup_{x \in B} \|(I - P_r)(Ax)\|_{\ell_p} \right). \quad (24)$$

Using (6), consider $y = (y_k) \in \ell_1$ as the associated sequence for all $x \in \ell_1(E_V^t B_{r,s}^{(\xi)})$. Since $A \in (\ell_1(E_V^t B_{r,s}^{(\xi)}), \ell_p)$, then $\bar{A} \in (\ell_1, \ell_p)$ i.e. $Ax = \bar{A}y$. As a result, one may obtain that

$$\begin{aligned} \|(I - P_r)(Ax)\|_{\ell_p} &= \|(I - P_r)(\bar{A}y)\|_{\ell_p} \\ &= \left(\sum_{n=r}^{\infty} |\bar{A}_n(y)|^p \right)^{\frac{1}{p}} \\ &\leq \sum_k \left(\sum_{n=r}^{\infty} |\bar{a}_{nk} y_k|^p \right)^{\frac{1}{p}} \\ &\leq \sup_k \left(\sum_{n=r}^{\infty} |\bar{a}_{nk}|^p \right)^{\frac{1}{p}} \left(\sum_k |y_k|^p \right)^{\frac{1}{p}} \\ &= \sup_k \left(\sum_{n=r}^{\infty} |\bar{a}_{nk}|^p \right)^{\frac{1}{p}} \|x\|_{\ell_1(E_V^t B_{r,s}^{(\xi)})}. \end{aligned}$$

Then, for every $r \in \mathbb{N}$, we also derive that,

$$\sup_{x \in iB} \|(I - P_r)(Ax)\|_{\ell_p} \leq \sup_k \left(\sum_{n=r}^{\infty} |\bar{a}_{nk}|^p \right)^{\frac{1}{p}}.$$

Now, using (24), we have

$$\|L_A\|_{\mathcal{X}} \leq \lim_{r \rightarrow \infty} \left(\sup_k \left(\sum_{n=r}^{\infty} |\bar{a}_{nk}|^p \right)^{\frac{1}{p}} \right).$$

For converse inequality, consider $e^{(l)} \in \ell_1(E_V^t B_{r,s}^{(\xi)})$ i.e. $E_V^t B_{r,s}^{(\xi)}(e^{(l)}) = \bar{e}^{(l)}$, the associated sequence of $e^{(l)}$ for each $l \in \mathbb{N}$, then we have $Ae^{(l)} = \bar{A}\bar{e}^{(l)}$. As a consequence $\mathcal{X}(AB_e) \leq \mathcal{X}(AB)$ since $AB_e \subset AB$ for $B_e = \{e^{(l)} : l \in \mathbb{N}\}$. With the help of Lemma 1, we conclude that

$$\begin{aligned} \mathcal{X}(AB_e) &= \lim_{r \rightarrow \infty} \left(\sup_i \left(\sum_{n=r}^{\infty} |A_n(e^{(l)})|^p \right)^{\frac{1}{p}} \right) \\ &= \lim_{r \rightarrow \infty} \left(\sup_l \left(\sum_{n=r}^{\infty} |\bar{a}_{nl}|^p \right)^{\frac{1}{p}} \right). \end{aligned}$$

Hence,

$$\lim_{r \rightarrow \infty} \left(\sup_l \left(\sum_{n=r}^{\infty} |\bar{a}_{nl}|^p \right)^{\frac{1}{p}} \right) \leq \|L_A\|_{\mathcal{X}}. \quad \square$$

LEMMA 6. ([36, Lemma 3.5]) *If $x \in \ell_1$, then for every $r \in \mathbb{N}_0$ the following inequality holds*

$$\begin{aligned} \sup_{N \in \mathcal{F}_r} \left| \sum_{n \in N} x_n \right| &\leq \sum_{n=r}^{\infty} |x_n| \\ &\leq 4 \cdot \sup_{N \in \mathcal{F}_r} \left| \sum_{n \in N} x_n \right|. \end{aligned}$$

LEMMA 7. ([36, Lemma 2.2]) *For $1 \leq p < \infty$, we have*

$$\|a\|_{\ell_p(E_V^t B_{r,s}^{(\xi)})}^* = \|\bar{a}\|_{\ell_q} = \begin{cases} \sup_k |\bar{a}_k|, & (p = 1) \\ \left(\sum_{k=0}^{\infty} |\bar{a}_k|^q \right)^{1/q}, & (1 < p < \infty) \end{cases} \quad (25)$$

for all $a = (a_k) \in (\ell_p(E_V^t B_{r,s}^{(\xi)}))^{\beta}$.

