Journal of Classical Analysis
Volume 28, Number 2 (2026), 277–291
doi:10.7153/jca-2026-28-18

ON ALMOST STATISTICAL ρ–CONVERGENCE IN MODULAR SPACES

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Abstract. This paper extends the notion of almost statistical ρ-convergence from real or complex sequences to the elements of a modular space equipped with a convex modular. We investigate the relationship between almost statistical ρ-convergence, strong almost p-Cesàro ρ-summability, and the Δ_2-type condition. Furthermore, we establish a Tauberian theorem for almost statistical ρ-convergence in this abstract setting.

1. Introduction

The concept of statistical convergence, which is based on the notion of asymptotic density of subsets of natural numbers, was originally introduced by Steinhaus [16] and Fast [3]. Since its inception, this method of summability has become a significant area of research in mathematical analysis, especially due to its applications in approximation theory, number theory, and Fourier analysis. In the classical sense, a sequence $\{x_k\}$ of real or complex numbers is said to be statistically convergent to L if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}$ has natural density zero, where the natural density of a set $A \subset \mathbb{N}$ is defined as

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \operatorname{card}\{k \leqslant n : k \in A\},$$

provided the limit exists. We refer the reader to [4, 15] for fundamental properties of statistical convergence.

Parallel to the development of statistical convergence, Lorentz [11] introduced the powerful concept of almost convergence by utilizing the theory of Banach limits. A bounded sequence $\{x_k\}$ is said to be almost convergent to a limit L if all its Banach limits coincide, which is equivalent to the condition that the sequence of its shift-means converges uniformly to L:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=j}^{j+n-1} x_k = L$$

uniformly in $j \in \mathbb{N}$. Almost convergence is a stronger condition than the classical Cesàro summability and has been extensively studied in various directions, including its strong version and its relationship with other summability methods.

Mathematics subject classification (2020): 40A35, 46A80, 40A05, 41A36.
Keywords and phrases: Almost convergence, statistical convergence, modular spaces, convex modular, Δ_2-type condition, Tauberian theorem.

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Paper JCA-28-18

Over the past few decades, these convergence notions have been extended from scalar sequences to more abstract structures, such as normed spaces and Banach spaces. However, in many applications – particularly in the study of Orlicz spaces, variable exponent Lebesgue spaces [2], and the modeling of electrorheological fluids – the natural modular need not satisfy the usual triangle inequality or be positively homogeneous. This necessitates the use of the "convex modular" technology, pioneered by Nakano [13] and Musielak [12]. A convex modular ρ acts as a generalized distance measure that does not necessarily satisfy the triangle inequality but provides a rich framework for defining convergence and completeness through the so-called modular spaces X_ρ.

In [14], statistical convergence of sequences was extended to the setting of modular spaces equipped with a convex modular. In the same work, the interplay between statistical ρ-convergence, the statistical ρ-Cauchy condition, and strong Cesàro summability was investigated under the assumption of the Δ_2-type condition. It was shown that the modular framework is sufficiently robust to handle non-normed convergence structures in abstract vector spaces.

Motivated by these recent developments and the increasing interest in approximation theory within modular spaces [7, 8, 10], the present paper aims to bridge the gap between almost convergence and modular spaces. The interest in convex modulars reflects their ability to measure proximity between complex objects where traditional norms are insufficient. For instance, in variable exponent Lebesgue spaces, ρ-convergence provides a natural framework for generalized convergence in mean and for the study of approximation processes in nonhomogeneous function spaces.

In this paper, we introduce and systematically study the notion of almost statistical ρ-convergence in modular spaces X_ρ. The main contribution of this work is threefold. First, we formally define ρ-almost convergence in the modular setting, thereby formulating Lorentz almost convergence in terms of a convex modular. Second, we establish fundamental equivalence results that clarify the relationship between almost statistical ρ-convergence, the almost statistical ρ-Cauchy condition, and strong almost p-Cesàro ρ-summability. Finally, we prove a Tauberian theorem showing that, under a natural difference condition formulated via a growth function associated with the Δ_2-type condition, almost statistical ρ-convergence reduces to genuine ρ-convergence. These results extend classical summability theory to a broader modular framework and provide a theoretical foundation for further applications in functional analysis and approximation theory.

