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GEOMETRIC AND MONOTONIC PROPERTIES OF THE BESSEL FUNCTIONS

JUGAL KISHORE PRAJAPAT

Abstract. In this paper, we obtain sufficient conditions for the parameters of the generalized Bessel functions to be univalent, convex, and starlike in the open unit disk. We also establish an inclusion relation and observe that the main results improve certain existing results.

1. Introduction

The Bessel functions are used in various fields, including mathematics, engineering, and applied sciences. They play a significant role in quantum physics, scattering theory, wave propagation, and related fields (see [5, 6, 10, 11, 12] and the references therein). Furthermore, it is important to note that there has been great interest in dealing with various inequalities and geometric properties like the univalency, starlikeness, convexity, and monotonicity properties (see [1, 2, 3, 4, 7, 8, 13, 14, 15] and the references therein).

The generalized Bessel function $w_{p,b,c}\,(p,b\in\mathbb{R},\ c\in\mathbb{C})$, unifies Bessel function of the first kind J_p, modified Bessel function of first kind I_p, and the spherical Bessel function of first kind K_p, and provides a broader framework for analysis, simplifying the study of their geometric and analytic properties. This function is a particular solution of the second-order linear homogeneous differential equation

$$z^2w''(z)+bzw'(z)+[cz^2-p^2+(1-b)p]w(z)=0,$$

and has the following familiar power series representation:

$$w_{p,b,c}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n c^n}{n!\,\Gamma\left(p+n+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{2n+p},\qquad (p,b\in\mathbb{R};\ c,z\in\mathbb{C}),\tag{1}$$

where Γ stands for the Euler gamma function. In addition, the series (1) converges everywhere. Furthermore, specific values of the parameters p,b, and c yield the Bessel, modified Bessel, and spherical Bessel functions, as follows:

$$w_{p,1,1}(z)=J_p(z),\quad w_{p,1,-1}(z)=I_p(z),\quad \text{and}\quad w_{p,2,1}(z)=K_p(z).$$

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Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$, $\mathcal{H}_0 \subset \mathcal{H}$ containing functions f such that $f(0) = 0$, and $\mathcal{A} \subset \mathcal{H}_0$ containing functions f such that $f'(0) = 1$. A function $f \in \mathcal{A}$ is said to be starlike (with respect to the origin 0), denoted by $f \in \mathcal{S}^*$, if $tw \in f(\Delta)$ whenever $w \in f(\Delta)$ and $t \in [0,1]$. A function $f \in \mathcal{A}$ is said to be starlike function if $zf'(z)/f(z)$ has a positive real part for $z \in \Delta$. A function $f \in \mathcal{H}$ is said to be convex if it is univalent and $f(\Delta)$ is convex. It is well known that $f \in \mathcal{H}$ is convex in Δ, if $f'(0) \neq 0$ and $(zf''(z)/f'(z)) + 1$ has positive real part for $z \in \Delta$. A function $f \in \mathcal{A}$ is said to be close-to-convex, denoted by $f \in \mathcal{C}$, if $\mathbb{C} \setminus f(\Delta)$ can be represented as a union of non-crossing half lines. A function $f \in \mathcal{H}$ is said to be close-to-convex function in Δ, if there exists a convex univalent function g in Δ and such that $f'(z)/g'(z)$ has a positive real part for $z \in \Delta$. The close-to-convex functions are univalent in Δ, which includes the classes of starlike functions and convex functions in Δ. For more details about the subclasses of $\mathcal{A}$, we refer to [9].

The generalized Bessel function $w_{p,b,c}$ does not belong to the class $\mathcal{A}$. Therefore, to investigate the functions $w_{p,b,c}$ from the perspective of geometric function theory, we shall need the following normalization:

$$\begin{aligned} u_{p,b,c}(z) &= 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{-p/2} w_{p,b,c}(\sqrt{z}) \\ &= \sum_{n=0}^{\infty} \frac{(-c/4)^n}{n!\,(p+(b+1)/2)_n} z^n, \quad z \in \Delta. \end{aligned} \tag{2}$$

where $\kappa := p + (b+1)/2 \in \tilde{\mathbb{C}} := \mathbb{C} \setminus \{0, -1, -2, \cdots\}$, and $(\lambda)_n$ denote the Pochhammer symbol defined by $(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1)$ for $n \geqslant 1$, and $(\lambda)_0 = 1$. We can easily verify that the relation $(\alpha)_n = \alpha(\alpha+1)_{n-1}$ $(n \in \mathbb{N})$ holds. In addition, we define

$$v_{p,b,c}(z) = z u_{p,b,c}(z), \quad z \in \Delta, \tag{3}$$

and observe that

$$u'_{p,b,c}(z) = -\frac{c}{4\kappa} u_{p+1,b,c}(z), \quad \kappa \neq 0. \tag{4}$$

Note that, both the function $u_{p,b,c}$ and $v_{p,b,c}$ are analytic on $\mathbb{C}$, and $u_{p,b,c}$ satisfies the second order linear differential equation

$$4z^2 u''(z) + 2(2p+b+1) z u'(z) + cz u(z) = 0.$$

Choosing the parameter values $c = b = 1$ and $-c = b = 1$, we have $u_{p,1,1}(z) = \mathcal{J}_p(\sqrt{z})$ and $u_{p,1,-1} = \mathcal{I}_p(\sqrt{z})$. These functions have the following forms:

$$\mathcal{J}_p(z) = 2^p \Gamma(p+1) z^{-p} J_p(z), \quad \text{and} \quad \mathcal{I}_p(z) = 2^p \Gamma(p+1) z^{-p} I_p(z).$$

