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APPROXIMATION FOR A NEW OPERATOR OF DURRMEYER TYPE

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Abstract. In the present research, we provide a new type of operator R_n, different from the actual exponential-type operator $\widehat{R}_n$ connected with $x^{4/3}$. Such operators are similar to exponential type operators, but do not satisfy the partial differential equation. We establish pointwise convergence using the characteristic function, error estimation in terms of modulus of continuity of second and first orders and a quantitative Voronovskaja type result for exponential type functions. Finally we estimate the difference estimate between the two operators.

1. Introduction

Several new operators have been appropriately modified and their approximation properties have been discussed in the last few years, we mention some of the works as [7], [10], [11] and [13] etc.

Recently Gupta and Anjali [9] introduced the operators of exponential-type connected to $x^{4/3}$, which are defined by

$$\begin{aligned}(\widehat{R}_n f)(x) &= \int_0^\infty \widehat{k}_n(x,t) f(t) dt \\ &= e^{-\frac{3n}{2}x^{2/3}} \left[f(0) + \sum_{m=1}^{\infty} \frac{(3n)^{3m}}{2^m\, m!\, \Gamma(2m)} \int_0^\infty \exp\left(\frac{-3nt}{x^{1/3}}\right) t^{2m-1} f(t)\, dt \right]. \end{aligned} \tag{1}$$

The operators $\widehat{R}_n$ defined above are of exponential-type and connected to $x^{4/3}$ as the kernel $k_n(x,t)$ of these operators satisfy the following partial differential equation

$$\frac{\partial}{\partial x}\widehat{k}_n(x,t) = \frac{n(t-x)}{x^{4/3}}\widehat{k}_n(x,t). \tag{2}$$

Here we consider a new operator for $x \in (0,\infty)$, which takes the following form:

$$\begin{aligned}(R_n f)(x) &= \int_0^\infty k_n(x,t) f(t) dt \\ &= \frac{3n}{x^{1/3}} \sum_{k=0}^{\infty} s_k\left(\frac{3n}{2}x^{2/3}\right) \int_0^\infty s_{2k}\left(\frac{3nt}{x^{1/3}}\right) f(t) dt, \end{aligned} \tag{3}$$

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where $s_k(y) = e^{-y}y^k/k!$ and this can be seen as the Durrmeyer type operators (see [8], [12]). The kernel $k_n(x,t)$ in (3) can also be expressed as

$$k_n(x,t) = \frac{3n}{x^{1/3}} \exp\left(-\frac{3n}{2}x^{2/3} - \frac{3nt}{x^{1/3}}\right) \; {}_0F_2\left(-;\frac{1}{2},1;\frac{(3n)^3t^2}{2}\right),$$

where ${}_0F_2$ is the generalized hypergeometric function. These operators are not of the exponential-type as the kernel of these operators do not satisfy any partial differential equation (2).

Further, the kernel $k_n(x,t)$ of the operators R_n satisfy different partial differential equation, which can be seen as below.

$$\log k_n(x,t) = \log(3n) - \frac{1}{3}\log x - \frac{3n}{2}x^{2/3} - \frac{3nt}{x^{1/3}} + \log\, {}_0F_2\left(-;\frac{1}{2},1;\frac{(3n)^3t^2}{2}\right).$$

Thus

$$\frac{\partial}{\partial x}\log k_n(x,t) = -\frac{1}{3x} - \frac{n}{x^{1/3}} + \frac{nt}{x^{4/3}}.$$

Multiplying by $k_n(x,t)$, we get

$$\frac{\partial}{\partial x}k_n(x,t) = \left[\frac{n(t-x)}{x^{4/3}} - \frac{1}{3x}\right] k_n(x,t).$$

REMARK 1. Using the identity ${}_0F_2\left(-;\frac{1}{2},1;y\right) = \sum_{k=0}^{\infty} \frac{1}{\left(\frac{1}{2}\right)_k (1)_k}\frac{y^k}{k!}$ with $\left(\frac{1}{2}\right)_k = \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$ and the Gamma duplication formula $\Gamma\left(k+\frac{1}{2}\right) = \frac{(2k)!}{4^k k!}\sqrt{\pi}$, the kernel $k_n(x,t)$ can be expressed alternatively as

$$k_n(x,t) = \frac{3n}{x^{1/3}} \exp\left(-\frac{3n}{2}x^{2/3} - \frac{3nt}{x^{1/3}}\right) \cosh(\sqrt{2}(3n)^{3/2}|t|).$$