We emphasize that almost statistical ρ-convergence implies statistical ρ-convergence studied in [14]. The essential difference lies in the uniformity with respect to the shift parameter, which reflects the philosophy of Lorentz almost convergence. This uniform block-wise control is crucial for the Tauberian arguments developed in Section 5.

2. Preliminaries

In this section, we provide the necessary background on modular spaces and the Δ_2-type condition, which are essential for our investigation of almost convergence. Throughout this paper, X will denote a real vector space. We use the term "modular

space" throughout, though note that "modular vector space" is sometimes used interchangeably in the literature. For a comprehensive treatment of modular function spaces and fixed point theory in this setting, we refer to [5, 6].

DEFINITION 1. [13] A functional $\rho : X \to [0,\infty]$ is called a convex modular on X if for all $x, y \in X$, the following conditions are satisfied

1. $\rho(x) = 0$ if and only if $x = 0$;
2. $\rho(-x) = \rho(x)$;
3. $\rho(\alpha x + \beta y) \leqslant \alpha\rho(x) + \beta\rho(y)$ for all $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$.

A convex modular ρ defines a modular space X_ρ as follows

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0^+\}.$$

The vector space X_ρ is a subspace of X. On this space, we define the Luxemburg norm by

$$\|x\|_\rho = \inf\{\lambda > 0 : \rho(x/\lambda) \leqslant 1\}.$$

DEFINITION 2. A sequence $\{x_k\}$ in X_ρ is said to be

1. ρ-convergent to $L \in X_\rho$ if $\lim_{k\to\infty} \rho(x_k - L) = 0$.
2. ρ-Cauchy if $\lim_{k,m\to\infty} \rho(x_k - x_m) = 0$.

DEFINITION 3. A sequence $\{x_n\} \subset X_\rho$ is said to be ρ-bounded if

$$\sup_{n\in\mathbb{N}} \rho(x_n) < \infty.$$

DEFINITION 4. We say that a convex modular ρ satisfies the Δ_2-type condition if there exists a constant $M > 0$ such that for every $x \in X_\rho$, the following inequality holds

$$\rho(2x) \leqslant M\rho(x).$$

PROPOSITION 1. [6, 8] *Let ρ be a convex modular satisfying the Δ_2-type condition. Then the following properties hold*

1. *$\rho(x) < \infty$ for all $x \in X_\rho$;*
2. *ρ-convergence is equivalent to the convergence with respect to the Luxemburg norm $\|x\|_\rho$;*
3. *The modular ρ is continuous, i.e., if $x_n \xrightarrow{\rho} x$, then $\rho(x_n) \to \rho(x)$;*
4. *Vector space algebraic operations in X_ρ are ρ-continuous;*
5. *ρ-convergence implies the ρ-Cauchy condition.*

DEFINITION 5. A modular space X_ρ is called ρ-complete if every ρ-Cauchy sequence is ρ-convergent to an element in X_ρ.

For further topological properties of modular spaces, including a modular version of the Goebel–Kirk theorem, we refer to [9].

In the context of almost convergence, we recall the concept of shift-means. For a sequence $\{x_k\}$ in X_ρ, we consider the means

$$t_{n,j} = \frac{1}{n}\sum_{k=j}^{j+n-1} x_k$$

where $n, j = 1,2,\ldots$. The classical Lorentz almost convergence requires that $t_{n,j} \to L$ uniformly in j as $n \to \infty$.

