

APPLICATIONS OF AN IDENTITY OF BATIR

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Abstract. Based on an interesting identity of Batir we derive new identities for double sums involving famous number sequences. We also prove some double sum identities for binomial transform pairs.

1. Introduction

As a reaction to Vălean's Master Theorem of Series, Batir [4] established a generalization of this theorem. His proof is based on the following identity which holds for any two sequences of complex numbers $(a_k)_{k \geq 1}$ and $(b_k)_{k \geq 1}$ [4, Lemma 1]:

$$\sum_{k=1}^n a_k \sum_{j=1}^k b_j = \sum_{p=0}^{n-1} \sum_{k=1}^{n-p} a_{p+k} b_k. \quad (1)$$

Our purpose is to derive various double summation identities based on the following identity which is a variation of (1):

$$\sum_{k=1}^n \sum_{j=0}^{k-1} a_{n-j} b_{k-j} = \sum_{k=1}^n a_k \sum_{j=1}^k b_j. \quad (2)$$

Throughout the paper we use the notation $H_n^{(s)}$, $O_n^{(s)}$, B_n , F_n and L_n for generalized harmonic numbers of order s with $H_n^{(1)} = H_n$ being the harmonic numbers, generalized odd harmonic numbers of order s with $O_n^{(1)} = O_n$ being the odd harmonic numbers, Bernoulli numbers, Fibonacci numbers and Lucas numbers, respectively. We also use $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ to denote Stirling numbers of the second kind, C_n the Catalan numbers, and G_n the generalized Fibonacci numbers, the so-called gibbonacci numbers.

2. General results

THEOREM 1. *If (a_k) , $k = 1, 2, 3, \dots$, is a sequence of complex numbers, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{a_{n-j}}{n-j} = \sum_{k=1}^n a_k. \quad (3)$$

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Proof. In (2), set $b_k = 1$ and replace a_k with a_k/k for $k = 1, 2, 3, \dots$ $\square$

THEOREM 2. If (a_k) , $k = 1, 2, 3, \dots$, is a sequence of complex numbers, then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} (-1)^{k-j+1} a_{n-j} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{2k+1}. \quad (4)$$

Proof. In (2), set $b_k = (-1)^k$ and use

$$\sum_{j=1}^k b_j = \frac{1}{2} \left((-1)^k - 1 \right). \quad \square$$

PROPOSITION 1. If n is a non-negative integer, then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{1}{(n-j)^2} = H_n, \quad (5)$$

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{1}{(n-j)(2(n-j)-1)} = O_n. \quad (6)$$

Proof. Use $a_k = 1/k$ and $a_k = (2k-1)^{-1}$ in Theorem 1. $\square$

PROPOSITION 2. If n is a non-negative integer, then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} (-1)^{k-j+1} (n-j) = \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \right)^2. \quad (7)$$

Proof. Use $a_k = k$ in Theorem 2 and

$$\sum_{k=0}^n (2k+1) = (n+1)^2. \quad \square$$

PROPOSITION 3. If n is a non-negative integer, then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} (-1)^{k-j+1} \frac{1}{n-j} = O_{\lfloor (n-1)/2 \rfloor}. \quad (8)$$

Proof. Use $a_k = 1/k$ in Theorem 2. $\square$

PROPOSITION 4. If n is a non-negative integer, then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} (-1)^{k-j+1} F_{n-j} = F_{2(\lfloor (n-1)/2 \rfloor + 1)} \quad (9)$$

and

$$\sum_{k=1}^n \sum_{j=0}^{k-1} (-1)^{k-j+1} L_{n-j} = L_{2(\lfloor (n-1)/2 \rfloor + 1)} - 2. \quad (10)$$

Proof. Use $a_k = F_k$ and $a_k = L_k$ in Theorem 2 and combine with

$$\sum_{k=0}^n F_{2k+1} = F_{2(n+1)} \quad \text{and} \quad \sum_{k=0}^n L_{2k+1} = L_{2(n+1)} - 2. \quad \square$$

PROPOSITION 5. Let $f(k)$ be any well defined function. If n is a non-negative integer, then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} (-1)^{k-j} f(n-j) B_{n-j} = \frac{f(1)}{2}. \quad (11)$$

In particular,

$$\sum_{k=1}^n \sum_{j=0}^{k-1} (-1)^{k-j} B_{n-j}^2 = -\frac{1}{4}. \quad (12)$$

Proof. Use $a_k = f(k)B_k$ in Theorem 2 and the fact that $B_{2k+1} = 0$ for all $k \geq 1$. $\square$

Our next result gives a double sum definition of Bernoulli numbers, expressed in terms of the Stirling numbers of the second kind.

