

FLETT-TYPE MEAN VALUE THEOREMS FOR GENERALIZED DERIVATIVES

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Abstract. This paper proves a Flett-type mean value theorem for several first-order generalized derivatives. We begin with the Laplace derivative. If $f \in C[a, b]$, the Laplace derivative exists on (a, b) , and the one-sided endpoint values satisfy

$$\text{LD}^+ f(a) = \text{LD}^- f(b),$$

then there is a point $c \in (a, b)$ such that

$$\text{LD}f(c) = \frac{f(c) - f(a)}{c - a}.$$

We then compare the first-order Borel, Cesàro, L^p , and Laplace derivatives. These comparisons give a unified Flett relation for the Borel derivative, the Cesàro derivative $C_r D$, and the first-order L^p derivative. Finally, we give examples of continuous functions showing that the main implications are not reversible in general.

1. Introduction

Flett's mean value theorem [7] is a refinement of Lagrange's mean value theorem. It says that if a differentiable function satisfies a suitable equality of derivatives at the two endpoints, then there is an interior point where the derivative equals the slope of the chord drawn from the left endpoint. More precisely, under the usual hypothesis $f'(a) = f'(b)$, one obtains a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(c) - f(a)}{c - a}.$$

This theorem has many variants and extensions [1, 9, 15], and it is natural to ask what happens when the ordinary derivative is replaced by a generalized derivative.

The aim of this paper is to study the same question for generalized derivatives. Such derivatives are useful when the ordinary derivative does not exist. Mukhopadhyay's monograph gives a systematic account of many such notions, including Cesàro, Borel, L^p , Laplace, and symmetric derivatives, together with their relations [12]. In the present paper we only use their first-order forms.

The Cesàro derivative has its classical origin in Burkill's work on the Cesàro–Perron integral and the Cesàro–Perron scale of integration [2, 3]. The L^p -derivative

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comes from the local L^p -approximation point of view developed in the work of Calderón and Zygmund and of Stein and Zygmund, Evans and later by Mukhopadhyay-Ray [4–6, 14, 21]. The Borel derivative goes back to work of Khintchine, Marcinkiewicz and Zygmund and Sargent [10, 11, 20].

The Laplace derivative is the main starting point of the paper. It was introduced by Svetic and later developed by Mukhopadhyay and Ray, and by Ray and Garai, as a generalized derivative with many familiar properties [13, 17–19, 22]. The definition of the Laplace derivative uses an exponential kernel, so it is well suited for local averaging near a point. We first prove a Flett-type theorem directly for this derivative.

We then pass from the Laplace derivative to other generalized derivatives. The paper considers the Borel derivative, the Cesàro derivative $C_r D$, and the first-order L^p -derivative. We use only the first-order forms needed for the present problem. The background definitions and comparison results are taken mainly from Mukhopadhyay's monograph, together with the first-order papers on the Laplace and Borel derivatives [16, 17].

Some of the implication results are already part of the standard hierarchy. For example, the first L^p -derivative implies the Borel derivative, and the Borel derivative implies the Laplace derivative, with the same value. The Cesàro derivatives also form a one-way scale. Other comparisons needed in the present paper are proved directly. In particular, we prove the first-order equivalence between the Borel derivative and $C_1 D$, the implication from $C_r D$ to the Laplace derivative, and the implication from the first L^p -derivative to $C_r D$. These comparisons allow us to derive Flett-type theorems for the Borel, Cesàro, and L^p derivatives from the Laplace case.

The first-order implication scheme used in the paper is summarized in Figure 1.

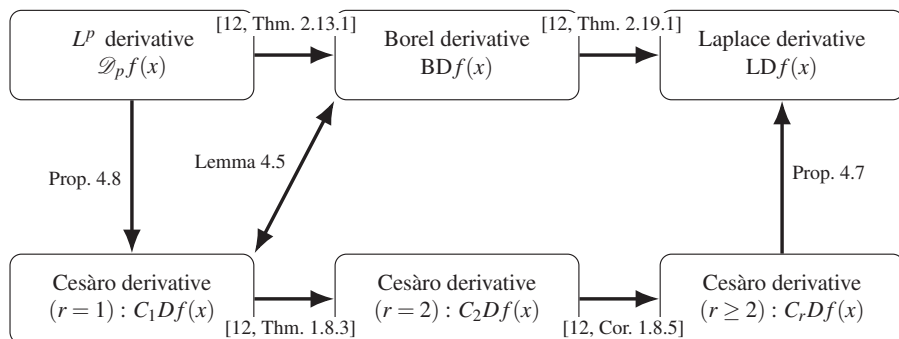


Figure 1: *Implications for the first-order generalized derivatives of a continuous function f at a fixed point x . The double arrow denotes equivalence; the other arrows denote one-way implications.*

The paper is organized as follows. Section 2 recalls the first-order Laplace derivative and the auxiliary facts used in the proof. Section 3 proves the Laplace–Flett theorem. Section 4 introduces the Borel, Cesàro, and L^p derivatives and records the comparison scheme among them. The unified Flett relation for these derivatives is obtained

there as a consequence of the Laplace theorem. Section 5 gives examples of continuous functions that separate the derivative notions and show that the implication arrows cannot be reversed in general. The paper ends with a short conclusion.