LEMMA 1. *Let ρ be a convex modular satisfying the Δ_2-type condition. Then there exists a constant $C > 0$ such that*

$$\rho(u+v) \leqslant C\big(\rho(u)+\rho(v)\big) \quad \textit{for all } u,v \in X_\rho.$$

Proof. Since ρ satisfies the Δ_2-type condition, there exists $M > 0$ such that $\rho(2x) \leqslant M\rho(x)$ for all $x \in X_\rho$. By convexity of ρ,

$$\rho(u+v) = \rho\left(2\cdot\frac{u+v}{2}\right) \leqslant M\rho\left(\frac{u+v}{2}\right) \leqslant \frac{M}{2}\big(\rho(u)+\rho(v)\big).$$

Thus the assertion holds with $C = M/2$. □

DEFINITION 6. Let ρ be a convex modular on X satisfying the Δ_2-type condition. A sequence $\{x_n\}$ in X_ρ is said to be *ρ-almost convergent* to $x \in X_\rho$ if

$$\lim_{n\to\infty} \rho\left(\frac{1}{n}\sum_{k=j}^{j+n-1} x_k - x\right) = 0 \quad \text{uniformly in } j \in \mathbb{N}.$$

REMARK 1. Definition 6 extends the classical Lorentz notion of almost convergence to modular spaces by replacing the norm with the modular ρ. Uniformity with respect to the shift parameter j reflects the Banach limit philosophy underlying almost convergence.

3. Almost statistical ρ-convergence

In this section, we introduce the concepts of almost statistical ρ-convergence and almost statistical ρ-Cauchy sequences in modular spaces. Throughout this section, ρ denotes a convex modular satisfying the Δ_2-type condition.

Throughout the paper, a set $A \subset \mathbb{N}$ is said to have uniform density zero if

$$\lim_{n\to\infty} \sup_{j\in\mathbb{N}} \frac{1}{n}\operatorname{card}\big(A \cap \{j, j+1, \ldots, j+n-1\}\big) = 0.$$

In this case, we write $f\text{-}\delta(A) = 0$.

DEFINITION 7. Let ρ be a convex modular defined on a vector space X.

(a) A sequence $\{x_n\}$ in X_ρ is said to be *almost statistically ρ-convergent* to $x \in X_\rho$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \sup_{j\in\mathbb{N}} \frac{1}{n} \operatorname{card}\{j \leqslant k \leqslant j+n-1 : \rho(x_k - x) > \varepsilon\} = 0.$$

In this case, we write $x_n \xrightarrow{f\text{-}st\text{-}\rho} x$.

(b) A sequence $\{x_n\}$ in X_ρ is called *almost statistically ρ-Cauchy* if for every $\varepsilon > 0$, there exists a positive integer N such that

$$\lim_{n\to\infty} \sup_{j\in\mathbb{N}} \frac{1}{n} \operatorname{card}\{j \leqslant k \leqslant j+n-1 : \rho(x_k - x_N) > \varepsilon\} = 0.$$

REMARK 2. The above definitions require the condition to hold uniformly for all $j \in \mathbb{N}$, consistent with the classical theory of almost convergence. This ensures that the frequency of elements outside any ε-ball is small in any sufficiently large block of the sequence.

THEOREM 2. (First Equivalence Theorem) *Let ρ be a convex modular satisfying the Δ_2-type condition, and let X_ρ be a ρ-complete modular space. Let*

$$C_0 := \max\{1, C\},$$

where C is the constant appearing in Lemma 1. For a sequence $\{x_n\} \subset X_\rho$, the following statements are equivalent:

1. *$\{x_n\}$ is almost statistically ρ-convergent to some $L \in X_\rho$;*
2. *$\{x_n\}$ is almost statistically ρ-Cauchy;*
3. *There exist a strictly increasing sequence of indices $\{N_m\}$, a sequence $\{r_m\}$ of positive real numbers satisfying $r_m \to 0$, and an element $L \in X_\rho$ such that*

$$x_{N_m} \xrightarrow{\rho} L$$

and

$$f\text{-}\delta\big(\{k \in \mathbb{N} : \rho(x_k - x_{N_m}) > r_m\}\big) = 0$$

for every $m \in \mathbb{N}$.