THEOREM 13. Let $1 < p < \infty$, $q = p/(p-1)$. If $A \in (\ell_p(E_v^t B_{r,s}^{(\xi)}), \ell_1)$, then L_A is compact if and only if

$$\lim_{r \rightarrow \infty} \left(\sup_{N \in \mathcal{F}_r} \left(\sum_k \left| \sum_{n \in N} \bar{a}_{nk} \right|^q \right)^{1/q} \right) = 0$$

and

$$\begin{aligned} \lim_{r \rightarrow \infty} \left[\sup_{N \in \mathcal{F}_r} \left(\sum_k \left| \sum_{n \in N} \bar{a}_{nk} \right|^q \right)^{1/q} \right] &\leq \|L_A\|_{\mathcal{X}} \\ &\leq 4 \cdot \lim_{r \rightarrow \infty} \left[\sup_{N \in \mathcal{F}_r} \left(\sum_k \left| \sum_{n \in N} \bar{a}_{nk} \right|^q \right)^{1/q} \right]. \end{aligned}$$

Proof. It follows the similar lines as of Theorem 12 using Lemmas 5, 6 and 7. $\square$

3. Korovkin approximation theorem

The present investigation involves the application aspects of the proposed generalized binomial Euler difference operator in the directions of approximation theories. However, approximation theories significantly glorify several achieved domains of mathematics, such as sequence spaces and summabilities, NLP, machine learning and artificial neural networks, fuzzy logic, and many more. For more clarification, we may refer the recent culprit of several research manuscripts, such as (see [25], [44], [28], etc.).

Moreover, this study encompasses the demonstration of the Korovkin-type approximation theorem along with the convergence analysis of rate of approximation via the modulus of smoothness of order one and Petree's K -functionals. In contrast, the greatest discovery of Bohman–Korovkin theorem, was initiated and studied by Popu-viciu [41], Bohman [19] and Korovkin [27] respectively, in the years 1951, 1952, and 1953.

THEOREM 14. (Bohmann–Korovkin theorem) Let us assume the sequence of positive linear operator $\{\tilde{T}_m\}_{m' \in \mathbb{N}_0}$ from $C[a_1, a_2]$ to itself, preserving the condition $\lim_{m' \rightarrow \infty} \tilde{T}_{m'} e_i = e_i$ (for $i = 0, 1, 2$). Then,

$$\lim_{m' \rightarrow \infty} \tilde{T}_{m'} f = f,$$

converges uniformly on $[a_1, a_2]$.

Furthermore, various studies regarding the extensions of certain specific positive linear operators such as, Bernstein, Baskov, Szász, Balázs and Gaussian operators via

various polynomials (see [38], [37]), the Stancu variant ([33], [17]), the summability method (see [20], [39]) and some fractional operator (see [15]) have been noticed.

Suppose that the domain $D \in \mathbb{R}^2$ and $C(K)$, represent the space all continuous real-valued bivariate functions on a fixed compact set $[a_1, a_2]$. However, $C(K)$ is endowed with a supremum norm

$$\|f\|_{C(K)} = \max_{(x_1, x_2) \in K} |f(x_1, x_2)|, \text{ for all } f \in C(K).$$

By modulus of continuity of function $f \in C(K)$ and $\delta > 0$, we mean the set defined below

$$w(f, \delta) = \max_{(u_1, u_2), (x_1, x_2) \in K} \left\{ |f(u_1, u_2) - f(x_1, x_2)| : \text{where } \sqrt{(u_1 - x_1)^2 + (u_2 - x_2)^2} < \delta \right\}.$$

In addition, a linear operator $\tilde{\mathcal{J}}$ is said to be positive if $\tilde{\mathcal{J}}g \geq 0$ for all $g \geq 0$. Let us set $(\tilde{\mathcal{J}}g)(x_1, x_2) = \tilde{\mathcal{J}}(g; x_1, x_2)$.