PROPOSITION 6. If n is a positive integer, then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} (-1)^{n-j} \frac{(n-j-1)!}{n-j+1} \left\{ \begin{matrix} n \\ n-j \end{matrix} \right\} = B_n. \quad (13)$$

Proof. Use the well-known result (see [8] for annotated references):

$$B_n = \sum_{k=1}^n \frac{(-1)^k k!}{k+1} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$$

in Theorem 1. $\square$

THEOREM 3. If (a_k) , $k = 1, 2, 3, \dots$, is a sequence of complex numbers, r is a complex number that is not a negative integer, then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{a_{n-j}}{k-j+r} = \sum_{k=1}^n a_k H_{k+r} - H_r \sum_{k=1}^n a_k. \quad (14)$$

In particular, we have the following double sum identities

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{a_{n-j}}{k-j} = \sum_{k=1}^n a_k H_k, \quad (15)$$

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{a_{n-j}}{2k-2j-1} = \sum_{k=1}^n a_k O_k. \quad (16)$$

Proof. Use $b_j = 1/(j+r)$ in (2). Setting $r = 0$ in (14) gives (15) while $r = -1/2$ produces (16). $\square$

PROPOSITION 7. *If n is a non-negative integer, x is a complex number with $x \neq 1$, and r is a complex number that is not a negative integer, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{x^{n-j}}{k-j+r} = \frac{1}{1-x} \left(\sum_{k=1}^n \frac{x^k}{k+r} - x^{n+1} (H_{n+r} - H_r) \right). \quad (17)$$

In particular,

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{x^{n-j}}{k-j} = \frac{1}{1-x} \left(\sum_{k=1}^n \frac{x^k}{k} - x^{n+1} H_n \right) \quad (18)$$

and

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{x^{n-j}}{2k-2j-1} = \frac{1}{1-x} \left(\sum_{k=1}^n \frac{x^k}{2k-1} - x^{n+1} O_n \right). \quad (19)$$

Proof. Work with $a_k = x^k$, $x \neq 1$, in (14). Then

$$\begin{aligned} \sum_{k=1}^n \sum_{j=0}^{k-1} \frac{x^{n-j}}{k-j+r} &= \sum_{k=1}^n x^k H_{k+r} - H_r \sum_{k=1}^n x^k \\ &= \sum_{k=1}^n x^k H_{k+r} - H_r \frac{x}{1-x} + H_r \frac{x^{n+1}}{1-x}. \end{aligned}$$

Applying summation by parts we get

$$\begin{aligned} \sum_{k=1}^n \frac{x^k}{k+r} &= \sum_{k=1}^n x^k (H_{k+r} - H_{k-1+r}) \\ &= x^{n+1} H_{n+r} - x H_r - \sum_{k=1}^n H_{k+r} (x^{k+1} - x^k) \\ &= x^{n+1} H_{n+r} - x H_r + (1-x) \sum_{k=1}^n x^k H_{k+r}. \end{aligned}$$

Gathering terms and simplifying proves the main statement. The particular cases are obtained for $r = 0$ and $r = -1/2$, respectively. $\square$

REMARK 1. The case $x = 1$ can be treated with l'Hospital's rule. However, this case is not interesting in the sense that it is not a double sum as we have for $r \geq 0$ an integer

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{1}{k-j+r} = \sum_{k=1}^n \sum_{j=1}^k \frac{1}{j+r} = \sum_{k=1}^n (H_{k+r} - H_r).$$

The evaluation of the last sum is elementary.

COROLLARY 1. If n is a non-negative integer and r is a complex number that is not a negative integer, then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{G_{n-j}}{k-j+r} = G_{n+2}(H_{n+r} - H_r) - \sum_{k=1}^n \frac{G_{k+1}}{k+r}, \quad (20)$$

where G_n is the n th gibbonacci number. In particular,

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{G_{n-j}}{k-j} = G_{n+2}H_n - \sum_{k=1}^n \frac{G_{k+1}}{k} \quad (21)$$

and

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{G_{n-j}}{2k-2j-1} = G_{n+2}O_n - \sum_{k=1}^n \frac{G_{k+1}}{2k-1}. \quad (22)$$

Proof. Set $x = \alpha$ and $x = \beta$ with $\alpha = (1 + \sqrt{5})/2$ and $\beta = -1/\alpha$ in (17), simplify, and combine according to the Binet forms for Fibonacci (Lucas) numbers. $\square$