It is worth noting that the generalized Bessel functions include numerous functions as special cases, for example,

$$u_{-\frac{1}{2},1,1}(z) = u_{-1,2,1}(z) = \cos\sqrt{z}, \qquad u_{\frac{1}{2},1,1}(z) = u_{0,2,1}(z) = \frac{\sin\sqrt{z}}{\sqrt{z}},$$

$$u_{\frac{3}{2},1,1}(z) = u_{1,2,1}(z) = \frac{3}{z}\left(\frac{\sin\sqrt{z}}{z} - \cos\sqrt{z}\right), \quad u_{-\frac{1}{2},1,-1}(z) = \cosh\sqrt{z},$$

$$u_{\frac{1}{2},1,-1}(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}}, \; u_{\frac{3}{2},1,-1}(z) = \frac{3}{z}\left(\frac{\sinh\sqrt{z}}{z} - \cosh\sqrt{z}\right).$$

In [3], Baricz and András proved a monotonicity result for $\mathcal{J}_p$ $(p \in \mathbb{R})$, that the inequalities $-1/4 < q < p$ implies $\mathcal{J}_p(\Delta) \subset \mathcal{J}_q(\Delta)$. Further, they conjectured that a weaker condition $-1 < q < p$ implies the inclusion $\mathcal{J}_p(\Delta) \subset \mathcal{J}_q(\Delta)$. In [4, Corollary 3], Baricz and Ponnusamy proved a generalized version of this conjecture for the generalized Bessel functions by using an integral representation, and this conjecture appeared as its special case. However, Cotîrlă and Szász [7] proved this conjecture by using the concept of subordination factor sequence. In this article, we establish a generalized version of this conjecture for generalized Bessel functions.

Recently, Baricz and Ponnusamy [4] used the following notations:

$$\kappa = p + \frac{b+1}{2}, \quad c_0 = \frac{|c|}{4} - 1, \quad c_1 = \frac{|c|}{4} + \frac{(\operatorname{Im}\kappa)^2}{6} + \frac{1}{2},$$

$$c_2 = \frac{|c|}{2} + \frac{(\operatorname{Im}\kappa)^2}{4} + 1, \quad c_3 = \frac{1}{8}\left(3|c| - 12 + \sqrt{7|c|^2 + 8|c| + 16}\right),$$

$$c_4 = \frac{1}{8}\left(4|c| - 4 + \sqrt{14|c|^2 + 16|c| + 16}\right),$$

and proved the following results, which are essential for the discussion in the next section.

LEMMA 1. ([4, Lemma 1]) *Let $p, b \in \mathbb{R}$ and $c \in \mathbb{C}\setminus\{0\}$ such that $\kappa \geqslant \max\{0, c_0\}$, then the function $u_{p,b,c}$ satisfies the following inequalities for all $z \in \Delta$:*

$$\frac{4\kappa(\kappa+1) - (2\kappa+1)|c| + |c|^2/8}{\kappa[4(\kappa+1) - |c|]} \leqslant |u_{p,b,c}(z)| \leqslant \frac{32\kappa^2 - |c|^2}{8\kappa(4\kappa - |c|)}.$$

LEMMA 2. ([4, Theorem 1]) *Let $p, b \in \mathbb{R}$ and $c \in \mathbb{C}\setminus\{0\}$ such that $\kappa \geqslant c_3\setminus\{0\}$. Then the function $u_{p,b,c}(z)$ is convex in Δ and is starlike of order $1/2$ with respect to 1. Moreover, the function $zu_{p+1,b,c}(z)$ is starlike in Δ.*

LEMMA 3. ([4, Corollary 3]) *Let $p, q, b \in \mathbb{R}$, $c \in \mathbb{C}\setminus\{0\}$ and $2\kappa_p = 2p + b + 1$, $2\kappa_q = 2q + b + 1$. If $\kappa_p \geqslant \kappa_q > \max\{0, c_3\}$. Then $\lambda_{p,b,c}(\Delta) \subseteq \lambda_{q,b,c}(\Delta)$, where $\lambda_{q,b,c}(z) = u_{p,b,c}(z^2)$. In particular, if $p \geqslant q > -1$, then $\mathcal{J}_p(\Delta) \subseteq \mathcal{J}_q(\Delta)$.*

Finally, we recall the following result of [2].

LEMMA 4. *Let $p, b, c \in \mathbb{C}$. Then Bessel functions satisfy the following properties in Δ.*

(i) If $\operatorname{Re}\kappa \geqslant c_0 + 2$, then $\operatorname{Re} u_{p,b,c}(z) \geqslant 0$ in Δ;

(ii) If $\operatorname{Re}\kappa \geqslant c_0+1$ *and* $c\neq 0$, *then* $u_{p,b,c}(z)$ *is univalent in* Δ;

(iii) If $\operatorname{Re}\kappa \geqslant c_1$ *and* $c\neq 0$, *then* $u_{p,b,c}(z)$ *is convex in* Δ;

(iv) If $\operatorname{Re}\kappa \geqslant c_1+1$ *and* $c\neq 0$, *then* $zu_{p+1,b,c}(z)$ *is starlike in* Δ.

Our primary motivation for writing this paper is to provide improvements on these results for the normalized form of generalized Bessel functions.

2. Main results

Our first main result is the following theorem, which gives the univalency for the function $u_{p,b,c}(z)$:

THEOREM 1. *Let* $p,b,c\in\mathbb{C}$. *If* $\operatorname{Re}\kappa \geqslant r_1(c)$, *where* $r_1(c)$ *is the unique positive root of the equation* $E_c(\kappa)=0$, *where*

$$E_c(\kappa)=2-\exp\left(\frac{|c|}{4\operatorname{Re}\kappa}\right), \tag{5}$$

then $\operatorname{Re}u_{p,b,c}(z)>0$ *in* Δ. *Moreover,* $u_{p,b,c}(z)$ *is univalent in* Δ *for all* κ *such that* $Re\,\kappa \geqslant r_1(c)-1,\ c\neq 0$.