2. Preliminaries

DEFINITION 2.1. Let f be defined at $x \in \mathbb{R}$. If f is Lebesgue integrable on a right neighborhood of x , say on $[x, x + \eta]$ for some $\eta > 0$, then the *right-hand Laplace derivative* of f at x is defined by

$$\text{LD}^+ f(x) := \lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} (f(x+t) - f(x)) dt, \quad 0 < \delta < \eta,$$

whenever this limit exists finitely. If f is Lebesgue integrable on a left neighborhood of x , say on $[x - \eta, x]$ for some $\eta > 0$, then the *left-hand Laplace derivative* of f at x is defined by

$$\text{LD}^- f(x) := \lim_{s \rightarrow \infty} (-s^2) \int_0^\delta e^{-st} (f(x-t) - f(x)) dt, \quad 0 < \delta < \eta,$$

whenever this limit exists finitely. If both one-sided derivatives exist finitely and are equal, the common value is called the *Laplace derivative* of f at x and is denoted by $\text{LD}f(x)$.

The value of each defining limit is independent of the particular $\delta > 0$; this is proved in [13, p. 133] and in [17, Lemma 1].

The defining integrals are linear in the function. Thus, whenever the one-sided limits exist,

$$\text{LD}^\pm(\alpha u + \beta v)(x) = \alpha \text{LD}^\pm u(x) + \beta \text{LD}^\pm v(x).$$

LEMMA 2.2. Let $\delta > 0$ and let $\alpha > -1$. Then

$$\lim_{s \rightarrow \infty} s^{\alpha+1} \int_0^\delta e^{-st} t^\alpha dt = \Gamma(\alpha + 1).$$

More generally, for any real β ,

$$s^\beta \int_0^\delta e^{-st} t^\alpha dt = \Gamma(\alpha + 1) s^{\beta-\alpha-1} + o(s^{\beta-\alpha-1}).$$

Proof. The proof is elementary. Put $u = st$. Then

$$s^{\alpha+1} \int_0^\delta e^{-st} t^\alpha dt = \int_0^{s\delta} e^{-u} u^\alpha du \rightarrow \int_0^\infty e^{-u} u^\alpha du = \Gamma(\alpha + 1).$$

The second formula follows by multiplying the first one by $s^{\beta-\alpha-1}$. $\square$

REMARK 2.3. Clearly

$$\lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} t \, dt = 1, \quad \lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} o(t) \, dt = 0.$$

If f is differentiable at x , then $f(x+t) - f(x) = f'(x)t + o(t)$, so the Laplace derivative recovers $f'(x)$.

LEMMA 2.4. Let $\eta > 0$. Suppose that f and g are continuous on $[c, c + \eta]$, and that $\text{LD}^+ f(c)$ and the right ordinary derivative $g'_+(c)$ exist finitely. Then $\text{LD}^+(fg)(c)$ exists and

$$\text{LD}^+(fg)(c) = g(c)\text{LD}^+ f(c) + f(c)g'_+(c).$$

The analogous left-hand statement holds when $[c, c + \eta]$, $\text{LD}^+ f(c)$, and $g'_+(c)$ are replaced by $[c - \eta, c]$, $\text{LD}^- f(c)$, and $g'_-(c)$, respectively. Consequently, if f and g are continuous in a neighborhood of c , $\text{LD}f(c)$ exists, and $g'(c)$ exists, then

$$\text{LD}(fg)(c) = g(c)\text{LD}f(c) + f(c)g'(c).$$

Proof. This product rule was proved in [17]. For completeness, we give a short proof of the right-hand local form.

Choose $0 < \delta < \eta$. For $0 < t < \delta$, write

$$f(c+t)g(c+t) - f(c)g(c) = g(c)(f(c+t) - f(c)) + f(c)(g(c+t) - g(c)) + R(t), \quad (1)$$

where

$$R(t) := (f(c+t) - f(c))(g(c+t) - g(c)).$$

Since f is continuous at c and $g'_+(c)$ exists, we have

$$f(c+t) - f(c) = o(1), \quad g(c+t) - g(c) = g'_+(c)t + o(t) \quad (t \downarrow 0).$$

In particular, $R(t) = o(t)$. Hence, by Remark 2.3,

$$\lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} R(t) \, dt = 0$$

and

$$\lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} (g(c+t) - g(c)) \, dt = g'_+(c).$$

Therefore, from (1),

$$\begin{aligned} \text{LD}^+(fg)(c) &= \lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} [f(c+t)g(c+t) - f(c)g(c)] \, dt \\ &= g(c) \lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} [f(c+t) - f(c)] \, dt \\ &\quad + f(c) \lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} [g(c+t) - g(c)] \, dt \\ &= g(c)\text{LD}^+ f(c) + f(c)g'_+(c). \end{aligned}$$

The left-hand formula is obtained in the same way, replacing $c+t$ by $c-t$ and using the factor $-s^2$. Finally, if c is an interior point, $\text{LD}f(c)$ exists, and $g'(c)$ exists, then

$$\text{LD}^+f(c) = \text{LD}^-f(c) = \text{LD}f(c), \quad g'_+(c) = g'_-(c) = g'(c).$$

Thus the two one-sided Laplace derivatives of fg agree, and the bilateral formula follows. $\square$

LEMMA 2.5. *Let $b \in \mathbb{R}$ and $\delta > 0$. Let $Y : [b - \delta, b] \rightarrow \mathbb{R}$ be continuous and suppose that $\text{LD}^-Y(b)$ exists. Then*

$$\lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} Y(b-t) dt = Y(b).$$

Proof. Define a function Φ on $[0, \delta]$ by

$$\Phi(t) := Y(b-t).$$

Then Φ is continuous on $[0, \delta]$. Since $\text{LD}^-Y(b)$ exists, $\text{LD}^+\Phi(0)$ exists and equals $-\text{LD}^-Y(b)$. Now let $g(t) = t$. Then g is continuous on $[0, \delta]$ and $g'_+(0) = 1$. Applying the right-hand form of the product rule in Lemma 2.4 to $\Phi(t)g(t) = t\Phi(t)$ at the endpoint 0, we get

$$\text{LD}^+(t\Phi)(0) = 0 \cdot \text{LD}^+\Phi(0) + \Phi(0)g'_+(0) = \Phi(0) = Y(b).$$