COROLLARY 2. If n is a non-negative integer, then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{k-j} = \begin{cases} \frac{1}{2}H_{n/2}, & \text{if } n \text{ is even;} \\ O_{(n+1)/2}, & \text{if } n \text{ is odd.} \end{cases} \quad (23)$$

Proof. When $x = -1$ in (18), then

$$\begin{aligned} \sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^{n-j}}{k-j} &= \frac{1}{2} \left(\sum_{k=1}^n \frac{(-1)^k}{k} + (-1)^n H_n \right) \\ &= \frac{1}{2} (-H_n + H_{\lfloor n/2 \rfloor} + (-1)^n H_n) \\ &= \begin{cases} \frac{1}{2} H_{\lfloor n/2 \rfloor}, & \text{if } n \text{ is even;} \\ -\left(H_n - \frac{1}{2} H_{\lfloor n/2 \rfloor}\right), & \text{if } n \text{ is odd.} \end{cases} \quad \square \end{aligned}$$

PROPOSITION 8. If n is a non-negative integer and p and x are complex numbers, then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{x^{n-j}}{k-j} \left(\frac{1}{n+p-j} - \frac{x}{n+1+p-j} \right) = \sum_{k=1}^n \frac{x^k}{k(k+p)} - H_n \frac{x^{n+1}}{n+1+p}. \quad (24)$$

Proof. Multiply (18) by $(1-x)x^{p-1}$, $p \in \mathbb{C}$, and integrate. $\square$

COROLLARY 3. If n is a non-negative integer and p is a complex number that is not a non-positive integer, then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{1}{(k-j)(n+p-j)(n+1+p-j)} = \frac{1}{p} (H_n + H_p - H_{n+p}) - \frac{H_n}{n+1+p}. \quad (25)$$

In particular,

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{1}{(k-j)(n+1-j)(n+2-j)} = \frac{n}{n+1} - \frac{H_n}{n+2} \quad (26)$$

and

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{1}{(k-j)(2(n-j)-1)(2(n-j)+1)} = O_n - \frac{n+1}{2n+1} H_n. \quad (27)$$

Proof. Set $x = 1$ in (24) and calculate

$$\sum_{k=1}^n \frac{1}{k(k+p)} = \frac{1}{p} \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+p} \right) = \frac{1}{p} (H_n + H_p - H_{n+p}).$$

Evaluate (25) at $p = 1$ and $p = -1/2$, respectively. $\square$

PROPOSITION 9. If r is a non-negative integer, then

$$\sum_{m=1}^r \sum_{n=1}^{m-1} \sum_{k=1}^{n-1} \frac{1}{(k-j)(n-j)(n-j+1)} = r - H_r - \frac{r+1}{2} (H_r^2 - H_r^{(2)}). \quad (28)$$

Proof. The sums

$$\sum_{m=1}^r H_m = (r+1)H_r - r, \quad (29)$$

$$\sum_{m=1}^r H_m^2 = (r+1)H_r^2 - (2r+1)H_r + 2r, \quad (30)$$

and

$$\sum_{m=1}^r H_m^{(2)} = (r+1)H_r^{(2)} - H_r \quad (31)$$

can be derived easily. Write $n-1$ for n in (26) and sum to obtain

$$\sum_{n=1}^{m-1} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} \frac{1}{(k-j)(n-j)(n-j+1)} = m - H_m - \frac{1}{2} (H_m^2 - H_m^{(2)}). \quad (32)$$

Summing (32) over m using (29), (30) and (31) produces (28). $\square$

PROPOSITION 10. *If n is a non-negative integer and p is a positive integer, then*

$$\begin{aligned} \sum_{k=1}^n \sum_{j=0}^{k-1} \frac{1}{(k-j)(n-j+p)} &= \frac{1}{2} \left(H_n^2 - H_n^{(2)} \right) - \frac{1}{2} \left(H_{p-1}^2 + H_{p-1}^{(2)} \right) \\ &\quad + H_{p-1} (H_{p+n-1} - H_n) + H_n (H_{p+n} - H_n) \\ &\quad - \sum_{k=1}^{p-1} \frac{H_{k-1}}{n+k}. \end{aligned} \quad (33)$$

In particular,

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{1}{(k-j)(n+1-j)} = \frac{1}{2} \left(H_n^2 - H_n^{(2)} \right) + \frac{H_n}{n+1}. \quad (34)$$