Proof. To prove first part, it is enough to establish that $\left|u_{p,b,c}(z)-1\right|<1,\ z\in\Delta$. Using the inequality $(x)_n \geqslant x^n\ \ (n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\})$, and the inequality

$$|(x)_n|=\left|\prod_{j=0}^{n-1}(x+j)\right| \geqslant \left|\prod_{j=0}^{n-1}(\operatorname{Re}x+j)\right| = |(\operatorname{Re}x)_n|, \quad n\in\mathbb{N}, \tag{6}$$

we obtain for all $z\in\Delta$, and $\operatorname{Re}\kappa>0$,

$$\begin{aligned}\left|u_{p,b,c}(z)-1\right| &= \left|\sum_{n=1}^{\infty}\frac{(-c)^n z^n}{4^n\,(\kappa)_n\,n!}\right| < \sum_{n=1}^{\infty}\frac{|c|^n}{4^n\,|(\operatorname{Re}\kappa)_n|\,n!}\\ &\leqslant \sum_{n=0}^{\infty}\left(\frac{|c|}{4\operatorname{Re}\kappa}\right)^n\frac{1}{n!}-1=\exp\left(\frac{|c|}{4\operatorname{Re}\kappa}\right)-1\leqslant 1,\end{aligned}$$

if $E_c(\kappa)\geqslant 0$, where $E_c(\kappa)$ is given by (5). Now, we prove that (5) has exactly one root for each fixed value of c. Let $\operatorname{Re}\kappa:=x>0$, then

$$E_c'(x)=\frac{|c|}{4x^2}\exp\left(\frac{|c|}{4x}\right)>0, \quad \text{and} \quad E_c''(x)=-\frac{|c|(|c|+8x)}{16x^4}\exp\left(\frac{|c|}{4x}\right)<0,$$

for each fixed value of c and $x>0$, showing that E_c is increasing and concave downward for $x>0$. In addition, $\lim_{x\to\infty}E_c(x)=1$ and $\lim_{x\to 0^+}E_c(x)=-\infty$, showing that $x=0$ and $y=1$ are asymptotes to the curve $y=E_c(x)$. Therefore, $E_c(x)$ must have an unique root $r_1(c)=|c|/(4\ln 2)$ in the interval $(0,\infty)$ and $E_c(x)>0$ in $(r_1(c),\infty)$.

Further, using the differentiation formula (4), we obtain $\operatorname{Re}\left(u'_{p,b,c}(z)/\phi'_{p,b,c}(z)\right) = \operatorname{Re} u_{p+1,b,c}(z)$. Since $\phi_{p,b,c}(z) = -(c/4\kappa)z$ is convex, we get the result. □

We now enhance the above result for certain values of c in the following manner, using a different inequality:

THEOREM 2. *Let $p,b,c \in \mathbb{C}$. If $\operatorname{Re}\kappa \geqslant r_2(c)$, where $r_2(c)$ is the unique root of the equation $F_c(\kappa) = 0$, where*

$$F_c(\kappa) = 1 - \frac{|c|}{4\operatorname{Re}\kappa}\left(1 + \frac{|c|}{2(4(\operatorname{Re}\kappa+1)-|c|)}\right), \tag{7}$$

then $\operatorname{Re} u_{p,b,c}(z) > 0$ in Δ. Moreover, $u_{p,b,c}(z)$ is univalent in Δ for all κ such that $\operatorname{Re}\kappa \geqslant r_2(c) - 1$, $c \neq 0$.

Proof. Using the inequality (6) and inequality $(x+1)_{n-1}\, n! \geqslant 2(x+1)^{n-1}$ $(x \geqslant 0,\ n \geqslant 2)$, we obtain for all $z \in \Delta$, and $\operatorname{Re}\kappa > 0$,

$$\begin{aligned}
\left|u_{p,b,c}(z) - 1\right| &< \frac{|c|}{4\operatorname{Re}\kappa}\sum_{n=1}^{\infty}\frac{|c|^{n-1}}{4^{n-1}(\operatorname{Re}\kappa+1)_{n-1}\, n!} \\
&\leqslant \frac{|c|}{4\operatorname{Re}\kappa}\left(1 + \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{|c|}{4(\operatorname{Re}\kappa+1)}\right)^{n-1}\right) \\
&= \frac{|c|}{4\operatorname{Re}\kappa}\left(1 + \frac{|c|}{2(4(\operatorname{Re}\kappa+1)-|c|)}\right) \leqslant 1, \quad \operatorname{Re}\kappa > d_1 := \max\{0, c_0\},
\end{aligned}$$

if $F_c(\kappa) \geqslant 0$, where $F_c(\kappa)$ is given by (7). Now, we prove that (7) has exactly one root. Let $\operatorname{Re}\kappa := x > d_1$, then

$$F'_c(x) = \frac{|c|}{4x^2} + \frac{8x - |c| + 4}{8x^2(4(x+1)-|c|)^2} > 0,$$

and

$$\begin{aligned}
F''_c(x) &= -\frac{|c|}{4x^2} - \frac{|c|^2 - 4|c|(3x+2) + 16(3x^2+3x+1)}{4x^3(4(x+1)-|c|)^3} \\
&= -\frac{|c|}{4x^2} - \frac{12x^2 + (6x+|c|-4)^2}{4x^3(4(x+1)-|c|)^3} < 0,
\end{aligned}$$

for every fixed value of c. This shows that F_c is increasing and concave downward in (d_1,∞). In addition $\lim_{x\to\infty} F_c(x) = 1$ and $\lim_{x\to d_1^+} F_c(x) = -\infty$ $(|c| \leqslant 8)$ are showing that the asymptotes to the curve $y = F_c(x)$ are $x = d_1$ and $y = 1$. This implies that $F_c(\kappa) = 0$ must have a unique root in the interval (d_1,∞). Further, we can obtain the second part of the result by using (4). □

REMARK 1. (i) If $\operatorname{Re}\kappa \geqslant \dfrac{|c|}{4\ln 2}$, then

$$F_c(\kappa) = 1 - \frac{|c|}{4\operatorname{Re}\kappa}\left(1 + \frac{|c|}{2(4(\operatorname{Re}\kappa+1)-|c|)}\right) \geqslant 1 - \ln 2 - \frac{|c|\ln 2}{2\left(\dfrac{|c|}{\ln 2} - |c| + 4\right)} \geqslant 0,$$

for all c such that $|c| \leqslant \alpha_1$, where $\alpha_1 \approx 5.82453$ is the root of the equation

$$2(1-\ln 2)\left(\frac{|c|}{\ln 2} - |c| + 4\right) - |c|\ln 2 = 0.$$

This shows that $r_1(c) \geqslant r_2(c)$ for all c such that $|c| \leqslant \alpha_1$. Therefore, Theorem 2 provides an improvement in Theorem 1 for all c such that $|c| \leqslant \alpha_1$.