Thus

$$\text{LD}^+(t\Phi)(0) = \lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} t\Phi(t) dt = \lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} tY(b-t) dt.$$

This completes the proof. $\square$

3. The Laplace–Flett Theorem

THEOREM 3.1. *Let f be continuous on $[a, b]$ and suppose $\text{LD}f$ exists on (a, b) with*

$$\text{LD}^+f(a) = \text{LD}^-f(b) =: M.$$

Then there exists $c \in (a, b)$ such that

$$\text{LD}f(c) = \frac{f(c) - f(a)}{c - a}.$$

Proof. Define

$$G(x) := f(x) - f(a) - M(x-a), \quad a \leq x \leq b.$$

Then G is continuous on $[a, b]$, $G(a) = 0$, $\text{LD}^+G(a) = 0$ and $\text{LD}^-G(b) = 0$. For $x \in (a, b]$ put

$$Y(x) := \frac{G(x)}{x-a} = \frac{f(x) - f(a)}{x-a} - M.$$

Then Y is continuous on $(a, b]$ and $G(x) = (x-a)Y(x)$. Since $x \mapsto (x-a)^{-1}$ is C^1 near b , the left-hand form of the product rule in Lemma 2.4, applied to $Y = G/(x-a)$ shows that $\text{LD}^-Y(b)$ exists and

$$\text{LD}^-Y(b) = \frac{\text{LD}^-G(b)}{b-a} - \frac{G(b)}{(b-a)^2} = -\frac{Y(b)}{b-a}.$$

Consequently, if $Y(b) > 0$, then $\text{LD}^-Y(b) < 0$, and if $Y(b) < 0$, then $\text{LD}^-Y(b) > 0$.

Step 1:

Case A: $Y(b) > 0$. We first show that there are arbitrarily small $t > 0$ such that

$$Y(a+t) < Y(b).$$

Indeed, if $Y(a+t) \geq Y(b)$ for all sufficiently small $t > 0$, then, choosing $\delta > 0$ inside that interval, we get

$$\text{LD}^+G(a) = \lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} G(a+t) dt \geq Y(b) \lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} t dt = Y(b) > 0,$$

contrary to $\text{LD}^+G(a) = 0$.

Next, since $\text{LD}^-Y(b) < 0$, there exists $\eta > 0$ such that

$$Y(b-\eta) > Y(b).$$

For if $Y(b-t) \leq Y(b)$ for all sufficiently small $t > 0$, then

$$\text{LD}^-Y(b) = \lim_{s \rightarrow \infty} (-s^2) \int_0^\delta e^{-st} \{Y(b-t) - Y(b)\} dt \geq 0,$$

which contradicts $\text{LD}^-Y(b) < 0$.

Now choose $\varepsilon > 0$ so small that

$$a + \varepsilon < b - \eta \quad \text{and} \quad Y(a + \varepsilon) < Y(b).$$

Let x_* be a point where Y attains its maximum on $[a + \varepsilon, b]$. The maximum is not attained at $a + \varepsilon$, because

$$Y(a + \varepsilon) < Y(b).$$

It is not attained at b , because

$$Y(b - \eta) > Y(b).$$

Therefore $x_* \in (a, b)$, and x_* is a local maximum point of Y .

Case B: $Y(b) < 0$. Apply Case A to $-G$ and $-Y$. This gives an interior local maximum of $-Y$, equivalently an interior local minimum of Y .

Case C: $Y(b) = 0$. If $Y \equiv 0$ on $(a, b]$, then

$$f(x) - f(a) = M(x - a), \quad x \in (a, b],$$

and the conclusion holds for every $c \in (a, b)$.

Otherwise choose $x_0 \in (a, b]$ such that $Y(x_0) \neq 0$. If $Y(x_0) > 0$, then the same argument used above at the left endpoint, with $Y(x_0)$ in place of $Y(b)$, gives some $\varepsilon \in (0, x_0 - a)$ such that

$$Y(a + \varepsilon) < Y(x_0).$$

Since $Y(b) = 0 < Y(x_0)$, the maximum of Y on $[a + \varepsilon, b]$ is attained at an interior point. If $Y(x_0) < 0$, apply the same argument to $-Y$. Thus, in every case, there exists $c \in (a, b)$ at which Y has a local extremum.

Step 2: We treat the case where c is a local maximum of Y . The local minimum case follows by applying the same argument to $-Y$.