Proof. Work with $a_k = 1/(k+p)$, $p \geq 1$, in (15). The particular case follows by setting $p = 1$. $\square$

PROPOSITION 11. *If n is a non-negative integer, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{H_{n-j}}{k-j} = (n+1)H_n^2 - (2n+1)H_n + 2n, \quad (35)$$

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{H_{n-j}^2}{k-j} = (n+1)H_n^3 - \frac{3}{2}(2n+1)H_n^2 + 3(2n+1)H_n + \frac{1}{2}H_n^{(2)} - 6n, \quad (36)$$

and

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{H_{n-j}^{(2)}}{k-j} = (n+1)H_n H_n^{(2)} - \frac{1}{2}(2n+1)H_n^{(2)} + H_n - \frac{1}{2}H_n^2. \quad (37)$$

Proof. Work with $a_k = H_k$, $a_k = H_k^2$, and $a_k = H_k^{(2)}$, in turn, in (15). The sums

$$\sum_{k=1}^n H_k^3 = (n+1)H_n^3 - \frac{3}{2}(2n+1)H_n^2 + 3(2n+1)H_n + \frac{1}{2}H_n^{(2)} - 6n$$

and

$$\sum_{k=1}^n H_k H_k^{(2)} = (n+1)H_n H_n^{(2)} - \frac{1}{2}(2n+1)H_n^{(2)} + H_n - \frac{1}{2}H_n^2$$

come from the paper [9]. $\square$

3. Miscellaneous results involving binomial coefficients

PROPOSITION 12. *If m and n are non-negative integers and p is a complex number, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{k-j} \binom{p}{j} \binom{n-j}{m} = \binom{n-p}{m-p} (H_{n-p} - H_{m-p} + H_m). \quad (38)$$

Proof. Use

$$a_k = (-1)^k \binom{p}{n-k} \binom{k}{m}$$

in (15), since ([11, Corollary 5])

$$\sum_{k=1}^n (-1)^k \binom{p}{n-k} \binom{k}{m} H_k = (-1)^n \binom{n-p}{m-p} (H_{n-p} - H_{m-p} + H_m). \quad \square$$

PROPOSITION 13. *If n is a non-negative integer, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \binom{n}{j} \frac{x^{n-j}}{k-j} = (1+x)^n H_n - \sum_{k=1}^n \frac{(1+x)^{n-k}}{k}. \quad (39)$$

In particular,

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{\binom{n}{j}}{k-j} = 2^n \left(H_n - \sum_{k=1}^n \frac{1}{2^k k} \right). \quad (40)$$

Proof. Work with $a_k = \binom{n}{k} x^k$ in (15) while making use of the Knuth-Boyadzhiev identity

$$\sum_{k=1}^n \binom{n}{k} H_k x^k = (1+x)^n H_n - \sum_{k=1}^n \frac{(1+x)^{n-k}}{k}. \quad \square \quad (41)$$

PROPOSITION 14. *If m and n are non-negative integers, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(n-j)!}{k-j} \binom{n}{j} \left\{ \begin{matrix} m \\ n-j \end{matrix} \right\} = n^m H_n - \sum_{k=1}^n \frac{(n-k)^m}{k}. \quad (42)$$

Proof. Write $-\exp(x)$ for x in (39) and differentiate m times through the resulting equation and evaluate at $x = 0$, using

$$\frac{d^m}{dx^m} (1 - e^x)^p \Big|_{x=0} = (-1)^p p! \left\{ \begin{matrix} m \\ p \end{matrix} \right\}. \quad \square$$

PROPOSITION 15. *If n is a non-negative integer, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{\binom{n}{j}^2}{k-j} = \binom{2n}{n} (2H_n - H_{2n}) \quad (43)$$

and

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{1}{k-j} \binom{n}{j} \binom{2n}{n+j} = \binom{3n}{n} H_n - \sum_{k=1}^n \frac{1}{k} \binom{3n-k}{n-k}. \quad (44)$$

Proof. To derive Identity (43), plug $a_k = \binom{n}{k}^2$ in (15) and use the fact that [6]:

$$\sum_{k=0}^n \binom{n}{k}^2 H_k = \binom{2n}{n} (2H_n - H_{2n}).$$

Identity (44) is obtained by setting $a_k = \binom{n}{k} \binom{2n}{k}$ in (15) and making use of [10]:

$$\sum_{k=0}^n \binom{n}{k} \binom{2n}{k} H_k = \binom{3n}{n} H_n - \sum_{k=1}^n \frac{1}{k} \binom{3n-k}{n-k}. \quad \square$$