Also, by using the relation ${}_0F_2\left(-;\frac{1}{2},1;z\right) = \frac{2}{\pi}\int_0^{\pi/2} I_0\left(2\sqrt{z}\sin\theta\right)d\theta$, this kernel can be expressed as

$$k_n(x,t) = \frac{6n}{\pi x^{1/3}} \exp\left(-\frac{3n}{2}x^{2/3} - \frac{3nt}{x^{1/3}}\right) \int_0^{\pi/2} I_0\left(\sqrt{2}(3n)^{3/2}t\sin\theta\right)d\theta,$$

where I_0 denotes the modified Bessel function of the first kind of the zeroth order.

Unlike the operators in (1), which reproduce linear functions, the operators R_n only reproduce the constant functions. In this paper, we establish certain direct results expressed in terms of the modulus of continuity. Also a convergence estimate is established for exponential type modulus of continuity.

2. Estimation of moments

LEMMA 1. *For $x \in (0,\infty)$ and a real parameter $A < \frac{3n}{x^{1/3}}$, we have*

$$(R_n \exp_A)(x) = \left(\frac{3n}{3n - Ax^{1/3}}\right) \exp\left(\frac{3nx^{2/3}}{2}\left\{\frac{(3n)^2}{(3n - Ax^{1/3})^2} - 1\right\}\right), \tag{4}$$

where $\exp_A(t) = e^{At}$. Also, if $(R_n e_m)(x)$; $e_m(t) = t^m$, $m = 0,1,2,\ldots$, then coefficient of $\frac{A^m}{m!}$ in the expansion of (4) will provide us the m-th order moment. Few moments are given by

$$\begin{aligned}
(R_n e_0)(x) &= 1\\
(R_n e_1)(x) &= x + \frac{x^{1/3}}{3n}\\
(R_n e_2)(x) &= x^2 + \frac{2x^{2/3} + 15nx^{4/3}}{9n^2}\\
(R_n e_3)(x) &= x^3 + \frac{2x + 27nx^{5/3} + 36n^2x^{7/3}}{9n^3}\\
(R_n e_4)(x) &= x^4 + \frac{8x^{4/3} + 168nx^2 + 369n^2x^{8/3} + 198n^3x^{10/3}}{27n^4}.
\end{aligned}$$

Proof follows using simple calculations on (4).

LEMMA 2. *If $\mu_{n,m}(x) = (R_n(e_1 - e_0 x)^m)(x)$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ (the set of non-negative integers), then we have*

$$\mu_{n,m}(x) = \left[\frac{\partial^m}{\partial A^m}\left(\frac{3n}{3n - Ax^{1/3}}\right) \exp\left(\frac{3nx^{2/3}}{2}\left\{\frac{(3n)^2}{(3n - Ax^{1/3})^2} - 1\right\} - Ax\right)\right]_{A=0}.$$

By basic computations, the few central moments are

$$\begin{aligned}
\mu_{n,0}(x) &= 1\\
\mu_{n,1}(x) &= \frac{x^{1/3}}{3n}\\
\mu_{n,2}(x) &= \frac{2x^{2/3} + 9nx^{4/3}}{9n^2}\\
\mu_{n,3}(x) &= \frac{2x + 21nx^{5/3}}{9n^3}\\
\mu_{n,4}(x) &= \frac{8x^{4/3} + 144nx^2 + 81n^2x^{8/3}}{27n^4}.
\end{aligned}$$

3. Convergence estimations

Let us consider $C_B(0,\infty)$ as the class of all real-valued functions that are continuous and bounded on interval $(0,\infty)$.

In this section, we establish point-wise convergence estimate, using the concept of characteristic function; error estimation in terms of modulus of continuity and a quantitative asymptotic formula in terms of modulus of continuity connected with exponential functions.