THEOREM 15. Suppose that $a, b \in \mathbb{R}$ and $c \in \mathbb{R}^+$ and $\{\tilde{\mathcal{J}}_n\}$ is a sequence of positive linear operators from $C(K)$ to $C(K)$ and the 2-D dimensional test function $e_{i,j}(u_1, u_2) = u_1^i u_2^j$, for all $i, j = 0, 1, 2$. If the following condition holds, i.e.,

$$\left\| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\mathcal{J}}(e_{0,0}(u_1, u_2); x_1, x_2) \right)_n - e_{0,0}(x_1, x_2) \right\|_{C(K)} = 0,$$

$$\left\| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\mathcal{J}}(e_{1,0}(u_1, u_2); x_1, x_2) \right)_n - e_{1,0}(x_1, x_2) \right\|_{C(K)} = 0,$$

$$\left\| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\mathcal{J}}(e_{0,1}(u_1, u_2); x_1, x_2) \right)_n - e_{0,1}(x_1, x_2) \right\|_{C(K)} = 0,$$

$$\left\| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\mathcal{J}}((e_{2,0} + e_{0,2})(u_1, u_2); x_1, x_2) \right)_n - (e_{2,0} + e_{0,2})(x_1, x_2) \right\|_{C(K)} = 0,$$

then, we can conclude the followings,

$$\left\| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\mathcal{J}}(f(u_1, u_2); x_1, x_2) \right)_n - f(x_1, x_2) \right\|_{C(K)} = 0.$$

Proof. Let us consider $f \in C(k)$. Then, from the definition $|f(u_1, u_2) - f(x_1, x_2)| < \varepsilon$, whenever $|u_1 - x_1| < \delta$, $|u_2 - x_2| < \delta$. On combining these inequalities, we obtain

$$|f(u_1, u_2) - f(x_1, x_2)| < \varepsilon + \frac{2M}{\delta^2} \{(u_1 - x_1)^2 + (u_2 - x_2)^2\},$$

where $M = \|f\|_{C(K)}$. Let $\{\tilde{\mathcal{J}}_n\}$ be the sequence of positive linear operators. Now

$$\begin{aligned} & \left| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\mathcal{J}}(f; x_1, x_2) \right)_n - f(x_1, x_2) \right| \\ &= \left| E_v^t B_{r,s}^{(\xi)} \tilde{\mathcal{J}}(|f(u_1, u_2) - f(x_1, x_2)|; x_1, x_2)_n \right| \\ & \quad + |f(x_1, x_2)| \left| E_v^t B_{r,s}^{(\xi)} \tilde{\mathcal{J}}(|1; x_1, x_2|)_n - 1 \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left| E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}} \left(\varepsilon + \frac{2M}{\delta^2} \{ (u_1 - x_1)^2 + (u_2 - x_2)^2 \}; x_1, x_2 \right)_n \right| \\
&\quad + M \left| E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{0,0}; x_1, x_2)_n - e_{0,0} \right| \\
&\leq \varepsilon \left| E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{0,0}; x_1, x_2)_n \right| \\
&\quad + \frac{2M}{\delta^2} \left\{ (H_1^2 + H_2^2 + H_3^2) \left| E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{0,0}(u, v, w); x_1, x_2)_n - e_{0,0} \right| \right\} \\
&\quad + \frac{4M}{\delta^2} H_1 \left| E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{1,0}(u_1, u_2); x_1, x_2)_n - e_{1,0} \right| \\
&\quad + \frac{4M}{\delta^2} H_2 \left| E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{0,1}(u_1, u_2); x_1, x_2)_n - e_{0,1} \right| \\
&\quad + \frac{2M}{\delta^2} \left| E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{2,0}(u_1, u_2) + e_{0,2}(u_1, u_2); x_1, x_2)_n - \{e_{2,0} + e_{0,2}\} \right| \\
&\quad + M \left| E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{0,0}; x_1, x_2)_n - e_{0,0} \right| + \varepsilon, \\
&\leq \left(\varepsilon + M + \frac{2M}{\delta^2} (H_1^2 + H_2^2) \right) \left| E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{0,0}(u_1, u_2); x_1, x_2)_n - e_{0,0} \right| \\
&\quad + \frac{4M}{\delta^2} H_1 \left| E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{1,0}(u_1, u_2); x_1, x_2)_n - e_{1,0} \right| \\
&\quad + \frac{4M}{\delta^2} H_2 \left| E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{0,1}(u_1, u_2); x_1, x_2)_n - e_{0,1} \right| \\
&\quad + \frac{2M}{\delta^2} \left| E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{2,0}(u_1, u_2) + e_{0,2}(u_1, u_2); x_1, x_2)_n - \{e_{2,0} + e_{0,2}\} \right| + \varepsilon,
\end{aligned}$$