Proof. Throughout the proof, for $N \in \mathbb{N}$ and $r > 0$, write

$$A(N, r) := \{k \in \mathbb{N} : \rho(x_k - x_N) > r\}.$$

(1) $\Rightarrow$ (2). Suppose that

$$x_n \xrightarrow{f\text{-}st\text{-}\rho} L.$$

Let $\varepsilon > 0$ and put

$$\eta := \frac{\varepsilon}{2C_0}.$$

Define

$$A := \{k \in \mathbb{N} : \rho(x_k - L) > \eta\}.$$

By almost statistical ρ-convergence,

$$f\text{-}\delta(A) = 0.$$

Since A has uniform density zero, its complement is nonempty. Choose $N \in \mathbb{N} \setminus A$. Then

$$\rho(x_N - L) \leqslant \eta.$$

Set

$$B := \{k \in \mathbb{N} : \rho(x_k - x_N) > \varepsilon\}.$$

If $k \notin A$, then

$$\rho(x_k - L) \leqslant \eta.$$

Consequently, Lemma 1 gives

$$\rho(x_k - x_N) \leqslant C_0\big(\rho(x_k - L) + \rho(L - x_N)\big) \leqslant 2C_0\eta = \varepsilon.$$

Thus $k \notin B$, and hence

$$B \subseteq A.$$

It follows that

$$f\text{-}\delta(B) = 0.$$

Therefore, $\{x_n\}$ is almost statistically ρ-Cauchy.

(2)$\Rightarrow$(3). Assume that $\{x_n\}$ is almost statistically ρ-Cauchy. Set

$$r_m := 2^{-m}, \qquad m \in \mathbb{N}.$$

For each $m \in \mathbb{N}$, apply the almost statistical ρ-Cauchy condition with

$$\eta_m := \frac{r_m}{2C_0}.$$

There exists an index $M_m \in \mathbb{N}$ such that

$$f\text{-}\delta\big(A(M_m, \eta_m)\big) = 0.$$

We construct inductively a strictly increasing sequence $\{N_m\}$ satisfying

$$f\text{-}\delta\big(A(N_m, r_m)\big) = 0$$

and

$$N_m \notin \bigcup_{q=1}^{m-1} A(N_q, r_q) \qquad (m \geqslant 2).$$

Since $A(M_1,\eta_1)$ has uniform density zero, its complement contains arbitrarily large integers. Choose $N_1 > M_1$ such that

$$N_1 \notin A(M_1,\eta_1).$$

Then

$$\rho(x_{N_1} - x_{M_1}) \leqslant \eta_1.$$

Now suppose that $N_1,\dots,N_{m-1}$ have been chosen for some $m \geqslant 2$. The finite union

$$D_m := A(M_m,\eta_m) \cup \bigcup_{q=1}^{m-1} A(N_q,r_q)$$

has uniform density zero. Hence its complement contains arbitrarily large integers. We may therefore choose

$$N_m > \max\{N_{m-1},M_m\}$$

such that

$$N_m \notin D_m.$$

It follows that

$$\rho(x_{N_m} - x_{M_m}) \leqslant \eta_m$$

and

$$N_m \notin A(N_q,r_q) \qquad (1 \leqslant q < m).$$

We next verify that

$$f\text{-}\delta\big(A(N_m,r_m)\big) = 0$$

for every $m \in \mathbb{N}$. If

$$k \notin A(M_m,\eta_m),$$

then

$$\rho(x_k - x_{M_m}) \leqslant \eta_m.$$

Since also

$$\rho(x_{N_m} - x_{M_m}) \leqslant \eta_m,$$

Lemma 1 yields

$$\rho(x_k - x_{N_m}) \leqslant C_0\big(\rho(x_k - x_{M_m}) + \rho(x_{M_m} - x_{N_m})\big) \leqslant 2C_0\eta_m = r_m.$$