(ii) If $\operatorname{Re}\kappa \geqslant \dfrac{|c|}{4} + 1$, then

$$E_c(\kappa) = 2 - \exp\left(\frac{|c|}{4\operatorname{Re}\kappa}\right) \geqslant 2 - \exp\left(\frac{|c|}{|c|+4}\right) \geqslant 0,$$

if $|c| \leqslant \alpha_2 \approx 9.03557$. This shows that, $c_0 + 2 \geqslant r_1(c)$ for all c such that $|c| \leqslant \alpha_2$. Therefore, Theorem 1 significantly improves Lemma 4(i)–(ii) for all c such that $|c| \leqslant \alpha_2$. Similarly,

$$\begin{aligned} F_c(\kappa) &= 1 - \frac{|c|}{4\operatorname{Re}\kappa}\left(1 + \frac{|c|}{2(4(\operatorname{Re}\kappa+1)-|c|)}\right) \\ &\geqslant 1 - \frac{|c|}{|c|+4}\left(1 + \frac{|c|}{16}\right) = \frac{64-|c|^2}{16(|c|+4)} \geqslant 0, \end{aligned}$$

if $|c| \leqslant 8$. This shows that $c_0 + 2 \geqslant r_2(c)$ for all c such that $|c| \leqslant 8$. Therefore, Theorem 2 improves Lemma 2.5(i)–(ii) for all c such that $|c| \leqslant 8$.

The next result provides inequalities for normalized Bessel functions $u_{p,b,c}$ and $v_{p,b,c}$.

THEOREM 3. *Let $p, b, c \in \mathbb{C}$. If $\operatorname{Re}\kappa \geqslant d_1 := \max\{0, c_0\}$, then $u_{p,b,c}$ satisfies the following inequalities for all $z \in \Delta$:*

$$1 - \frac{|c|}{4\operatorname{Re}\kappa}\left(1 + \frac{|c|}{2(4(\operatorname{Re}\kappa+1)-|c|)}\right) \leqslant \left|u_{p,b,c}(z)\right| \leqslant \exp\left(\frac{|c|}{4\operatorname{Re}\kappa}\right), \tag{8}$$

$$\left|\frac{zv'_{p,b,c}(z)}{v_{p,b,c}(z)} - 1\right| \leqslant \frac{\dfrac{|c|}{4\operatorname{Re}\kappa}\exp\left(\dfrac{|c|}{4(\operatorname{Re}\kappa+1)}\right)}{1 - \dfrac{|c|}{4\operatorname{Re}\kappa}\left(1 + \dfrac{|c|}{2(4(\operatorname{Re}\kappa+1)-|c|)}\right)}, \tag{9}$$

$$\left|zv'_{p,b,c}(z)\right| \leqslant 1 + \frac{2|c|(\operatorname{Re}\kappa+1)}{\operatorname{Re}\kappa[4(\operatorname{Re}\kappa+1)-|c|]}. \tag{10}$$

Proof. Using the triangle inequality, the inequality (6), and $(x)_n \geqslant x^n$ $(n \geqslant 0)$, we obtain for all $z \in \Delta$ and $\operatorname{Re}\kappa > 0$,

$$\left|u_{p,b,c}(z)\right| \leqslant \sum_{n=0}^{\infty} \frac{|c|^n}{4^n\,|(\kappa)_n|\,n!} = \sum_{n=0}^{\infty} \frac{|c|^n}{4^n\,|(\operatorname{Re}\kappa)_n|\,n!} \leqslant \exp\left(\frac{|c|}{4\operatorname{Re}\kappa}\right). \tag{11}$$

Similarly, by using inequalities (6) and $(x+1)_{n-1}\,n! \geqslant 2(x+1)^{n-1}$ $(x \geqslant 0, n \geqslant 2)$, we obtain for all $z \in \Delta$ and $\operatorname{Re}\kappa > d_1 := \max\{0, c_0\}$,

$$\begin{aligned}
\left|u_{p,b,c}(z)\right| &= \left|1 + \sum_{n=1}^{\infty} \frac{(-c)^n z^n}{4^n\,|(\kappa)_n|\,n!}\right| \\
&> 1 - \frac{|c|}{4\operatorname{Re}\kappa} \sum_{n=1}^{\infty} \frac{|c|^{n-1}}{4^{n-1}\,(\operatorname{Re}\kappa+1)_{n-1}\,n!} \\
&> 1 - \frac{|c|}{4\operatorname{Re}\kappa}\left(1 + \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{|c|}{4(\operatorname{Re}\kappa+1)}\right)^{n-1}\right) \\
&= 1 - \frac{|c|}{4\operatorname{Re}\kappa}\left(1 + \frac{|c|}{2(4(\operatorname{Re}\kappa+1) - |c|)}\right).
\end{aligned} \tag{12}$$

Now, combining the inequalities (11) and (12), we obtain the result.