PROPOSITION 16. *If n is a non-negative integer, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{n-j} \binom{n}{j}^3 = \begin{cases} (-1)^{n/2} \binom{n}{n/2} \binom{3n/2}{n} - 1, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases} \quad (45)$$

Proof. Set $a_k = \binom{n}{k}^3$ in (3) and use Dixon's identity [7]:

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 = \begin{cases} (-1)^{n/2} \binom{n}{n/2} \binom{3n/2}{n}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \quad \square$$

PROPOSITION 17. *If n is a non-negative integer, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{2^{2j}}{k-j} \binom{2(n-j)}{n-j} = 2^{2n+1} + (H_n - 2)(2n+1) \binom{2n}{n} \quad (46)$$

and

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{2^{2j}}{2k-2j-1} \binom{2(n-j)}{n-j} = (O_{n+1} - 1)(2n+1) \binom{2n}{n}. \quad (47)$$

Proof. Use $a_k = 2^{-2k} \binom{2k}{k}$ in (15) and also in (16) and employ the following identities from [3]:

$$\begin{aligned} \sum_{k=1}^n 2^{-2k} \binom{2k}{k} H_k &= 2 + (H_n - 2)(2n+1) 2^{-2n} \binom{2n}{n}, \\ \sum_{k=1}^n 2^{-2k} \binom{2k}{k} O_k &= (O_{n+1} - 1)(2n+1) 2^{-2n} \binom{2n}{n}. \quad \square \end{aligned}$$

PROPOSITION 18. *If n is a non-negative integer, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{2^{2j}}{k-j} \binom{2(n-j)}{n-j} O_{n-j} = \sum_{k=1}^n \sum_{j=0}^{k-1} \frac{2^{2j}}{2k-2j-1} \binom{2(n-j)}{n-j} H_{n-j}. \quad (48)$$

Proof. This result follows from (15) and (16) and the following identity [5]:

$$\begin{aligned} \sum_{k=1}^n 2^{-2k} \binom{2k}{k} O_k H_k &= (2n+1) 2^{-2n} \binom{2n}{n} (H_n - 2) \left(O_n - \frac{2n}{2n+1} \right) \\ &\quad + \frac{2n+1}{2(4n-1)} \binom{2n}{n} - 2. \quad \square \end{aligned}$$

PROPOSITION 19. *If n is a non-negative integer and s is a complex number that is not a negative integer, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j s}{k-j} \binom{s}{n-j} = s \binom{s-1}{n} H_n + \binom{s-1}{n} - (-1)^n. \quad (49)$$

Proof. Batir derived the identity [5, Corollary 3]

$$\sum_{k=1}^n \binom{n+s}{k} (-1)^k H_k = (-1)^n \binom{n+s-1}{n} H_n + \frac{(-1)^n}{n+s} \binom{n+s-1}{n} - \frac{1}{n+s}.$$

Using the transformation $s \mapsto s - n$ this identity can be stated equivalently as

$$\sum_{k=1}^n \binom{s}{k} (-1)^k s H_k = s (-1)^n \binom{s-1}{n} H_n + (-1)^n \binom{s-1}{n} - \frac{1}{s}.$$

Now, set $a_k = (-1)^k \binom{s}{k}$ in (15). $\square$

COROLLARY 4. *If n is a non-negative integer then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{1}{k-j} \frac{2^{2j}}{2(n-j)-1} \binom{2(n-j)}{n-j} = 2^{2n+1} - \binom{2n}{n} (H_n + 2) \quad (50)$$

and

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{2^{2j}}{k-j} \binom{2(n-j)}{n-j} = 2^{2n+1} + (2n+1) \binom{2n}{n} (H_n - 2). \quad (51)$$

Proof. Evaluate (49) at $s = -1/2$ and $s = 1/2$, and keep in mind that

$$\binom{1/2}{r} = (-1)^{r+1} \frac{2^{-2r}}{2r-1} \binom{2r}{r}, \quad \binom{-1/2}{r} = (-1)^r 2^{-2r} \binom{2r}{r},$$

and

$$\binom{-3/2}{r} = (-1)^r (2r+1) 2^{-2r} \binom{2r}{r}. \quad \square$$

PROPOSITION 20. *If n is a non-negative integer, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{k-j} \binom{n}{j}^{-1} = (n+1) \left(\frac{H_{n+1}}{n+2} - \frac{(-1)^n + 1}{(n+2)^2} \right). \quad (52)$$