Therefore,

$$A(N_m,r_m) \subseteq A(M_m,\eta_m),$$

and consequently

$$f\text{-}\delta\big(A(N_m,r_m)\big) = 0.$$

It remains to prove that $\{x_{N_m}\}$ is ρ-Cauchy. If $p > m$, then, by the construction of N_p,

$$N_p \notin A(N_m,r_m).$$

Therefore,

$$\rho(x_{N_p} - x_{N_m}) \leqslant r_m = 2^{-m}.$$

Since $r_m \to 0$, the sequence $\{x_{N_m}\}$ is ρ-Cauchy. By the ρ-completeness of X_ρ, there exists $L \in X_\rho$ such that

$$x_{N_m} \xrightarrow{\rho} L.$$

This proves (3).

(3)$\Rightarrow$(1). Assume that there exist $\{N_m\}$, $\{r_m\}$, and $L \in X_\rho$ satisfying (3). Fix $\varepsilon > 0$. Since

$$r_m \longrightarrow 0 \qquad \text{and} \qquad \rho(x_{N_m} - L) \longrightarrow 0,$$

we may choose m sufficiently large that

$$C_0\big(r_m + \rho(x_{N_m} - L)\big) < \varepsilon.$$

Set

$$A_m := \{k \in \mathbb{N} : \rho(x_k - x_{N_m}) > r_m\}.$$

By assumption,

$$f\text{-}\delta(A_m) = 0.$$

If $k \notin A_m$, then

$$\rho(x_k - x_{N_m}) \leqslant r_m.$$

Hence, by Lemma 1,

$$\rho(x_k - L) \leqslant C_0\big(\rho(x_k - x_{N_m}) + \rho(x_{N_m} - L)\big) < \varepsilon.$$

Therefore,

$$\{k \in \mathbb{N} : \rho(x_k - L) \geqslant \varepsilon\} \subseteq A_m.$$

It follows that

$$f\text{-}\delta\big(\{k \in \mathbb{N} : \rho(x_k - L) \geqslant \varepsilon\}\big) = 0.$$

Since $\varepsilon > 0$ was arbitrary,

$$x_n \xrightarrow{f\text{-}st\text{-}\rho} L.$$

Thus (1), (2), and (3) are equivalent. □

REMARK 3. The almost statistical ρ-limit is unique. Indeed, if $x_n \xrightarrow{f\text{-}st\text{-}\rho} L_1$ and $x_n \xrightarrow{f\text{-}st\text{-}\rho} L_2$, then for every $\varepsilon > 0$, the sets

$$A_i := \{k : \rho(x_k - L_i) > \varepsilon\}, \quad i = 1, 2,$$

have uniform density zero. Hence their union also has uniform density zero, so its complement is nonempty. Pick k outside this union. Then

$$\rho(L_1 - L_2) \leqslant C_0\big(\rho(L_1 - x_k) + \rho(x_k - L_2)\big) \leqslant 2C_0\varepsilon.$$

Letting $\varepsilon \to 0$ yields $\rho(L_1 - L_2) = 0$, hence $L_1 = L_2$.

4. Strong almost summability

In this section, we investigate the relationship between almost statistical convergence and strong almost p-Cesàro ρ-summability. The following result is a modular counterpart of the classical relationship between statistical convergence and strong p-Cesàro summability; see [1].

DEFINITION 8. Let ρ be a convex modular defined on a vector space X and p be a positive real number. A sequence $\{x_n\}$ in X_ρ is said to be *strongly almost p-Cesàro ρ-summable* to $x \in X_\rho$ if

$$\lim_{n\to\infty} \sup_{j\in\mathbb{N}} \frac{1}{n} \sum_{k=j}^{j+n-1} [\rho(x_k - x)]^p = 0.$$

THEOREM 3. (Second Equivalence Theorem (Strong almost summability)) *Let ρ be a convex modular satisfying the Δ_2-type condition and $p \in \mathbb{R}$ with $0 < p < \infty$.*

(i) If a sequence $\{x_n\}$ in X_ρ is strongly almost p-Cesàro ρ-summable to x, then it is almost statistically ρ-convergent to x.