THEOREM 1. *For $f \in C_B(0,\infty)$, we have*

$$\lim_{n\to\infty} (R_{rn} f(nt)) \left(\frac{x}{n}\right) = f(x),$$

and

$$\lim_{n\to\infty} \left(R_r f\left(\frac{t}{n}\right)\right)(nx) = f(x).$$

Proof. By applying Lemma 1 for each $r \geqslant 1$, $s \in R$ and $x > 0$ the limit of characteristic functions are given by

$$\lim_{n\to\infty} (R_{rn} \exp_{isn}) \left(\frac{x}{n}\right) = e^{isx} = (Id \exp_{is})(x)$$

and

$$\lim_{n\to\infty} (R_n \exp_{is/n})(nx) = e^{isx} = (Id \exp_{is})(x),$$

where Id denotes the identity operator. According to [2, Th. 1.1] and the works referenced therein, the result follows. □

We denote

$$C_B^2(0,\infty) = \{h \in C_B(0,\infty) : h', h'' \in C_B(0,\infty)\}.$$

For any $g \in C_B(0,\infty)$ and $\tau > 0$, the Peetre's K-functional is given by,

$$K_2(g,\tau) = \inf\{\|g-h\| + \tau\|h''\| : h \in C_B^2(0,\infty)\}.$$

It follows from [5], there exists an absolute constant $c_1 > 0$, for which

$$K_2(g,\tau) \leqslant c_1 \omega_2(g;\sqrt{\tau}), \tag{5}$$

with the second-order modulus of continuity expressed by

$$\omega_2(g,\sqrt{\tau}) = \sup_{\substack{0<h\leqslant\sqrt{\tau} \\ x\in(0,\infty)}} |g(x+2h) - 2g(x+h) + g(x)|.$$

THEOREM 2. *For $f \in C_B(0,\infty)$ and $x > 0$, there exists an absolute constant c_1 independent of f, x, n, for which*

$$|(R_n f)(x) - f(x)| \leqslant c_1\ \omega_2\left(f, \sqrt{\frac{x^{2/3} + 3nx^{4/3}}{3n^2}}\right) + \omega\left(f, \frac{x^{1/3}}{3n}\right).$$

Proof. We start by considering the following operators, which reproduces the linear functions

$$(\overline{R}_n f)(x) = (R_n f)(x) + f(x) - f\left(x + \frac{x^{1/3}}{3n}\right). \tag{6}$$

Let $g \in C_B^2(0,\infty)$. By Taylor's expansion, we can write

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-\eta)g''(\eta)d\eta.$$

Consequently,

$$\begin{aligned}(\overline{R}_n g)(x) - g(x) &= g'(x)(\overline{R}_n(t-x))(x) + \left(\overline{R}_n \int_x^t (t-\eta)g''(\eta)d\eta\right)(x)\\ &= \left(\overline{R}_n \int_x^t (t-\eta)g''(\eta)d\eta\right)(x)\\ &= \left(R_n \int_x^t (t-\eta)\, g''(\eta)d\eta\right)(x) - \int_x^{x+\frac{x^{1/3}}{3n}} \left(x + \frac{x^{1/3}}{3n} - \eta\right) g''(\eta)d\eta.\end{aligned}$$

Therefore,

$$|(\overline{R}_n g)(x) - g(x)| \leqslant \left(R_n \left|\int_x^t (t-\eta)\, g''(\eta)d\eta\right|\right)(x) + \left|\int_x^{x+\frac{x^{1/3}}{3n}} \left(x + \frac{x^{1/3}}{3n} - \eta\right) g''(\eta)d\eta\right|.$$

Obviously, we have

$$\left|\int_x^t (t-\eta)\, g''(\eta)d\eta\right| \leqslant (t-x)^2 \|g''\|$$

and

$$\left|\int_x^{x+\frac{x^{1/3}}{3n}} \left(x + \frac{x^{1/3}}{3n} - \eta\right) g''(\eta)d\eta\right| \leqslant \left(\frac{x^{1/3}}{3n}\right)^2 \|g''\|,$$

therefore, it follows that

$$\begin{aligned}|(\overline{R}_n g)(x) - g(x)| &\leqslant \left(\mu_{n,2}(x) + \left(\frac{x^{1/3}}{3n}\right)^2\right)\|g''\|\\ &= \left(\frac{x^{2/3} + 3nx^{4/3}}{3n^2}\right)\|g''\|. \end{aligned}\tag{7}$$