Choose $\rho > 0$ such that $[c - \rho, c + \rho] \subset (a, b)$ and for $0 < t \leq \rho$,

$$Y(c + t) \leq Y(c), \quad Y(c - t) \leq Y(c).$$

Put

$$H(x) := (x - a)Y(x) = G(x).$$

So $\text{LDH}(c)$ exists and

$$\text{LDH}(c) = \text{LD}f(c) - M. \quad (2)$$

Choose $\delta \in (0, \rho)$. Since c is a local maximum of Y , for $0 < t \leq \delta$,

$$H(c + t) - H(c) = (c - a)\{Y(c + t) - Y(c)\} + tY(c + t) \leq tY(c).$$

It follows that

$$\text{LD}^+H(c) \leq Y(c) \lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} t dt = Y(c).$$

On the other hand,

$$H(c) - H(c - t) = (c - a)\{Y(c) - Y(c - t)\} + tY(c - t) \geq tY(c - t).$$

Since $Y = H/(x - a)$ near c and $x \mapsto (x - a)^{-1}$ is C^1 in a neighborhood of c , the product rule in Lemma 2.4, applied to $Y = H/(x - a)$ shows that $\text{LD}Y(c)$ exists. In particular, $\text{LD}^-Y(c)$ exists. Also, Y is continuous on $[c - \delta, c]$. Hence by Lemma 2.5,

$$\text{LD}^-H(c) = \lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} \{H(c) - H(c - t)\} dt \geq \lim_{s \rightarrow \infty} s^2 \int_0^\delta e^{-st} tY(c - t) dt = Y(c).$$

Since $\text{LDH}(c)$ exists, combining the two inequalities and using (2), we get

$$\text{LD}f(c) - M = \text{LDH}(c) = Y(c).$$

Therefore

$$\text{LD}f(c) - M = \frac{f(c) - f(a)}{c - a} - M,$$

and hence

$$\text{LD}f(c) = \frac{f(c) - f(a)}{c - a}.$$

This proves the theorem. $\square$

EXAMPLE 3.2. Let

$$h(x) = \begin{cases} x^3 - 3x - \frac{x}{\sin 1} \sin(1/x^2), & x \neq 0, \\ 0, & x = 0, \end{cases} \quad x \in [-1, 1].$$

Then h is continuous on $[-1, 1]$ and classically differentiable on $[-1, 1] \setminus \{0\}$. Hence, at every point $x \neq 0$, the Laplace derivative exists and agrees with the ordinary derivative.

For $x \neq 0$,

$$h'(x) = 3x^2 - 3 - \frac{1}{\sin 1} \sin(1/x^2) + \frac{2}{x^2 \sin 1} \cos(1/x^2).$$

Therefore

$$h'(-1) = h'(1) = -1 + 2 \cot 1.$$

Thus

$$\text{LD}^+h(-1) = \text{LD}^-h(1).$$

The ordinary derivative of h at 0 does not exist, because

$$\frac{h(t) - h(0)}{t} = t^2 - 3 - \frac{1}{\sin 1} \sin(1/t^2)$$

has no limit as $t \rightarrow 0$.

Note that $(x^3 - 3x)$ is differentiable so for the right Laplace derivative,

$$\text{LD}^+h(0) = -3 - \lim_{s \rightarrow \infty} \frac{s^2}{\sin 1} \int_0^\delta e^{-st} t \sin(1/t^2) dt.$$

We will show

$$J_s := s^2 \int_0^\delta e^{-st} t \sin(1/t^2) dt \rightarrow 0.$$

Since

$$\frac{d}{dt} \cos(1/t^2) = \frac{2}{t^3} \sin(1/t^2),$$

we have

$$t \sin(1/t^2) = \frac{1}{2} t^4 \frac{d}{dt} \cos(1/t^2).$$

Hence integration by parts gives

$$J_s = \frac{s^2}{2} \left[e^{-st} t^4 \cos(1/t^2) \right]_0^\delta - \frac{s^2}{2} \int_0^\delta e^{-st} (4t^3 - st^4) \cos(1/t^2) dt.$$

So

$$|J_s| \leq \frac{s^2 \delta^4}{2} e^{-s\delta} + s^2 \int_0^\delta e^{-st} \left(2t^3 + \frac{s}{2} t^4 \right) dt.$$

Again by Lemma 2.2,

$$s^2 \int_0^\delta e^{-st} t^3 dt = \frac{6 + o(1)}{s^2} \rightarrow 0 \quad \text{and} \quad s^3 \int_0^\delta e^{-st} t^4 dt = \frac{24 + o(1)}{s^2} \rightarrow 0.$$

Consequently,

$$J_s \rightarrow 0.$$

So $\text{LD}^+ h(0) = -3$. For the left derivative, observe that for $t > 0$,

$$h(0) - h(-t) = t^3 - 3t - \frac{t}{\sin 1} \sin(1/t^2) = h(t).$$

Hence the same calculation gives

$$\text{LD}^- h(0) = -3.$$

Therefore

$$\text{LD} h(0) = -3.$$

Thus h satisfies the hypotheses of Theorem 3.1. We have

$$h(-1) = (-1)^3 - 3(-1) - \frac{-1}{\sin 1} \sin 1 = 3, \quad h(0) = 0,$$

So

$$\frac{h(0) - h(-1)}{0 - (-1)} = -3 = \text{LD} h(0).$$

Therefore the Flett point may be chosen as $c = 0$.

4. Implications among first-order generalized derivatives and inherited Flett theorems

DEFINITION 4.1. Let f be defined at $x \in \mathbb{R}$. The first-order one-sided Borel derivatives of f at x are defined by

$$\begin{aligned} \text{BD}^+ f(x) &= \lim_{h \downarrow 0} \frac{1}{h} \int_0^h \frac{f(x+t) - f(x)}{t} dt, \\ \text{BD}^- f(x) &= \lim_{h \downarrow 0} \frac{1}{h} \int_0^h \frac{f(x) - f(x-t)}{t} dt, \end{aligned}$$

provided, in the first formula, f is integrable on a right neighborhood of x , and, in the second formula, f is integrable on a left neighborhood of x ; the improper integrals are assumed to exist and the limits are finite. If both one-sided Borel derivatives exist and are equal, then the common value is denoted by $\text{BD}f(x)$ and is called the *Borel derivative* of f at x .