(ii) If a ρ-bounded sequence $\{x_n\}$ in X_ρ is almost statistically ρ-convergent to x, then it is strongly almost p-Cesàro ρ-summable to x.

Proof. (i) For any $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{n} \sum_{k=j}^{j+n-1} [\rho(x_k - x)]^p &= \frac{1}{n} \left(\sum_{\substack{k=j \\ \rho(x_k-x)>\varepsilon}}^{j+n-1} [\rho(x_k - x)]^p + \sum_{\substack{k=j \\ \rho(x_k-x)\leqslant\varepsilon}}^{j+n-1} [\rho(x_k - x)]^p \right) \\ &\geqslant \frac{1}{n} \operatorname{card}\{j \leqslant k \leqslant j+n-1 : \rho(x_k - x) > \varepsilon\} \varepsilon^p.\end{aligned}$$

Taking supremum over j and letting $n \to \infty$, the left-hand side tends to zero uniformly in j, hence the cardinality term also vanishes uniformly, proving (i).

(ii) Suppose $\{x_n\}$ is ρ-bounded and $x_n \xrightarrow{f\text{-}st\text{-}\rho} x$. Let $C > 0$ be the constant from Lemma 1. Since $\{x_n\}$ is ρ-bounded and $\rho(x) < \infty$, there exists $Q > 0$ such that

$$\rho(x_n - x) \leqslant C\big(\rho(x_n) + \rho(x)\big) \leqslant Q$$

for every $n \in \mathbb{N}$. For $\varepsilon > 0$, select n_ε such that for $n > n_\varepsilon$ and for each $j \in \mathbb{N}$,

$$\frac{1}{n} \operatorname{card}\{j \leqslant k \leqslant j+n-1 : \rho(x_k - x) > (\varepsilon/2)^{1/p}\} < \frac{\varepsilon}{2Q^p}.$$

Then,

$$\frac{1}{n}\sum_{k=j}^{j+n-1}[\rho(x_k-x)]^p = \frac{1}{n}\left(\sum_{k\in S_{n,j}}[\rho(x_k-x)]^p + \sum_{\substack{j\leqslant k\leqslant j+n-1\\ k\notin S_{n,j}}}[\rho(x_k-x)]^p\right)$$
$$< \frac{1}{n}\left(\frac{n\varepsilon}{2Q^p}\right)Q^p + \frac{\varepsilon}{2} = \varepsilon,$$

where $S_{n,j} = \{j \leqslant k \leqslant j+n-1 : \rho(x_k-x) > (\varepsilon/2)^{1/p}\}$. This uniform estimate proves (ii). □

5. Tauberian properties

In this section, we establish a Tauberian theorem for almost statistical ρ-convergence. Under an order-type condition on the forward differences, we show that almost statistical ρ-convergence implies ρ-convergence.

THEOREM 4. (Tauberian Theorem) *Let ρ be a convex modular on X satisfying the Δ_2-type condition, and let $\{x_n\} \subset X_\rho$ be almost statistically ρ-convergent to $x \in X_\rho$.*

Let $C > 0$ be the constant from Lemma 1, and set

$$C_0 := \max\{1, C\}, \qquad \alpha := \log_2 C_0, \qquad w(m) := m^{1+\alpha}, \quad m \in \mathbb{N}.$$

If

$$\sup_{k\in\mathbb{N}} w(k)\rho(\Delta x_k) < \infty, \tag{1}$$

where

$$\Delta x_k := x_k - x_{k+1},$$

then

$$x_n \xrightarrow{\rho} x.$$

Proof. For each $m \in \mathbb{N}$, set

$$r_m := 2^{-m} \qquad \text{and} \qquad \eta_m := \frac{r_m}{2C_0}.$$

Since

$$x_n \xrightarrow{f\text{-}st\text{-}\rho} x,$$

the set

$$E_m := \{k \in \mathbb{N} : \rho(x_k - x) > \eta_m\}$$

has uniform density zero.