where $H_1 = \max |x_1|$, and $H_2 = \max |x_2|$. Now, applying supremum over both side we obtain,

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \left\| \left(E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(f; x_1, x_2) \right)_n - f(x, y, z) \right\|_{C(K)} \\
&\leq \lim_{n \rightarrow \infty} H^* \left\{ \left\| \left(E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{0,0}(u_1, u_2); x_1, x_2) \right)_n - e_{0,0} \right\|_{C(K)} \right. \\
&\quad + \left\| \left(E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{1,0}(u_1, u_2); x_1, x_2) \right)_n - e_{1,0} \right\|_{C(K)} \\
&\quad + \left\| \left(E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{0,1}(u_1, u_2); x_1, x_2) \right)_n - e_{0,1} \right\|_{C(K)} \\
&\quad + \left\| \left(E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{0,0,1}(u_1, u_2); x_1, x_2) \right)_n - e_{0,0,1} \right\|_{C(K)} \\
&\quad + \left\| \left(E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}(e_{2,0}(u_1, u_2) + e_{0,2}(u_1, u_2)) \right)_n - (e_{0,0,2}(u_1, u_2); x_1, x_2)_n \right. \\
&\quad \left. \left. - \{e_{2,0} + e_{0,2}\} \right\|_{C(K)} \right\} + \varepsilon.
\end{aligned}$$

where $H^* = \max \left\{ \varepsilon + M + \frac{2M}{\delta^2} (H_1^2 + H_2^2), \frac{4M}{\delta^2} H_1, \frac{4M}{\delta^2} H_2, \frac{2M}{\delta^2} \right\}$. Now, replacing $\tilde{\mathcal{P}}_n$ by $E_{\nu}^t B_{r,s}^{(\xi)} \tilde{\mathcal{P}}_n$ and taking the limit on both sides, obtain the following results.

By using the conditions of assumptions, we reach at our desired result i.e.,

$$\lim_{n \rightarrow \infty} \left\| \left(E_{\nu}^t B_{r,s}^{(\alpha)} \tilde{\mathcal{P}}(f; x_1, x_2) \right)_n - f(x_1, x_2) \right\|_{C(K)} = 0.$$

So, this completes the proof. $\square$

We want to illustrate the stated theorem with an interesting example presented below.

EXAMPLE 1. $T_n(f; y) = \frac{1}{E_{\nu}^t B_{r,s}^{(\xi)}(e)} E_{\nu}^t B_{r,s}^{(\xi)}(B_{n,n}(f; x_1, x_2))$, where $B_{n,n}$ is the bivariate Bernstein operator and the series in the denominator converges to a nonzero term. It can be observed that this sequence operator will satisfy the condition of Theorem 15.

Without loosing any generality, we can evaluate that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\| E_{\nu}^t B_{r,s}^{(\xi)}(T(1; x_1, x_2))_n - 1 \right\|_{C(K)} &= 0, \\ \lim_{n \rightarrow \infty} \left\| E_{\nu}^t B_{r,s}^{(\xi)}(T(e_{1,0}; x_1, x_2))_n - x_1 \right\|_{C(K)} &= 0, \\ \lim_{n \rightarrow \infty} \left\| E_{\nu}^t B_{r,s}^{(\xi)}(T(e_{0,1}; x_1, x_2))_n - x_2 \right\|_{C(K)} &= 0, \\ \lim_{n \rightarrow \infty} \left\| E_{\nu}^t B_{r,s}^{(\xi)}(T(e_{2,0} + e_{0,2}; x_1, x_2))_n - (x_1^2 + x_2^2) \right\|_{C(K)} &= 0. \end{aligned}$$