Further, using the triangle inequality, the inequality (6), and $(x)_n \geqslant x^n$ $(n \geqslant 0)$, we obtain for all $z \in \Delta$ and $\operatorname{Re}\kappa > 0$,

$$\begin{aligned}
\left|v'_{p,b,c}(z) - \frac{v_{p,b,c}(z)}{z}\right| &< \sum_{n=1}^{\infty} \frac{|c|^n}{4^n\,|(\kappa)_n|\,(n-1)!} \\
&\leqslant \frac{|c|}{4\operatorname{Re}\kappa} \sum_{n=0}^{\infty} \frac{|c|^n}{4^n\,(\operatorname{Re}\kappa+1)_n\,n!} \\
&\leqslant \frac{|c|}{4\operatorname{Re}\kappa} \sum_{n=0}^{\infty} \left(\frac{|c|}{4(\operatorname{Re}\kappa+1)}\right)^n \frac{1}{n!} \\
&= \frac{|c|}{4\operatorname{Re}\kappa} \exp\left(\frac{|c|}{4(\operatorname{Re}\kappa+1)}\right).
\end{aligned} \tag{13}$$

Now, combining the inequalities (12) and (13), we obtain (9).

Moreover, using the triangle inequality, the inequality (6), and the inequality

$$2n!(x+1)_{n-1} \geqslant (n+1)(x+1)^{n-1} \quad (n \in \mathbb{N}), \tag{14}$$

we obtain for all $z \in \Delta$ and $\operatorname{Re}\kappa > d_1$,

$$\begin{aligned}\left|zv'_{p,b,c}(z)\right| &< 1+\sum_{n=1}^{\infty}\frac{(n+1)|c|^n}{4^n\,|(\kappa)_n|\,n!}\\ &\leqslant 1+\frac{|c|}{4\operatorname{Re}\kappa}\sum_{n=1}^{\infty}\frac{(n+1)|c|^{n-1}}{4^{n-1}\,(\operatorname{Re}\kappa+1)_{n-1}\,n!}\\ &\leqslant 1+\frac{|c|}{2\operatorname{Re}\kappa}\sum_{n=1}^{\infty}\left(\frac{|c|}{4(\operatorname{Re}\kappa+1)}\right)^{n-1}\\ &= 1+\frac{2|c|(\operatorname{Re}\kappa+1)}{\operatorname{Re}\kappa[4(\operatorname{Re}\kappa+1)-|c|]}. \end{aligned} \tag{15}$$

This proves the result. □

REMARK 2. Note that

$$\left|u_{p,b,c}(z)\right| = \left|1+\sum_{n=1}^{\infty}\frac{(-c)^n}{4^n\,|(\kappa)_n|\,n!}\right| > 1-\sum_{n=1}^{\infty}\frac{|c|^n}{4^n\,(\operatorname{Re}\kappa)_n\,n!} \geqslant 2-\exp\left(\frac{|c|}{4\operatorname{Re}\kappa}\right),$$

for $\operatorname{Re}\kappa > 0$ and $z \in \Delta$. Now using this inequality, along with the inequality (11), we obtain the following inequality for all $z \in \Delta$ and $\operatorname{Re}\kappa > 0$,

$$2-\exp\left(\frac{|c|}{4\operatorname{Re}\kappa}\right) \leqslant \left|u_{p,b,c}(z)\right| \leqslant \exp\left(\frac{|c|}{4\operatorname{Re}\kappa}\right). \tag{16}$$

Now in view of Remark 1(i), we observe that

$$2-\exp\left(\frac{|c|}{4\operatorname{Re}\kappa}\right) \geqslant 1-\frac{|c|}{4\operatorname{Re}\kappa}\left(1+\frac{|c|}{2(4(\operatorname{Re}\kappa+1)-|c|)}\right)$$

for all $c \in \mathbb{C}$ such that $|c| \leqslant \alpha_1$, where $\alpha_1 \approx 5.82453$. Thus, the inequality (8) is sharper than the inequality (16) for all $|c| \leqslant \alpha_1$, and the reverse case will happen if $|c| > \alpha_1$.

THEOREM 4. *Let $p,b,c \in \mathbb{C}$, and α_1 be root of the equation*

$$2(1-\ln 2)\left(\frac{x}{\ln 2}-x+4\right)-x\ln 2 = 0. \tag{17}$$

If $|c| \in (0,\alpha_1]$ and $\operatorname{Re}\kappa \geqslant r_4(c)$, where $r_4(c)$ is greatest root of the equation $G_c(\kappa) = 0$, where

$$G_c(\kappa) = 1-\frac{|c|}{4(\operatorname{Re}\kappa+1)}\left[1+\frac{|c|}{2(4(\operatorname{Re}\kappa+2)-|c|)}+\exp\left(\frac{|c|}{4(\operatorname{Re}\kappa+2)}\right)\right], \tag{18}$$

or if $|c| \in (\alpha_1,\infty)$ and $\operatorname{Re}\kappa \geqslant r_5(c)$, where $r_5(c)$ is the greatest root of the equation $H_c(\kappa) = 0$, where

$$H_c(\kappa) = 2-\exp\left(\frac{|c|}{4(\operatorname{Re}\kappa+1)}\right)-\frac{|c|}{4(\operatorname{Re}\kappa+1)}\exp\left(\frac{|c|}{4(\operatorname{Re}\kappa+2)}\right). \tag{19}$$

Then $u_{p,b,c}(z)$ is convex in Δ. Moreover, $zu_{p+1,b,c}(z)$ is starlike in Δ.