DEFINITION 4.2. Let $1 \leq p < \infty$, and let f be defined at $x \in \mathbb{R}$. For $A \in \mathbb{R}$, the first-order one-sided L^p derivatives of f at x are defined by

$$\begin{aligned}\mathcal{D}_p^+ f(x) = A &\iff \lim_{h \downarrow 0} \left(\frac{1}{h^{p+1}} \int_0^h |f(x+t) - f(x) - At|^p dt \right)^{1/p} = 0, \\ \mathcal{D}_p^- f(x) = A &\iff \lim_{h \downarrow 0} \left(\frac{1}{h^{p+1}} \int_0^h |f(x-t) - f(x) + At|^p dt \right)^{1/p} = 0,\end{aligned}$$

provided, in the first formula, $f \in L^p$ on a right neighborhood of x , and, in the second formula, $f \in L^p$ on a left neighborhood of x . If both one-sided L^p derivatives exist and are equal, then the common value is denoted by $\mathcal{D}_p f(x)$ and is called the first-order L^p derivative of f at x .

DEFINITION 4.3. Let r be a positive integer; this convention is used throughout whenever a Cesàro derivative $C_r D$ is considered. Let f be defined at $x \in \mathbb{R}$. For $h > 0$, define the right and left Cesàro means of order r by

$$\begin{aligned}\mathcal{C}_{r,h}^+ f(x) &:= \frac{r}{h^r} \int_0^h (h-t)^{r-1} f(x+t) dt, \\ \mathcal{C}_{r,h}^- f(x) &:= \frac{r}{h^r} \int_0^h (h-t)^{r-1} f(x-t) dt,\end{aligned}$$

provided, in the first formula, f is integrable on a right neighborhood of x , and, in the second formula, f is integrable on a left neighborhood of x . For the continuous functions considered in this paper these are ordinary integrals; in the general Cesàro–Perron setting they are understood as $C_{r-1}P$ -integrals.

The right and left first-order Cesàro derivatives are defined by

$$\begin{aligned}C_r D^+ f(x) &:= \lim_{h \downarrow 0} \frac{r+1}{h} \left(\mathcal{C}_{r,h}^+ f(x) - f(x) \right), \\ C_r D^- f(x) &:= \lim_{h \downarrow 0} \frac{r+1}{h} \left(f(x) - \mathcal{C}_{r,h}^- f(x) \right),\end{aligned}$$

whenever the limits exist and are finite. If both one-sided Cesàro derivatives exist and are equal, then the common value is denoted by $C_r Df(x)$ and is called the first-order Cesàro derivative of order r at x .

The case $r = 0$ is interpreted as the ordinary first derivative.

REMARK 4.4. The implication results from Mukhopadhyay's book used here are as follows. First, if the first L^p derivative of f exists at x , then the first Borel derivative

of f exists at x with the same value [12, Theorem 2.13.1, pp. 140–143]. Second, the Borel–Laplace comparison is one-sided: if $\text{BD}^+f(x)$ exists, then $\text{LD}^+f(x)$ exists and has the same value [12, Theorem 2.19.1, pp. 169–172]. The corresponding left-hand implication follows by applying the right-hand result to the reflected function $\tilde{f}(t) := f(2x-t)$. Consequently, if $\text{BD}f(x)$ exists, then $\text{LD}f(x)$ exists and $\text{LD}f(x) = \text{BD}f(x)$. In the first-order bilateral case this is also stated by Ray and Garai [16, Theorem 2.5]. Third, the Cesàro derivatives form a one-way scale: if $C_r Df(x)$ exists, then $C_{r+1} Df(x)$ exists with the same value [12, Theorem 1.8.3 and Corollary 1.8.5, pp. 36–37].

The Cesàro–Borel comparison in Mukhopadhyay’s book is of a different type. It is formulated through an r -fold indefinite integral Φ of the original function f : it compares $C_r Df(x)$ with $\text{BD}_{r+1}\Phi(x)$ [12, Theorem 2.15.1, pp. 146–147]. The special first-order relation between the Borel derivative and $C_1 D$ is recorded next.

LEMMA 4.5. *Suppose that f is continuous on a right neighborhood of x . Then*

$$\text{BD}^+f(x) \text{ exists} \iff C_1 D^+f(x) \text{ exists},$$

with equal values. The analogous left-hand statement holds when f is continuous on a left neighborhood of x , with $\text{BD}^-f(x)$ and $C_1 D^-f(x)$ in place of $\text{BD}^+f(x)$ and $C_1 D^+f(x)$, respectively. Consequently, if f is continuous in a neighborhood of x , then

$$\text{BD}f(x) \text{ exists} \iff C_1 Df(x) \text{ exists},$$

and the two derivatives have the same value.

Proof. We prove the lemma for right-hand derivative. The left-hand case is obtained in the same way by replacing $f(x+t) - f(x)$ with $f(x) - f(x-t)$.