Therefore we may conclude from Theorem 15 that

$$\lim_{n \rightarrow \infty} \left\| E_{\nu}^t B_{r,s}^{(\xi)}(T(f; x_1, x_2))_n - f(x_1, x_2) \right\|_{C(K)} = 0.$$

3.1. Discussion of i.e. parameters

In order to stabilize the theoretical obtained result, we have performed some graphical numerical experiments with the help of few special function, parameters, and iterated n values. For doing the same we summarized the error estimation via maximum

ξ	r	s	t	Maximum absolute error	Mean square error
1.0	0.5	0.5	0.5	$5.559370e-03$	$2.793379e-04$
0.001	0.4	0.25	0.05	$3.59859e-03$	$1.224467e-04$
0.001	0.6	0.8	0.022	$7.003950e-03$	$4.284522e-04$
0.5	0.5	0.5	0.5	$2.896758e-03$	$8.026831e-05$

Table 2: Error estimations for the operator $E_{\nu}^t B(r, s)^{(\xi)}$, for functions $f(x, y) = e^{-x} \sin(y)$ with fixed $n = 50$. From the above conducted we may observed that for the suitable parameters, such as $\xi = r = s = t = 0.5$, we reach at Maximum absolute error = $2.896758e-03$ & Mean square error = $8.026831e-05$, respectively.

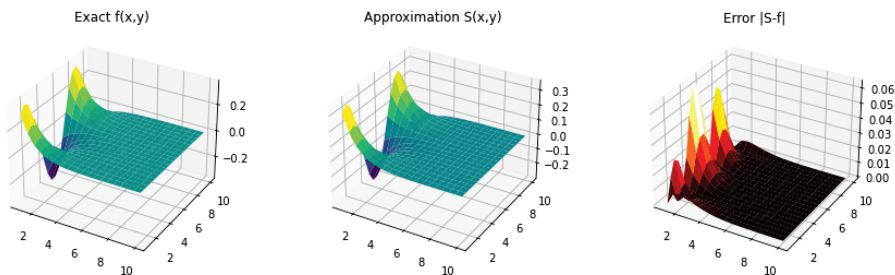


Figure 1: Approximation of $\left(E_v^t B_{r,s}^{(\xi)}\left(T\left(e^{-x_1} \sin x_2; x_1, x_2\right)\right)\right)_n$

mean squared error and maximum absolute error for our established result as in Theorem 15, in the following Table 2.

In the same contrast, we can examine other three cases as mentioned above, i.e., from the Table 1, Table 2, Table 3, as well.

3.2. Rate of convergence

At this stage, we want to proposed a theorem, presenting the rate of convergence of f via the modulus of continuity $w(f, \delta)$, for $\delta > 0$.

THEOREM 16. *Suppose that the sequence $\{\tilde{\wp}_n\}$ of positive linear operators of three variables from $C(K)$ to $C(K)$. Suppose that the following conditions hold:*

$$\begin{aligned} & \left\| E_v^t B_{r,s}^{(\xi)}\left(T(f(u_1, u_2); x_1, x_2)\right)_n - f(x_1, x_2) \right\|_{C(K)} \\ & \leq H \left\{ w(f, \mathfrak{I}_k) + \left\| \left(E_v^t B_{r,s}^{(\xi)}\left(T(1; x_1, x_2)\right)\right)_n - 1 \right\|_{C(K)} \right\}, \end{aligned}$$

where $H = \max\{2, (1 + M)\}$, and $M = \|f\|_{C(K)}$.

Proof. Suppose that $f \in C(K)$ and $(x_1, x_2) \in K$ is any fixed point in the domain $[a_1, a_2]$. Now, We obtain the following estimations.