Using (6), we have

$$|(R_nf)(x)-f(x)| \leqslant |(\overline{R}_n(f-g))(x)|+|(f-g)(x)| \\ +|(\overline{R}_ng)(x)-g(x)|+\left|f\left(x+\frac{x^{1/3}}{3n}\right)-f(x)\right|.$$

Since, $|(\overline{R}_nf)(x)| \leqslant 3\|f\|$, it follows from (7) that

$$|(R_nf)(x)-f(x)| \leqslant 4\|f-g\|+\left(\frac{x^{2/3}+3nx^{4/3}}{3n^2}\right)\|g''\|+\omega\left(f,\frac{x^{1/3}}{3n}\right). \tag{8}$$

Taking the infimum over $g \in C_B^2(0,\infty)$ on the right-hand side of (8) yields

$$|(R_nf)(x)-f(x)| \leqslant 4K_2\left(f,\frac{x^{2/3}+3nx^{4/3}}{3n^2}\right)+\omega\left(f,\frac{x^{1/3}}{3n}\right),$$

where K_2 is Peetre's K-functional. It follows from (5) that

$$|(R_nf)(x)-f(x)| \leqslant c_1\omega_2\left(f,\sqrt{\frac{x^{2/3}+3nx^{4/3}}{3n^2}}\right)+\omega\left(f,\frac{x^{1/3}}{3n}\right).$$

This completes the proof of the theorem. □

A Korovkin-type theorem for exponential functions has been produced in [4] and generalized by Altomare-Campiti in [3]. Holhoş [14] added to the work of [3] and obtained an estimation for the rate of convergence of the operators satisfying the general Korovkin-type result for exponential functions.

Let $\widehat{C}(0,\infty)$ denote the space of all continuous real-valued functions $f(t)$ that have a finite limit as $t \to \infty$. Next, we consider the following moduli:

$$\widehat{\omega}(f,\xi)=\sup_{u,v>0}\{|f(u)-f(v)| : |e^{-u}-e^{-v}| \leqslant \xi\}, \qquad \xi \geqslant 0.$$

THEOREM 3. *If f, f'' belong to $\widehat{C}(0,\infty)$, then for $x \in (0,\infty)$, the following inequality exists:*

$$\left|n[(R_nf)(x)-f(x)]-\frac{x^{1/3}}{3}f'(x)-\frac{x^{4/3}}{2}f''(x)\right| \\ \leqslant \frac{x^{2/3}}{9n}|f''(x)|+2\widehat{\omega}\left(f'',\frac{1}{\sqrt{n}}\right)\left[\frac{2x^{2/3}+9nx^{4/3}}{9n}+\left(\frac{8x^{4/3}+144nx^2+81n^2x^{8/3}}{27n^2}\right)^{1/2}\right. \\ \left.\left[n^2\left(R_n\left(e^{-x}-e^{-t}\right)^4\right)(x)\right]^{1/2}\right].$$

Proof. Applying Taylor's expansion, we find a point δ in the interval $[x,t]$ such that

$$f(t) = \sum_{k=0}^{2} \frac{(t-x)^k}{k!} f^{(k)}(x) + \varepsilon(t,x)(e_1 - xe_0)^2, \tag{9}$$

where

$$\varepsilon(t,x) := \frac{f''(\delta) - f''(x)}{2}.$$

Application of the operator R_n to (9) and using Lemma 2, we have the following

$$\begin{aligned} &\left| n\left[(R_n f)(x) - f(x)\right] - \frac{x^{1/3}}{3} f'(x) - \frac{x^{4/3}}{2} f''(x) \right| \\ &\leqslant \frac{x^{2/3}}{9n} |f''(x)| + n\left(R_n\left|\varepsilon(t,x)\right|(e_1 - e_0 x)^2\right)(x). \end{aligned}$$