Proof. (i) To prove the convexity of $u_{p,b,c}$, it is enough to establish that $\left|zu''_{p,b,c}(z)/u'_{p,b,c}(z)\right| < 1,\ z \in \Delta$. First suppose that $|c| \in (0, \alpha_1]$, then the inequality (8) is sharper than the inequality (16). Now, using the differentiation formula (4) and inequality (8), we obtain for $z \in \Delta$ and $\operatorname{Re}\kappa > d_2 := \max\{-1, c_0 - 1\}$,

$$\left|\frac{zu''_{p,b,c}(z)}{u'_{p,b,c}(z)}\right| = \left|\frac{zu'_{p+1,b,c}(z)}{u_{p+1,b,c}(z)} - 1\right| = \left|-\frac{|c|}{4(\operatorname{Re}\kappa+1)}\frac{u_{p+2,b,c}(z)}{u_{p+1,b,c}(z)}\right|$$

$$< \frac{|c|}{4(\operatorname{Re}\kappa+1)}\frac{\exp\left(\dfrac{|c|}{4(\operatorname{Re}\kappa+2)}\right)}{1-\dfrac{|c|}{4(\operatorname{Re}\kappa+1)}\left(1+\dfrac{|c|}{2(4(\operatorname{Re}\kappa+2)-|c|)}\right)}. \tag{20}$$

If

$$1-\frac{|c|}{4(\operatorname{Re}\kappa+1)}\left(1+\frac{|c|}{2(4(\operatorname{Re}\kappa+2)-|c|)}\right) > 0, \tag{21}$$

and $G_c(\kappa) \geqslant 0$, then $\left|zu''_{p,b,c}(z)/u'_{p,b,c}(z)\right| < 1$. Note that $G_c(\kappa) \geqslant 0$ implies the inequality (21). Now, we need to show that the equation $G_c(\kappa) = 0$ has at least one root in (d_2, ∞) for every c. Suppose that $\operatorname{Re}\kappa := x > d_2$, then

$$G'_c(x) = \frac{|c|}{4(x+1)^2} + \frac{|c|^2(8x+12-|c|)}{8(x+1)^2(4(x+2)-|c|)^2}$$
$$+\frac{|c|[|c|(x+1)+4(x+2)^2]}{16(x+1)^2(x+2)^2}\exp\left(\frac{|c|}{4(x+2)}\right) > 0$$

shows that G_c is increasing. Further,

$$G''_c(x) = -\frac{|c|}{2(x+1)^3} - \frac{|c|^2\left[|c|^2 - 4|c|(3x+5) + 16(3x^2+9x+7)\right]}{4(x+1)^3(4(x+2)-|c|)^3}$$
$$-\frac{|c|\left[|c|^2(x+1)^2 + 8|c|(2x^3+9x^2+13x+6) + 32(x+1)^4\right]}{64(x+1)^3(x+2)^4}\exp\left(\frac{1}{4(x+2)}\right)$$
$$= -\frac{|c|}{2(x+1)^3} - \frac{|c|^2\left[48x^2 + (12x+20)(4-|c|) + 96x + 32\right]}{4(x+1)^3(4(x+2)-|c|)^3}$$
$$-\frac{|c|\left[|c|^2(x+1)^2 + 8|c|(2x^3+9x^2+13x+6) + 32(x+1)^4\right]}{64(x+1)^3(x+2)^4}$$
$$\times\exp\left(\frac{1}{4(x+2)}\right) < 0,$$

and $\lim_{x\to\infty} G_c(x) = 1,\ \lim_{x\to d_2^+} G_c(x) = -\infty$, are showing that $x = d_2,\ y = 1$ are asymptotes to the curve $y = G_c(x)$. Therefore, G_c is concave downward in (d_2, ∞). This implies that G_c must have a unique root in the interval (d_2, ∞). Further, we can prove the starlikeness of $zu_{p+1,b,c}(z)$ by using (4).

Next, suppose that $|c| \in (\alpha_1, \infty)$, then we obtain for $\operatorname{Re}\kappa > -1$,

$$\left|\frac{zu''_{p,b,c}(z)}{u'_{p,b,c}(z)}\right| < \frac{|c|}{4(\operatorname{Re}\kappa+1)} \frac{\exp\left(\dfrac{|c|}{4(\operatorname{Re}\kappa+2)}\right)}{2-\exp\left(\dfrac{|c|}{4\operatorname{Re}(\kappa+1)}\right)}. \tag{22}$$

If

$$2-\exp\left(\frac{|c|}{4\operatorname{Re}(\kappa+1)}\right) > 0, \tag{23}$$

and $H_c(\kappa) \geqslant 0$, then $u_{p,b,c}$ is convex in Δ. Note that $H_c(\kappa) \geqslant 0$ implies the inequality (23), and as above, we can show that the lines $x = -1$ and $y = 1$ are asymptotes to the curve $y = H_c(x)$. Therefore, H_c is concave downward in $(-1, \infty)$. This implies that H_c must have a unique root in the interval $(-1, \infty)$. Further, the starlikeness of $zu_{p+1,b,c}(z)$ follows by (4). □

REMARK 3. (i). Note that $\operatorname{Re}\kappa \geqslant |c|/4 + 1/2$ implies that

$$\begin{aligned} G_c(\kappa) &= 1 - \frac{|c|}{4(\operatorname{Re}\kappa+1)}\left[1 + \frac{|c|}{2(4(\operatorname{Re}\kappa+2)-|c|)} + \exp\left(\frac{|c|}{4(\operatorname{Re}\kappa+2)}\right)\right] \\ &\geqslant 1 - \frac{|c|}{|c|+6}\left(1 + \frac{|c|}{20} + \exp\left(\frac{|c|}{|c|+10}\right)\right) \geqslant 0, \end{aligned}$$

if $|c| \leqslant \alpha_3$, where $\alpha_3 \approx 3.93845$ is root of the equation

$$\frac{|c|}{|c|+6}\left(1 + \frac{|c|}{20} + \exp\left(\frac{|c|}{|c|+10}\right)\right) = 1.$$

Therefore, for all c such that $|c| \leqslant \alpha_3$, we have $c_1 \geqslant r_4(c)$. Hence, first part of Theorem 4 improves Lemma 4(iii)–(iv) for all c such that $|c| \leqslant \alpha_3$.