Proof. Use $a_k = (-1)^k \binom{n}{k}^{-1}$ in (15) together with the following identity [5]:

$$\sum_{k=1}^n (-1)^k \binom{n}{k}^{-1} H_k = (n+1) \left(\frac{H_{n+1}}{n+2} - \frac{(-1)^n + 1}{(n+2)^2} \right). \quad \square$$

LEMMA 1. *If n is a non-negative integer and p and q are complex numbers that are not negative integers, then*

$$\sum_{k=1}^n \binom{k+q}{p} = \frac{n+q+1}{p+1} \binom{n+q}{p} - \binom{q}{p+1} - \binom{q}{p} \quad (53)$$

and

$$\begin{aligned} \sum_{k=1}^n \binom{k+q}{p} H_{k+q} &= \binom{n+q}{p} \left(\frac{1}{p+1} + \frac{n+q+1}{p+1} \left(H_{n+q} - \frac{1}{p+1} \right) \right) \\ &\quad - \binom{q}{p+1} \left(H_q - \frac{1}{p+1} \right) - \binom{q}{p} H_q. \end{aligned} \quad (54)$$

Proof. Identity (53) is a consequence of Pascal's formula:

$$\binom{k+q}{p} = \binom{k+q+1}{p+1} - \binom{k+q}{p+1}.$$

Identity (54) is obtained by differentiating (53) first with respect to p , and then with respect to q and eliminating H_{q-p-1} between the two resulting equations. $\square$

LEMMA 2. *If k is a non-negative integer and r is a complex number that is not a negative integer, then*

$$\binom{k-1/2}{k+r} = (-1)^r r \binom{2r}{r} \frac{C_k}{2^{2k+2r}} \binom{k+r}{k+1}^{-1}, \quad (55)$$

where C_j is the j^{th} Catalan number, defined for every non-negative integer j by

$$C_j = \frac{1}{j+1} \binom{2j}{j}.$$

Proof. A consequence of the generalized binomial coefficient, expressed in terms of the Gamma function, $\Gamma(z)$, namely,

$$\binom{r}{s} = \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)}. \quad \square$$

PROPOSITION 21. *If n is a non-negative integer and p and q are complex numbers that are not negative integers, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{1}{n-j} \binom{n-j+q}{p} = \frac{n+q+1}{p+1} \binom{n+q}{p} - \binom{q}{p+1} - \binom{q}{p} \quad (56)$$

and

$$\begin{aligned} \sum_{k=1}^n \sum_{j=0}^{k-1} \frac{\binom{n-j+q}{p}}{k-j+q} &= \binom{n+q}{p} \left(\frac{1}{p+1} + \frac{n+q+1}{p+1} \left(H_{n+q} - H_q - \frac{1}{p+1} \right) \right) \\ &\quad + \binom{q}{p+1} \frac{1}{p+1}. \end{aligned} \quad (57)$$

Proof. Identity (56) is obtained by putting $a_k = \binom{k+q}{p}$ in Theorem 1 and using (53). Set $a_k = \binom{k+q}{p}$ in (14) with $r=q$ and make use of (53) and (54) to get (57). $\square$

Our next result is a double sum identity that involves binomial coefficients, Catalan numbers, and odd harmonic numbers.

PROPOSITION 22. *If n is a non-negative integer and $r \neq 1/2$ is a complex number that is not a non-positive integer, then*

$$\begin{aligned} \sum_{k=1}^n \sum_{j=0}^{k-1} 2^{2j} \binom{n-j+r}{r-1}^{-1} \frac{C_{n-j}}{2k-2j-1} \\ = - \binom{n+r}{r-1}^{-1} \frac{C_n}{2r-1} \left(1 + (2n+1) \left(O_n + \frac{1}{2r-1} \right) \right) + \frac{2^{2n+1}}{(2r-1)^2}. \end{aligned} \quad (58)$$

In particular,

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{2^{2j} C_{n-j}}{2k-2j-1} = -C_n (1 + (2n+1)(O_n + 1)) + 2^{2n+1}. \quad (59)$$

Proof. With Lemma 2 in mind, set $q = -1/2$ and $p = -r - 1/2$ in (57). $\square$