Put

$$u(t) := f(x+t) - f(x), \quad A(h) := \int_0^h u(t) dt,$$

and, whenever the improper integral exists,

$$B(h) := \int_0^h \frac{u(t)}{t} dt.$$

Since f is right continuous at x , we have

$$u(t) \rightarrow 0 \quad \text{as} \quad (t \downarrow 0).$$

The definitions give

$$C_1 D^+f(x) = \lim_{h \downarrow 0} \frac{2A(h)}{h^2}, \quad \text{BD}^+f(x) = \lim_{h \downarrow 0} \frac{B(h)}{h}.$$

First suppose that $\text{BD}^+f(x) = L$. Then

$$B(h) = Lh + o(h) \quad (h \downarrow 0).$$

For $0 < \varepsilon < h$, the function $u(t)/t$ is continuous on $[\varepsilon, h]$, and hence

$$B'(t) = \frac{u(t)}{t} \quad (\varepsilon \leq t \leq h).$$

Thus

$$A(h) - A(\varepsilon) = \int_{\varepsilon}^h u(t) dt = \int_{\varepsilon}^h t B'(t) dt.$$

Integration by parts on $[\varepsilon, h]$ gives

$$A(h) - A(\varepsilon) = hB(h) - \varepsilon B(\varepsilon) - \int_{\varepsilon}^h B(t) dt.$$

Now

$$|A(\varepsilon)| \leq \varepsilon \sup_{0 < t < \varepsilon} |u(t)| = o(\varepsilon),$$

so $A(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Also, from $B(\varepsilon) = L\varepsilon + o(\varepsilon)$, $\varepsilon B(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Moreover $B(t) = Lt + o(t)$, so B is integrable near 0. We use the elementary fact that if $r(t) = o(t)$ as $t \downarrow 0$, then

$$\int_0^h r(t) dt = o(h^2).$$

Indeed, for every $\varepsilon > 0$ and all sufficiently small $t > 0$, $|r(t)| \leq \varepsilon t$, and hence

$$\left| \int_0^h r(t) dt \right| \leq \varepsilon \int_0^h t dt = \frac{\varepsilon h^2}{2}.$$

Therefore

$$\begin{aligned} A(h) &= hB(h) - \int_0^h B(t) dt \\ &= h\{Lh + o(h)\} - \int_0^h \{Lt + o(t)\} dt \\ &= Lh^2 + o(h^2) - \left(\frac{Lh^2}{2} + o(h^2) \right) \\ &= \frac{Lh^2}{2} + o(h^2). \end{aligned}$$

Hence

$$C_1 D^+ f(x) = \lim_{h \downarrow 0} \frac{2A(h)}{h^2} = L.$$

Conversely, suppose that $C_1 D^+ f(x) = L$. Then

$$A(h) = \frac{Lh^2}{2} + o(h^2) \quad (h \downarrow 0).$$

For $0 < \varepsilon < h$, since $A'(t) = u(t)$ on $[\varepsilon, h]$, integration by parts gives

$$\int_{\varepsilon}^h \frac{u(t)}{t} dt = \int_{\varepsilon}^h \frac{A'(t)}{t} dt = \frac{A(h)}{h} - \frac{A(\varepsilon)}{\varepsilon} + \int_{\varepsilon}^h \frac{A(t)}{t^2} dt.$$

Now

$$\frac{A(\varepsilon)}{\varepsilon} = \frac{L\varepsilon}{2} + o(\varepsilon) \rightarrow 0 \quad \text{and} \quad \frac{A(t)}{t^2} \rightarrow \frac{L}{2} \quad (t \downarrow 0).$$

Therefore the improper integral is an ordinary integral and letting $\varepsilon \downarrow 0$, we obtain

$$B(h) = \frac{A(h)}{h} + \int_0^h \frac{A(t)}{t^2} dt.$$

Dividing by h gives

$$\frac{B(h)}{h} = \frac{A(h)}{h^2} + \frac{1}{h} \int_0^h \frac{A(t)}{t^2} dt.$$

Since

$$\frac{A(h)}{h^2} \rightarrow \frac{L}{2}$$

we get

$$\frac{1}{h} \int_0^h \frac{A(t)}{t^2} dt \rightarrow \frac{L}{2},$$

that is

$$\frac{B(h)}{h} \rightarrow L.$$

Thus

$$\text{BD}^+ f(x) = L.$$

The left-hand proof is identical, with

$$u(t) := f(x) - f(x-t).$$

Thus the right- and left-hand equivalences hold with equal values. Combining the two one-sided assertions gives

$$\text{BD}f(x) \text{ exists} \iff C_1 Df(x) \text{ exists},$$

and, whenever these derivatives exist, $\text{BD}f(x) = C_1 Df(x)$. $\square$

LEMMA 4.6. *Let $m > -1$, let $\rho > 0$, and let F be continuous on $[0, \rho]$. If*

$$F(h) = Kh^m + o(h^m) \quad (h \downarrow 0),$$

then

$$\lim_{s \rightarrow \infty} s^{m+1} \int_0^\rho e^{-sh} F(h) dh = K\Gamma(m+1).$$

Proof. Write

$$F(h) = Kh^m + h^m \eta(h), \quad \eta(h) \rightarrow 0 \quad (h \downarrow 0).$$

By Lemma 2.2,

$$s^{m+1} \int_0^\rho e^{-sh} Kh^m dh \rightarrow K\Gamma(m+1).$$

It remains to show that

$$s^{m+1} \int_0^\rho e^{-sh} h^m \eta(h) dh \rightarrow 0.$$

Let $\varepsilon > 0$. Choose $\rho_0 \in (0, \rho)$ such that

$$|\eta(h)| < \varepsilon \quad (0 < h < \rho_0).$$

Then, again by Lemma 2.2,

$$s^{m+1} \left| \int_0^{\rho_0} e^{-sh} h^m \eta(h) dh \right| \leq \varepsilon s^{m+1} \int_0^{\rho_0} e^{-sh} h^m dh \rightarrow \varepsilon \Gamma(m+1).$$