$$\begin{aligned} & \left| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\wp}(f(u_1, u_2); x_1, x_2) \right)_n - f(x_1, x_2) \right| \\ & \leq \left(E_v^t B_{r,s}^{(\xi)} \tilde{\wp}(|f(u_1, u_2) - f(x_1, x_2)|; x_1, x_2) \right)_n \\ & \quad + |f(x_1, x_2)| \left| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\wp}(e_{0,0}(u_1, u_2); x_1, x_2) \right)_n - e_{0,0} \right| \\ & \leq w(f, \delta) \left| E_v^t B_{r,s}^{(\xi)} \left(\tilde{\wp} \left(1 + \left(\frac{(u_1 - x_1)^2 + (u_2 - x_2)^2}{\delta^2} \right); x_1, x_2 \right) \right)_n \right| \\ & \quad + |f(x_1, x_2)| \left| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\wp}(e_{0,0}(u_1, u_2); x_1, x_2) \right)_n - e_{0,0} \right| \end{aligned}$$

$$\leq w(f, \delta) \left\{ \left(E_v^t B_{r,s}^{(\xi)} \tilde{\rho}(1; x_1, x_2) \right)_n + \frac{1}{\delta^2} \left(E_v^t B_{r,s}^{(\xi)} \tilde{\rho}(\rho; x_1, x_2) \right)_n \right\} \\ + M \left| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\rho}(1; x_1, x_2) \right)_n - 1 \right|,$$

where $M = \|f\|_{C(K)}$, and $\rho = \frac{(u_1 - x_1)^2 + (u_2 - x_2)^2}{\delta^2}$. Now, taking the supremum over all $(x_1, x_2) \in K$, on both sides, we obtain followings.

$$\left\| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\rho}(f(u_1, u_2); x_1, x_2) \right)_n - f(x_1, x_2) \right\|_{C(K)} \\ \leq w(f, \delta) \left\{ \left\| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\rho}(1; x_1, x_2) \right)_n - 1 \right\|_{C(K)} + \frac{1}{\delta^2} \left\| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\rho}(\rho; x_1, x_2) \right)_n \right\|_{C(K)} + 1 \right\} \\ + M \left\| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\rho}(1; x_1, x_2) \right)_n - 1 \right\|_{C(K)} \\ \leq \max \{2, (1 + M)\} \left\{ w(f, \mathfrak{I}_k) + \left\| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\rho}(1; x_1, x_2) \right)_n - 1 \right\|_{C(K)} \right\}.$$

Let us choose $\delta = \sqrt{\left\| \left(E_v^t B_{r,s}^{(\xi)} (\tilde{\rho}(\rho; x_1, x_2)) \right)_n \right\|_{C(K)}} = \mathfrak{I}_k$. Then, the above inequality is being reduced to

$$\left\| E_v^t B_{r,s}^{(\xi)} (\tilde{\rho}(f(u_1, u_2); x_1, x_2))_n - f(x_1, x_2) \right\|_{C(K)} \\ \leq H \left\{ w(f, \mathfrak{I}_k) + \left\| \left(E_v^t B_{r,s}^{(\xi)} (T(1; x_1, x_2)) \right)_n - 1 \right\|_{C(K)} \right\},$$

where $H = \max \{2, (1 + M)\}$. So, this completes the proof. $\square$

In this final section, we intend to establish some error estimates via Lipschitz type maximal function, for more details, we may refer ([2], [40]), for the bivariate functions. Before proceeding further let us memorize the following.

$$Lip_M(\kappa) = \left\{ f \in \mathfrak{S} : |f(x_1, x_2) - f(y_1, y_2)| \right. \\ \leq M \frac{|x_1 - y_1|^{\kappa_1}}{(\alpha_1^2 y_1^2 + \alpha_1^* y_1 + x_1)^{\frac{\kappa_1}{2}}} \frac{|x_2 - y_2|^{\kappa_2}}{(\alpha_2^2 y_2^2 + \alpha_2^* y_2 + x_2)^{\frac{\kappa_2}{2}}}, \\ \left. \text{for } \alpha_1, \alpha_2, \alpha_1^*, \alpha_2^* > 0, \kappa_1, \kappa_2 \in (0, 1] \text{ and } M > 0 \right\}.$$

THEOREM 17. *Let us take $f \in Lip_M(\kappa)$. Then for every $\tau_1, \tau_2, \tau_3 \in (0, 1]$, we have the following inequality.*

$$\left| \left(E_v^t B_{r,s}^{(\xi)} \tilde{\rho}(f(u_1, u_2); x_1, x_2) \right)_n - f(x_1, x_2) \right| \\ \leq M \sqrt[4]{\frac{[\tilde{\Phi}_n^4(x_1)]^{\tau_1}}{[\alpha_1 x_1^2 + \alpha_1^* x_1]^{2\tau_1}} \cdot \frac{[\tilde{\Phi}_n^4(x_2)]^{\tau_2}}{[\alpha_2 x_2^2 + \alpha_2^* x_2]^{2\tau_2}}} \tilde{\Psi}(x_1, x_2).$$