LEMMA 3. ([1, Section 13]) *If n is a non-negative integer and r is a complex number that is not a negative integer, then*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k}{k+r} = \begin{cases} \frac{\sin(\pi r)}{\pi(n+r)}, & \text{if } r \neq 0; \\ 0, & \text{if } r = 0, n \neq 0; \end{cases} \quad (60)$$

and

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k}{k+r} H_{k+r} = \begin{cases} \frac{\sin(\pi r)}{\pi(n+r)} \left(H_{r-1} + \frac{1}{n+r} \right), & \text{if } r \neq 0; \\ -\frac{1}{n}, & \text{if } r = 0, n \neq 0. \end{cases} \quad (61)$$

PROPOSITION 23. *If n is a non-negative integer and r is a complex number that is not a negative integer, then*

$$\begin{aligned} & \sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{k-j+r} \binom{n+r}{j} \\ &= (-1)^n \binom{n}{n+r}^{-1} \begin{cases} \frac{\sin(\pi r)}{\pi(n+r)} \left(\frac{1}{n+r} - \frac{1}{r} \right), & \text{if } r \neq 0; \\ -\frac{1}{n}, & \text{if } r = 0, n \neq 0. \end{cases} \end{aligned} \quad (62)$$

Proof. Set $a_k = (-1)^k \binom{n}{k} \binom{k}{k+r}$ in (14) and use (60) and (61). $\square$

COROLLARY 5. *If n is a non-negative integer, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{2(k-j)+1} \frac{\binom{2n+1}{2j} \binom{2j}{j}}{\binom{n}{j}} 2^{-2j} = (-1)^{n+1} \frac{n(n+1)}{(2n+1)^2} 2^{-2n} \binom{2(n+1)}{n+1}. \quad (63)$$

Proof. Set $r = 1/2$ in (62) and simplify using

$$\binom{n+1/2}{j} = \frac{\binom{2n+1}{2j} \binom{2j}{j}}{\binom{n}{j}} 2^{-2j} \quad \text{and} \quad \binom{n}{n+1/2} = \frac{1}{\pi} \frac{2^{2n+2}}{n+1} \binom{2(n+1)}{n+1}^{-1}. \quad \square$$

4. Applications to binomial transforms

Two sequences of complex numbers (s_n) and (σ_n) are called a binomial transform pair if these sequences are connected by the relations

$$\sigma_n = \sum_{k=0}^n \binom{n}{k} (-1)^k s_k \quad \Longleftrightarrow \quad s_n = \sum_{k=0}^n \binom{n}{k} (-1)^k \sigma_k.$$

THEOREM 4. *Let (a_k) be a sequence of complex numbers. Let (s_k) and (σ_k) be a binomial transform pair. Then*

$$\sum_{k=1}^n (-1)^k \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} a_{n-j} s_{k-j} = \sum_{k=1}^n a_k \sigma_k - s_0 \sum_{k=1}^n a_k. \quad (64)$$

Proof. Work with (2) and apply it to $b_j = \binom{k}{j} (-1)^j s_j$. Note that

$$\sum_{j=1}^k b_j = \sum_{j=1}^k \binom{k}{j} (-1)^j s_j = \sum_{j=0}^k \binom{k}{j} (-1)^j s_j - s_0 = \sigma_k - s_0. \quad \square$$

PROPOSITION 24. Let (a_k) be a sequence of complex numbers. Let F_n be the Fibonacci sequence and L_n be the Lucas sequence, respectively. Then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \binom{k}{j} a_{n-j} F_{k-j} = \sum_{k=1}^n a_k F_{2k} \quad (65)$$

and

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \binom{k}{j} a_{n-j} L_{k-j} = \sum_{k=1}^n a_k (L_{2k} - 2). \quad (66)$$

Proof. Apply Theorem 4 in turn to the binomial transform pairs $\{s_n, \sigma_n\} = \{(-1)^n F_n, F_{2n}\}$ and $\{s_n, \sigma_n\} = \{(-1)^n L_n, L_{2n}\}$. $\square$

PROPOSITION 25. Let (a_k) be a sequence of complex numbers. Let B_n be the sequence Bernoulli numbers. Then

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \binom{k}{j} a_{n-j} B_{k-j} = \sum_{k=1}^n a_k \left((-1)^k B_k - 1 \right). \quad (67)$$

Proof. Apply Theorem 4 to the binomial transform pair

$$\{s_n, \sigma_n\} = \{(-1)^n B_n, (-1)^n B_n\}. \quad \square$$

THEOREM 5. Let (s_k) and (σ_k) be a binomial transform pair. Let m , n , and r be non-negative integers. Then

$$\begin{aligned} \sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{n-j+1} \binom{n}{j} \sum_{p=0}^r (-1)^p \binom{r}{p} s_{k+p+m-j} \\ = \frac{(-1)^n}{n+1} \sum_{p=0}^m (-1)^p \binom{m}{p} (\sigma_{n+p+r} - \sigma_{p+r}). \end{aligned} \quad (68)$$