(ii) Note that for all $p, b, c \in \mathbb{C}$, Theorem 4 improves Lemma 4. In particular, if $p, b \in \mathbb{R}$, $c \in \mathbb{C}$, and suppose that $\kappa \geqslant c_1 \setminus \{0\}$, then

$$\frac{|c|}{4(\kappa+1)} \leqslant \frac{2|c|}{3|c|-4+\sqrt{7|c|^2+8|c|+16}}$$

and

$$\frac{|c|}{4(\kappa+2)} \leqslant \frac{2|c|}{3|c|+4+\sqrt{7|c|^2+8|c|+16}}.$$

Therefore

$$\begin{aligned} G_c(\kappa) \geqslant 1 - \frac{2|c|}{3|c|-4+\sqrt{7|c|^2+8|c|+16}}\Bigg(1 + \frac{|c|}{|c|+4+\sqrt{7|c|^2+8|c|+16}} \\ + \exp\left(\frac{2|c|}{3|c|+4+\sqrt{7|c|^2+8|c|+16}}\right)\Bigg) > 0 \end{aligned}$$

for all c. Therefore, Theorem 4 improves Lemma 4.

(iii) For $b=c=1$ and $b=-c=1$, Theorem 4 shows that the functions $\mathcal{J}_p(\sqrt{z})$ and $\mathcal{I}_p(\sqrt{z})$ are convex in Δ and starlike of order $1/2$ with respect to 1, if $\operatorname{Re} p \geqslant p_4 \backslash \{-2,-3\}$, where $p_4 \approx -1.433028$ is the greatest root of the equation

$$1-\frac{1}{4(\operatorname{Re} p+2)}\left[1+\frac{1}{2(4(\operatorname{Re} p+3)-1)}+\exp\left(\frac{1}{4(\operatorname{Re} p+3)}\right)\right]=0.$$

Moreover, the functions $\mathcal{J}_{p+1}(\sqrt{z})$ and $\mathcal{I}_{p+1}(\sqrt{z})$ are starlike in Δ.

THEOREM 5. *Let $p,b,c\in\mathbb{C}$, and α_1 be the root of the equation (17). If $|c|\in(0,\alpha_1]$ and $\operatorname{Re}\kappa \geqslant r_6(c)$, where $r_6(c)$ is the greatest root of the equation $I_c(\kappa)=0$, where*

$$I_c(z)=1-\frac{|c|}{4\operatorname{Re}\kappa}\left[1+\frac{|c|}{2(4(\operatorname{Re}\kappa+1)-|c|)}+2\exp\left(\frac{|c|}{4(\operatorname{Re}\kappa+1)}\right)\right], \tag{24}$$

or if $|c|\in(\alpha_1,\infty)$ and $\operatorname{Re}\kappa \geqslant r_7(c)$, where $r_7(c)$ is the greatest root of the equation $J_c(\kappa)=0$, where

$$J_c(\kappa)=2-\exp\left(\frac{|c|}{4\operatorname{Re}\kappa}\right)-\frac{|c|}{2\operatorname{Re}\kappa}\exp\left(\frac{|c|}{4(\operatorname{Re}\kappa+1)}\right), \tag{25}$$

Then the function $zu_{p,b,c}(z)$ is starlike of order $1/2$ in Δ and consequently the function $z^{1-p}w_{p,b,c}(z)$ is starlike in Δ.

Proof. Let $f(z)=zu_{p,b,c}(z^2)$, then

$$f(z)=\frac{v_{p,b,c}(z^2)}{z}=2^p\Gamma(\kappa)z^{1-p}w_{p,b,c}(z)$$

and

$$\frac{zf'(z)}{f(z)}-1=2\left(\frac{z^2v'_{p,b,c}(z^2)}{v_{p,b,c}(z^2)}-1\right)=2\frac{z^2u'_{p,b,c}(z^2)}{u_{p,b,c}(z^2)}.$$

This implies that f is starlike in Δ if $|z^2u'_{p,b,c}(z^2)/u_{p,b,c}(z^2)|<1/2$ for all $z\in\Delta$. Hence $v_{p,b,c}$ is starlike of order $1/2$ if f is starlike in Δ. First suppose that $|c|\in(0,\alpha_1]$, then using (4) and inequality (8), we obtain for $\operatorname{Re}\kappa>d_2:=\max\{0,c_0\}$,

$$\begin{aligned}\left|\frac{z^2u'_{p,b,c}(z^2)}{u_{p,b,c}(z^2)}\right| &\leqslant \frac{|c|}{4\operatorname{Re}\kappa}\left|\frac{z^2u_{p+1,b,c}(z^2)}{u_{p,b,c}(z^2)}\right| \\ &< \frac{|c|}{4\operatorname{Re}\kappa}\frac{\exp\left(\dfrac{|c|}{4(\operatorname{Re}\kappa+1)}\right)}{1-\dfrac{|c|}{4\operatorname{Re}\kappa}\left(1+\dfrac{|c|}{2(4(\operatorname{Re}\kappa+1)-|c|)}\right)}.\end{aligned} \tag{26}$$

If

$$1-\frac{|c|}{4\operatorname{Re}\kappa}\left(1+\frac{|c|}{2(4(\operatorname{Re}\kappa+1)-|c|)}\right)>0 \tag{27}$$

and $I_c(\kappa)\geqslant 0$, then $|z^2u'_{p,b,c}(z^2)/u_{p,b,c}(z^2)|<1/2$ for all $z\in\Delta$. Note that $I_c(\kappa)\geqslant 0$ implies that the inequality (27) holds. Let $\operatorname{Re}\kappa:=x>d_2$, then

$$I'_c(x)=\frac{|c|}{4x^2}+\frac{|c|^2(8x+4-|c|)}{8x^2(4(x+1)-|c|)^2}+\frac{|c|(4x^2+(|c|+8)x+4)}{8x^2(x+1)^2}\exp\left(\frac{|c|}{4(x+1)}\right)>0,$$