The complement of E_m contains arbitrarily large integers. Hence we may choose a strictly increasing sequence of indices $\{N_m\}$ such that

$$N_m \notin E_m.$$

Thus

$$\rho(x_{N_m} - x) \leqslant \eta_m,$$

and consequently

$$x_{N_m} \xrightarrow{\rho} x.$$

Define

$$A_m := \{k \in \mathbb{N} : \rho(x_k - x_{N_m}) > r_m\}.$$

We claim that

$$A_m \subseteq E_m.$$

Indeed, if $k \notin E_m$, then

$$\rho(x_k - x) \leqslant \eta_m$$

and

$$\rho(x_{N_m} - x) \leqslant \eta_m.$$

By Lemma 1,

$$\rho(x_k - x_{N_m}) \leqslant C_0\big(\rho(x_k - x) + \rho(x - x_{N_m})\big) \leqslant 2C_0\eta_m = r_m.$$

Hence $k \notin A_m$, proving the claimed inclusion. Therefore,

$$f\text{-}\delta(A_m) = 0 \qquad \text{for every } m \in \mathbb{N}.$$

Since A_m has uniform density zero, for every $m \in \mathbb{N}$ there exists an integer $h_m \geqslant 1$ such that

$$\frac{1}{h_m}\operatorname{card}\big(A_m \cap \{j, j+1, \ldots, j+h_m-1\}\big) < \frac{1}{2} \tag{2}$$

for every $j \in \mathbb{N}$. In particular, no interval of h_m consecutive integers is contained entirely in A_m.

Choose a strictly increasing sequence of positive integers $\{K_m\}$ such that

$$K_1 \geqslant 2h_1, \qquad K_{m+1} > \max\{K_m, (m+2)h_{m+1}\} \quad \text{for } m \geqslant 1.$$

Then $K_m \geqslant (m+1)h_m$ for all m, and for each $k \geqslant K_1$ there is a unique m with $K_m \leqslant k < K_{m+1}$.

For each such k, let $m = m(k)$ be that unique integer. By (2), the interval

$$\{k - h_m + 1, \ldots, k\}$$

contains at least one integer outside A_m. Choose such an integer and denote it by q_k. Thus

$$k - h_m + 1 \leqslant q_k \leqslant k$$

and

$$q_k \notin A_m.$$

Set

$$p_k := k - q_k.$$

Then

$$0 \leqslant p_k < h_m.$$

Moreover, by the choice of K_m,

$$q_k \geqslant k - h_m + 1 \geqslant K_m - h_m + 1 \geqslant mh_m + 1.$$

Therefore,

$$0 \leqslant \frac{p_k}{q_k} \leqslant \frac{h_m}{mh_m + 1} \leqslant \frac{1}{m}. \tag{3}$$

Since $m(k) \to \infty$ as $k \to \infty$, it follows that

$$\frac{p_k}{q_k} \longrightarrow 0.$$

Since $q_k \notin A_m$, we have

$$\rho(x_{q_k} - x_{N_m}) \leqslant r_m.$$

Hence

$$\rho(x_{q_k} - x) \leqslant C_0\big(\rho(x_{q_k} - x_{N_m}) + \rho(x_{N_m} - x)\big) \leqslant C_0(r_m + \eta_m).$$

Because $r_m \to 0$ and $\eta_m \to 0$, we obtain

$$\rho(x_{q_k} - x) \longrightarrow 0. \tag{4}$$

We next estimate $\rho(x_k - x_{q_k})$. We first note that for every $s \in \mathbb{N}$ and every $u_1, \ldots, u_s \in X_\rho$,

$$\rho\left(\sum_{i=1}^{s} u_i\right) \leqslant C_0 s^\alpha \sum_{i=1}^{s} \rho(u_i). \tag{5}$$

Indeed, by padding the sum with zero terms up to the next power of two and iterating Lemma 1, we obtain

$$\rho\left(\sum_{i=1}^{s} u_i\right) \leqslant C_0^{\lceil \log_2 s \rceil} \sum_{i=1}^{s} \rho(u_i).$$

Since

$$\lceil \log_2 s \rceil < \log_2 s + 1,$$

we have

$$C_0^{\lceil \log_2 s \rceil} \leqslant C_0 s^{\log_2 C_0} = C_0 s^\alpha,$$

which proves (5).