On the interval $[\rho_0, \rho]$, the function $h^m \eta(h)$ is bounded. Hence

$$s^{m+1} \left| \int_{\rho_0}^\rho e^{-sh} h^m \eta(h) dh \right| \leq C s^{m+1} e^{-s\rho_0} \rightarrow 0.$$

Therefore

$$\limsup_{s \rightarrow \infty} \left| s^{m+1} \int_0^\rho e^{-sh} h^m \eta(h) dh \right| \leq \varepsilon \Gamma(m+1).$$

Since $\varepsilon > 0$ is arbitrary, the error term tends to 0. This proves the result. $\square$

PROPOSITION 4.7. *Let $r \geq 1$, and suppose that f is continuous on a right neighborhood of x . If $C_r D^+ f(x)$ exists, then $\text{LD}^+ f(x)$ exists and*

$$\text{LD}^+ f(x) = C_r D^+ f(x).$$

The analogous left-hand statement holds when f is continuous on a left neighborhood of x , with $C_r D^- f(x)$ and $\text{LD}^- f(x)$ in place of $C_r D^+ f(x)$ and $\text{LD}^+ f(x)$, respectively. Consequently, if f is continuous in a neighborhood of x and $C_r D f(x)$ exists, then $\text{LD} f(x)$ exists and

$$\text{LD} f(x) = C_r D f(x).$$

Proof. We prove the right-hand case. Put

$$A := C_r D^+ f(x), \quad U(t) := f(x+t) - f(x).$$

By the definition of $C_r D^+ f(x)$,

$$\frac{r(r+1)}{h^{r+1}} \int_0^h (h-t)^{r-1} U(t) dt \longrightarrow A \quad (h \downarrow 0).$$

Thus, if

$$F(h) := \int_0^h (h-t)^{r-1} U(t) dt,$$

then

$$F(h) = \frac{A}{r(r+1)} h^{r+1} + o(h^{r+1}) \quad (h \downarrow 0). \quad (3)$$

Choose $\rho > 0$ so that f is continuous on $[x, x + \rho]$. We now define a convolution extension of F . For $h \geq 0$, put

$$F_*(h) := \int_0^\rho (h-t)_+^{r-1} U(t) dt,$$

where

$$(h-t)_+^{r-1} = \begin{cases} (h-t)^{r-1}, & t < h, \\ 0, & t \geq h. \end{cases}$$

Then $F_*(h) = F(h)$ for $0 < h < \rho$. Hence (3) also holds for F_* as $h \downarrow 0$.

We next compute the Laplace transform of F_* . Since U is bounded on $[0, \rho]$, Fubini's theorem gives

$$\begin{aligned} \int_0^\infty e^{-sh} F_*(h) dh &= \int_0^\infty e^{-sh} \int_0^\rho (h-t)_+^{r-1} U(t) dt dh \\ &= \int_0^\rho U(t) \int_t^\infty e^{-sh} (h-t)^{r-1} dh dt. \end{aligned}$$

In the inner integral, put $u = h - t$. Then

$$\int_t^\infty e^{-sh} (h-t)^{r-1} dh = e^{-st} \int_0^\infty e^{-su} u^{r-1} du = e^{-st} \frac{(r-1)!}{s^r}.$$

Therefore

$$\int_0^\infty e^{-sh} F_*(h) dh = \frac{(r-1)!}{s^r} \int_0^\rho e^{-st} U(t) dt. \quad (4)$$

Now we apply Lemma 4.6 to F_* near zero. From (3), with

$$K = \frac{A}{r(r+1)} \quad \text{and} \quad m = r+1,$$

we get

$$s^{r+2} \int_0^\rho e^{-sh} F_*(h) dh \longrightarrow \frac{A}{r(r+1)} \Gamma(r+2).$$

Since U is bounded on $[0, \rho]$, there is a constant $C > 0$ such that

$$|F_*(h)| \leq Ch^{r-1} \quad (h \geq \rho).$$

Hence

$$s^{r+2} \int_\rho^\infty e^{-sh} |F_*(h)| dh \leq Cs^{r+2} \int_\rho^\infty e^{-sh} h^{r-1} dh \longrightarrow 0.$$

Consequently,

$$s^{r+2} \int_0^\infty e^{-sh} F_*(h) dh \longrightarrow \frac{A}{r(r+1)} \Gamma(r+2).$$

Using (4), we obtain

$$\begin{aligned} s^2 \int_0^p e^{-st} \{f(x+t) - f(x)\} dt &= \frac{s^{r+2}}{(r-1)!} \int_0^\infty e^{-sh} F_*(h) dh \\ &\longrightarrow \frac{1}{(r-1)!} \cdot \frac{A}{r(r+1)} \Gamma(r+2) = A. \end{aligned}$$

Thus

$$\text{LD}^+ f(x) = A = C_r D^+ f(x).$$

The left-hand assertion follows in the same way, with

$$A := C_r D^- f(x), \quad U(t) := f(x) - f(x-t).$$

Thus the corresponding left Laplace derivative exists and equals $C_r D^- f(x)$. If $C_r D f(x)$ exists, the two one-sided Cesàro derivatives are equal; therefore the two one-sided Laplace derivatives are equal, and $\text{LD} f(x) = C_r D f(x)$. This proves the proposition. $\square$