Proof. By definition of the Lipschitz maximal function, we obtain the following for the $\tau_1, \tau_2 \in (0, 1]$,

$$\begin{aligned} & \left| \left(E_{\nu}^t B_{r,s}^{(\alpha)}(f(u_1, u_2); x_1, x_2) \right)_n - f(x_1, x_2) \right| \\ & \leq \left| \left(E_{\nu}^t B_{r,s}^{(\alpha)}(\tilde{\rho}(f(u_1, u_2) - f(x_1, x_2); x_1, x_2))_n \right) \right| \\ & \leq M \left(E_{\nu}^t B_{r,s}^{(\alpha)} \left(\tilde{\rho} \left(\left(\frac{|u_1 - x_1|^{\tau_1}}{(\alpha_1 x_1^2 + \alpha_1^* x_1)^{\frac{\tau_1}{2}}} ; x_1, x_2 \right) \right) \right) \right)_n \\ & \quad \times \left(E_{\nu}^t B_{r,s}^{(\alpha)} \left(\tilde{\rho} \left(\left(\frac{|u_2 - x_2|^{\tau_2}}{(\alpha_2 x_2^2 + \alpha_2^* x_2)^{\frac{\tau_2}{2}}} ; x_1, x_2 \right) \right) \right) \right)_n \end{aligned}$$

inserting Holder's inequality on both sides,

$$\begin{aligned} & \leq M \left(E_{\nu}^t B_{r,s}^{(\alpha)} \left(\tilde{\rho} \left(\left(\frac{|u_1 - x_1|^4}{(\alpha_1 y_1^2 + \alpha_1^* y_1)^2} \right)^{\frac{\tau_1}{4}} \cdot \left(\frac{|u_2 - x_2|^4}{(\alpha_2 x_2^2 + \alpha_2^* x_2)^2} \right)^{\frac{\tau_2}{4}} ; x_1, x_2 \right) \right) \right)_n \\ & \quad \times \left(E_{\nu}^t B_{r,s}^{(\alpha)}(\tilde{\rho}(e_{1,0}; x_1, x_2))_n \right)^{\frac{4-\tau_1}{4}} \cdot \left(E_{\nu}^t B_{r,s}^{(\alpha)}(\tilde{\rho}(e_{0,1}; x_1, x_2 x_1, x_2))_n \right)^{\frac{4-\tau_2}{4}} \\ & \quad \times \left(E_{\nu}^t B_{r,s}^{(\alpha)}(\tilde{\rho}(e_{0,0,1}; x_1, x_2))_n \right)^{\frac{4-\tau_3}{4}} \\ & \leq M \sqrt[4]{\frac{[\tilde{\Phi}_n^4(x_1)]^{\tau_1}}{[\alpha_1 x_1^2 + \alpha_1^* x_1]^2 \tau_1} \cdot \frac{[\tilde{\Phi}_n^4(x_2)]^{\tau_2}}{[\alpha_2 x_2^2 + \alpha_2^* x_2]^2 \tau_2}} \tilde{\psi}(x_1, x_2) \end{aligned}$$

where $\tilde{\Phi}_n^4(x_i) = \left(E_{\nu}^t B_{r,s}^{(\alpha)} \left(\tilde{\rho} \left(\frac{|u_i - x_i|^4}{(\alpha_i x_i^2 + \alpha_i^* x_i)^2} \right) \right) \right)_n$ and

$$\tilde{\psi}(x_1, x_2) = \left(E_{\nu}^t B_{r,s}^{(\alpha)}(\tilde{\rho}(e_{1,0}; x_1, x_2))_n \right)^{\frac{4-\tau_1}{4}} \left(E_{\nu}^t B_{r,s}^{(\alpha)}(\tilde{\rho}(e_{0,1}; x_1, x_2))_n \right)^{\frac{4-\tau_2}{4}}.$$

Hence, we reach at the desired result. $\square$

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