By (1), there exists $Q > 0$ such that

$$\rho(\Delta x_i) \leqslant \frac{Q}{w(i)} = \frac{Q}{i^{1+\alpha}}$$

for every $i \in \mathbb{N}$.

If $p_k = 0$, then $x_k = x_{q_k}$. Suppose that $p_k \geqslant 1$. Since

$$x_{q_k} - x_k = \sum_{i=q_k}^{k-1} \Delta x_i,$$

inequality (5) gives

$$\rho(x_k - x_{q_k}) \leqslant C_0 p_k^{\alpha} \sum_{i=q_k}^{k-1} \rho(\Delta x_i).$$

Because $i \geqslant q_k$ and w is increasing,

$$\sum_{i=q_k}^{k-1} \rho(\Delta x_i) \leqslant \sum_{i=q_k}^{k-1} \frac{Q}{w(i)} \leqslant \frac{Qp_k}{w(q_k)}.$$

Consequently,

$$\rho(x_k - x_{q_k}) \leqslant C_0 Q \frac{p_k^{1+\alpha}}{q_k^{1+\alpha}} = C_0 Q \left(\frac{p_k}{q_k}\right)^{1+\alpha}.$$

By (3),

$$\rho(x_k - x_{q_k}) \longrightarrow 0. \tag{6}$$

Finally, Lemma 1, together with (4) and (6), yields

$$\rho(x_k - x) \leqslant C_0 \big(\rho(x_k - x_{q_k}) + \rho(x_{q_k} - x)\big) \longrightarrow 0.$$

Therefore,

$$x_k \xrightarrow{\rho} x,$$

as required. □

REMARK 4. The normalization $C_0 := \max\{1, C\}$ is introduced only to ensure that $\alpha = \log_2 C_0 \geqslant 0$. No additional restriction is imposed on the modular.

REMARK 5. The Tauberian condition (1) depends only on the local behavior of successive differences. The uniform density assumption ensures that every sufficiently long block contains an index at which the sequence is modularly close to the limit. The difference condition then transfers this local closeness to all sufficiently large terms of the sequence.

6. Conclusion

In this paper, we have extended the theory of almost convergence to the framework of modular spaces. By utilizing the properties of convex modulars and the Δ_2-type condition, we have introduced the concepts of almost statistical ρ-convergence and almost statistical ρ-Cauchy sequences. Our investigations have led to several fundamental results.

First, we established the First Equivalence Theorem, which shows that almost statistical ρ-convergence is equivalent to the almost statistical ρ-Cauchy condition. We also characterized these notions by the existence of a ρ-convergent subsequence $\{x_{N_m}\}$ and a sequence of positive radii $r_m \to 0$ such that, for every m, the set

$$\{k \in \mathbb{N} : \rho(x_k - x_{N_m}) > r_m\}$$

has uniform density zero.

Second, we demonstrated the relationship between strong almost p-Cesàro ρ-summability and almost statistical ρ-convergence. We showed that for ρ-bounded sequences, these two notions of summability are equivalent under the Δ_2-type condition. This result provides a modular analog to the classical results of Lorentz.

Third, we established a Tauberian Theorem, which provides an order-type condition involving the growth function w of the modular. This theorem allows for the recovery of classical ρ-convergence from almost statistical ρ-convergence, further strengthening the connection between these summability methods.

The results presented in this paper broaden the application area of almost convergence, particularly to settings where the modular need not satisfy the usual triangle inequality, such as variable exponent Lebesgue spaces [2]. Future research may focus on relaxing the Δ_2-type assumption or investigating lacunary versions of almost convergence within modular spaces.

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(Received February 1, 2026)

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