In particular,

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{n-j+1} \binom{n}{j} \sum_{p=0}^r (-1)^p \binom{r}{p} s_{k+p-j} = \frac{(-1)^n}{n+1} (\sigma_{n+r} - \sigma_r), \quad (69)$$

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{n-j+1} \binom{n}{j} s_{k+m-j} = \frac{(-1)^n}{n+1} \sum_{p=0}^m (-1)^p \binom{m}{p} (\sigma_{n+p} - \sigma_p), \quad (70)$$

and

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{n-j+1} \binom{n}{j} s_{k-j} = \frac{(-1)^n}{n+1} (\sigma_n - \sigma_0). \quad (71)$$

Proof. Set $a_k = (-1)^k \binom{n}{k} (k+1)^{-1}$ in (2) and apply [1, Theorems 6.3 and 7.9]. $\square$

PROPOSITION 26. *If n is a non-negative integer and x is a complex number, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{n-j+1} \binom{n}{j} x^{k-j} H_{k-j} = \frac{(-1)^n}{n+1} \left((1-x)^n H_n - \sum_{k=1}^n \frac{(1-x)^{n-k}}{k} \right). \quad (72)$$

In particular,

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{n-j+1} \binom{n}{j} H_{k-j} = \frac{(-1)^{n+1}}{n(n+1)}. \quad (73)$$

Proof. Work in (71) with the binomial transform pair

$$s_n = x^n H_n \quad \text{and} \quad \sigma_n = (1-x)^n H_n - \sum_{k=1}^n \frac{(1-x)^{n-k}}{k},$$

which follows from the Knuth-Boyadzhiev identity (41). $\square$

PROPOSITION 27. *If n is a positive integer and r is a non-negative integer such that $r < n$, then*

$$\sum_{k=1}^n \sum_{j=0}^k \frac{(-1)^j}{n-j+1} \binom{n}{j} (k-j)^r H_{k-j} = \frac{(-1)^{n+1}}{n+1} \sum_{k=0}^r \frac{(-1)^k k!}{n-k} \left\{ \begin{matrix} r \\ k \end{matrix} \right\}. \quad (74)$$

Proof. Write $\exp x$ for x in (72) and proceed as in the proof of Proposition 14. $\square$

PROPOSITION 28. *If n is a non-negative integer, then*

$$\begin{aligned} & \sum_{k=1}^n \sum_{j=0}^{k-1} (-1)^j 2^{j-2k} \frac{2(k-j)+1}{(n-j+1)(k-j+1)} \binom{n}{j} C_{k-j} (H_{k-j+1} - O_{k-j+1}) \\ &= (-1)^{n+1} 2^{-2n} \frac{2n+1}{(n+1)^2} C_n (H_{n+1} - O_{n+1}). \end{aligned} \quad (75)$$

Proof. Use the following known result ([1, Prop. 13.22]):

$$\sum_{k=0}^n (-1)^k \binom{n}{k} 2^{-2k} \frac{2k+1}{k+1} C_k (H_{k+1} - O_{k+1}) = -2^{-2n} \frac{2n+1}{n+1} C_n (H_{n+1} - O_{n+1})$$

in (71). $\square$

PROPOSITION 29. *If n is a non-negative integer, then*

$$\sum_{k=1}^n (-1)^k \sum_{j=0}^{k-1} \frac{B_{k-j+1}}{(k-j+1)(n-j+1)} \binom{n}{j} = \frac{(-1)^n}{(n+1)^2} \left((-1)^{n+1} B_{n+1} + \frac{n-1}{2} \right). \quad (76)$$

Proof. Use the identity ([1, Prop. 13.33]):

$$\sum_{k=0}^n \binom{n}{k} \frac{B_{k+1}}{k+1} = (-1)^{n+1} \frac{B_{n+1}}{n+1} - \frac{1}{n+1}$$

in (71). $\square$

PROPOSITION 30. *If n is a non-negative integer, then*

$$\sum_{k=1}^n \sum_{j=0}^{k-1} \frac{(-1)^j}{n-j+1} \binom{n}{j} \binom{2(k-j)}{k-j} = \frac{1}{n+1} \left(\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k} - (-1)^n \right). \quad (77)$$

Proof. It is known that [2, Prop. 6]:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k} = (-1)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k}.$$

Use this result in (71). $\square$

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