$$\begin{aligned}
I''_c(x)=&-\frac{|c|}{2x^3}-\frac{|c|^2(|c|^2-4|c|(3x+2)+16(3x^2+3x+1))}{4x^3(4(x+1)-|c|)^3}\\
&-\frac{|c|(32x^4+16(|c|+8)x^3+(|c|^2+24|c|+192)x^2+8(|c|+16)x+32)}{32x^3(4x+3)^2}\\
&\times\exp\left(\frac{|c|}{4(x+1)}\right)\\
=&-\frac{|c|}{2x^3}-\frac{|c|^2[(|c|-2(3x+2))^2+12x^2]}{4x^3(4(x+1)-|c|)^3}-\frac{|c|(32x^4+16(|c|+8)x^3)}{32x^3(4x+3)^2}\\
&\times\exp\left(\frac{|c|}{4(x+1)}\right)\\
&-\frac{|c|[(|c|^2+24|c|+192)x^2+8(|c|+16)x+32]}{32x^3(4x+3)^2}\exp\left(\frac{|c|}{4(x+1)}\right)<0
\end{aligned}$$

and $\lim_{x\to\infty}I_c(x)=1$, $\lim_{x\to d_2}I_c(x)=-\infty$. Therefore $I_c(x)$ is increasing and concave downward in (d_2,∞). This implies that I_c must have a unique root in the interval (d_2,∞). Thus, the proof is complete.

Next, suppose that $|c|\in[\alpha_1,\infty)$, then we obtain for $\operatorname{Re}\kappa>d_2$,

$$\left|\frac{z^2u'_{p,b,c}(z^2)}{u_{p,b,c}(z^2)}\right|<\frac{|c|}{4\operatorname{Re}\kappa}\frac{\exp\left(\frac{|c|}{4(\operatorname{Re}\kappa+1)}\right)}{2-\exp\left(\frac{|c|}{4\operatorname{Re}\kappa}\right)}. \tag{28}$$

If

$$2-\exp\left(\frac{|c|}{4\operatorname{Re}\kappa}\right)>0 \tag{29}$$

and $J_c(\kappa)\geqslant 0$, then $|z^2u'_{p,b,c}(z^2)/u_{p,b,c}(z^2)|<1/2$ for all $z\in\Delta$. Note that $J_c(\kappa)\geqslant 0$ implies that the inequality (29) holds, we can show that the lines $x=d_2, y=1$ are asymptotes to the curve $y=J_c(x)$. Therefore, J_c is concave downward in (d_2,∞). This implies that J_c must have a unique root in the interval (d_2,∞). □

Finally, we conclude this section by providing an improved version of Lemma 2.3.

THEOREM 6. *Let* $p,q,b \in \mathbb{R}$, $\kappa_p = p+(b+1)/2$, $\kappa_q = q+(b+1)/2$, *and* α_1 *be the root of equation* (17). *If* $|c| \in (0,\alpha_1]$ *and* $\operatorname{Re}\kappa_p \geqslant \operatorname{Re}\kappa_q > \max\{0, r_4(c)\}$, *where* $r_4(c)$ *is the greatest root of the equation* (18), *and if* $|c| \in [\alpha_1,\infty)$, $\operatorname{Re}\kappa_p \geqslant \operatorname{Re}\kappa_q > \max\{0, r_5(c)\}$, *where* $r_5(c)$ *is the greatest root of the equation* (19). *Then*

$$u_{p,b,c}(\Delta) \subseteq u_{q,b,c}(\Delta).$$

Moreover,

$$\lambda_{p,b,c}(\Delta) \subseteq \lambda_{q,b,c}(\Delta).$$

Proof. If we use (2) to represent $u_{q,b,c}(tz)$ and multiply both sides of (2) by $t^{\kappa_q-1}(1-t)^{\kappa_p-\kappa_p-1}$, and integrating on $[0,1]$, we obtain

$$\sum_{n=0}^{\infty} \frac{(-c/4)^n}{n!\,(\kappa_p)_n} z^n \int_0^1 t^{\kappa_q+n-1}(1-t)^{\kappa_p-\kappa_q-1}\,dt = \int_0^1 u_{p,b,c}(tz)\,t^{\kappa_q-1}(1-t)^{\kappa_p-\kappa_q-1}\,dt.$$

Since $\operatorname{Re}\kappa_p > \operatorname{Re}\kappa_q > 0$, the integral on the right-hand side of the above equation is the beta function $B(k_q,k_p)$, we have

$$\sum_{n=0}^{\infty} \frac{(-c/4)^n}{n!\,(\kappa_p)_n} z^n = \int_0^1 u_{q,b,c}(tz)\,\frac{t^{\kappa_q-1}(1-t)^{\kappa_p-\kappa_q-1}}{B(\kappa_q,\kappa_p-\kappa_q)}\,dt,$$

which implies that

$$u_{p,b,c}(z) = \int_0^1 u_{q,b,c}(tz)\,d\mu(t)\,dt,$$

where

$$d\mu(t) = \frac{\Gamma(\kappa_p)}{\Gamma(\kappa_q)\Gamma(\kappa_p-\kappa_q)} t^{\kappa_q-1}(1-t)^{\kappa_p-\kappa_q-1}$$

is a probability measure on $[0,1]$. The above integral implies that $u_{p,b,c}(\Delta)$ contained in hull of

$$\{u_{q,b,c}(t\Delta) : t \in [0,1]\} = \{u_{q,b,c}(tz) : t \in [0,1], z \in \Delta\}.$$

By Theorem 4, $u_{q,b,c}$ is a convex function, hence $u_{p,b,c}(\Delta) \subseteq u_{q,b,c}(\Delta)$, and consequently $\lambda_{p,b,c}(\Delta) \subseteq \lambda_{q,b,c}(\Delta)$. □

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Jugal Kishore Prajapat
Department of Mathematics
Central University of Rajasthan
Bandarsindri, Kishangarh-305817, Rajasthan, India
e-mail: jkprajapat@curaj.ac.in