Using the inequality due to Acar et al. [1, pp. 6] given by

$$|\varepsilon(t,x)| \leqslant 2\left(1 + \frac{1}{\xi^2}\left(\frac{1}{e^x} - \frac{1}{e^t}\right)^2\right) \widehat{\omega}\left(f'', \xi\right)$$

and applying Cauchy-Schwarz inequality by choosing $\xi = \frac{1}{\sqrt{n}}$, we get

$$\begin{aligned} &n\left(R_n\left|\varepsilon(t,x)\right|(e_1 - xe_0)^2\right)(x) \\ &\leqslant 2\widehat{\omega}\left(f'', n^{-1/2}\right)\left[n\mu_{n,2}(x) + \sqrt{n^2\left(R_n\left(e^{-x} - e^{-t}\right)^4\right)(x)}\sqrt{n^2\mu_{n,4}(x)}\right]. \end{aligned}$$

The required result follows by using Lemma 2. This completes the proof of theorem. □

REMARK 2. The convergence of the operator R_n in the preceding theorem takes place for n large enough. Further, for $A = -1, -2, -3, -4$ in (4), we get

$$\begin{aligned} &\lim_{n\to\infty} n^2\left(R_n\left(e^{-x} - e^{-t}\right)^4\right)(x) \\ &= \lim_{n\to\infty} n^2\Big[e^{-4x} - 4e^{-3x}(R_n e^{-t})(x) + 6e^{-2x}(R_n e^{-2t})(x) \\ &\quad -4e^{-x}(R_n e^{-3t})(x) + (R_n e^{-4t})(x)\Big] \\ &= 3e^{-4x}x^{8/3}. \end{aligned}$$

4. Difference between $\widehat{R}_n$ and R_n

In the following result, we estimate the difference between the two operators.

THEOREM 4. *Let $f^{(m)} \in C_B(0,\infty)$, $m \in \{0,1,2\}$ and $x > 0$, then we have*

$$|((\widehat{R}_n - R_n)f)(x)| = \left(\frac{x^{4/3}}{3n} + \frac{x^{2/3}}{18n^2}\right)||f''|| + \omega\left(f, \frac{x^{1/3}}{3n}\right),$$

where $||f|| = \sup\{|f(x)| : x \in (0,\infty)\} < \infty$.

Proof. The operators (1) can be expressed as

$$(\widehat{R}_n f)(x) = \sum_{k=0}^{\infty} s_k\left(\frac{3n}{2}x^{2/3}\right) H_{n,k}(f),$$

where

$$H_{n,k}(f) = \frac{3n}{x^{1/3}} \int_0^{\infty} s_{2k-1}\left(\frac{3nt}{x^{1/3}}\right) f(t)dt, \quad H_{n,0}(f) = f(0).$$

Obviously, we get

$$H_{n,k}(e_r) = \frac{(2k+r-1)!}{(2k-1)!} \frac{x^{r/3}}{(3n)^r}, \quad \mu_2^{H_{n,k}} = H_{n,k}(e_2) - [H_{n,k}(e_1)]^2 = \frac{2kx^{2/3}}{9n^2}.$$

In a similar manner the operators (3) can be expressed as

$$(R_n f)(x) = \sum_{k=0}^{\infty} s_k\left(\frac{3n}{2}x^{2/3}\right) G_{n,k}(f),$$

where

$$G_{n,k}(f) = \frac{3n}{x^{1/3}} \int_0^{\infty} s_{2k}\left(\frac{3nt}{x^{1/3}}\right) f(t)dt.$$

By simple computation, we get

$$G_{n,k}(e_r) = \frac{(2k+r)!}{(2k)!} \frac{x^{r/3}}{(3n)^r}, \quad \mu_2^{G_{n,k}} = G_{n,k}(e_2) - [G_{n,k}(e_1)]^2 = \frac{(2k+1)x^{2/3}}{9n^2}.$$

Therefore

$$A(x) = \frac{1}{2}\sum_{k=0}^{\infty} s_k\left(\frac{3n}{2}x^{2/3}\right)\left(\mu_2^{H_{n,k}} + \mu_2^{G_{n,k}}\right) = \frac{x^{4/3}}{3n} + \frac{x^{2/3}}{18n^2}.$$

Next

$$\delta^2(x) = \sum_{k=0}^{\infty} s_k\left(\frac{3n}{2}x^{2/3}\right)\left(H_{n,k}(e_1) - H_{n,k}(e_1)\right)^2 = \frac{x^{2/3}}{9n^2}.$$

Following [6, Prop.1], the result follows. □

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