PROPOSITION 4.8. *Let $1 \leq p < \infty$ and $r \geq 1$, and suppose that f is continuous on a right neighborhood of x . If $\mathcal{D}_p^+ f(x)$ exists, then $C_r D^+ f(x)$ exists and*

$$C_r D^+ f(x) = \mathcal{D}_p^+ f(x).$$

The analogous left-hand statement holds when f is continuous on a left neighborhood of x , with $\mathcal{D}_p^- f(x)$ and $C_r D^- f(x)$ in place of $\mathcal{D}_p^+ f(x)$ and $C_r D^+ f(x)$, respectively. Consequently, if f is continuous in a neighborhood of x and $\mathcal{D}_p f(x)$ exists, then $C_r D f(x)$ exists for every $r \geq 1$, and

$$C_r D f(x) = \mathcal{D}_p f(x).$$

Proof. We prove the right-hand case. Put $A := \mathcal{D}_p^+ f(x)$ and

$$R(t) := f(x+t) - f(x) - At.$$

By the definition of $\mathcal{D}_p^+ f(x)$,

$$\left(\int_0^h |R(t)|^p dt \right)^{1/p} = o(h^{1+1/p}).$$

Using the definition of the right Cesàro mean,

$$\mathcal{C}_{r,h}^+ f(x) - f(x) = \frac{r}{h^r} \int_0^h (h-t)^{r-1} \{At + R(t)\} dt.$$

The polynomial part is

$$\frac{r}{h^r} \int_0^h (h-t)^{r-1} At dt = \frac{Ah}{r+1}.$$

It remains to estimate the remainder. If $p > 1$, let $q = p/(p-1)$. Hölder's inequality gives

$$\left| \frac{r}{h^r} \int_0^h (h-t)^{r-1} R(t) dt \right| \leq \frac{r}{h^r} \left(\int_0^h |R(t)|^p dt \right)^{1/p} \left(\int_0^h (h-t)^{(r-1)q} dt \right)^{1/q} = o(h).$$

For $p = 1$, the same conclusion follows from

$$\left| \frac{r}{h^r} \int_0^h (h-t)^{r-1} R(t) dt \right| \leq \frac{rh^{r-1}}{h^r} \int_0^h |R(t)| dt = o(h).$$

Therefore

$$\mathcal{C}_{r,h}^+ f(x) - f(x) = \frac{Ah}{r+1} + o(h),$$

and hence

$$C_r D^+ f(x) = \lim_{h \downarrow 0} \frac{r+1}{h} (\mathcal{C}_{r,h}^+ f(x) - f(x)) = A.$$

The left-hand proof is identical after taking $A := \mathcal{D}_p^- f(x)$ and replacing $f(x+t) - f(x)$ by $f(x) - f(x-t)$. If $\mathcal{D}_p f(x)$ exists, then the two one-sided L^p derivatives agree; hence the two one-sided Cesàro derivatives agree and $C_r Df(x) = \mathcal{D}_p f(x)$. $\square$

COROLLARY 4.9. *Let $\mathfrak{D}$ denote one of the following bilateral first-order generalized derivatives:*

$$\mathfrak{D} \in \{ \text{BD}, C_r D \ (r \geq 1), \mathcal{D}_p \ (1 \leq p < \infty) \}.$$

Let $\mathfrak{D}^+$ and $\mathfrak{D}^-$ denote the corresponding right and left one-sided derivatives. Let f be continuous on $[a, b]$. Suppose that $\mathfrak{D}f(x)$ exists for every $x \in (a, b)$ and that

$$\mathfrak{D}^+ f(a) = \mathfrak{D}^- f(b) =: M.$$

Then there exists $c \in (a, b)$ such that

$$\mathfrak{D}f(c) = \frac{f(c) - f(a)}{c - a}.$$

Proof. The one-sided comparison results imply the endpoint Laplace conditions. Indeed, according to the choice of $\mathfrak{D}$, Lemma 4.5, Remark 4.4, and Propositions 4.7 and 4.8 give

$$\text{LD}^+ f(a) = \mathfrak{D}^+ f(a) = M, \quad \text{LD}^- f(b) = \mathfrak{D}^- f(b) = M.$$

At each interior point where $\mathfrak{D}f$ exists, the corresponding bilateral comparison also gives $\text{LD}f = \mathfrak{D}f$. Applying the Laplace–Flett theorem, Theorem 3.1, we obtain a point $c \in (a, b)$ such that

$$\text{LD}f(c) = \frac{f(c) - f(a)}{c - a}.$$

Using again the equality $\text{LD}f(c) = \mathfrak{D}f(c)$ gives the desired relation. $\square$

5. Counterexamples and strictness of the implications

The following table lists the converse implications which fail. The details are proved in the examples below.

Forward implication	Converse that fails	Example proving failure
$\mathcal{D}_p f(x) \Rightarrow BDf(x)$	$BDf(x) \not\Rightarrow \mathcal{D}_p f(x)$	Example 5.2
$\mathcal{D}_p f(x) \Rightarrow C_r Df(x), r \geq 1$	$C_r Df(x) \not\Rightarrow \mathcal{D}_p f(x)$	Example 5.2
$BDf(x) \Rightarrow LDf(x)$	$LDf(x) \not\Rightarrow BDf(x)$	Example 5.5
$BDf(x) \Rightarrow C_r Df(x), r \geq 2$	$C_r Df(x) \not\Rightarrow BDf(x)$	Example 5.3
$C_1 Df(x) \Rightarrow C_r Df(x), r \geq 2$	$C_r Df(x) \not\Rightarrow C_1 Df(x)$	Example 5.3, since $C_1 D \iff BD$
$C_r Df(x) \Rightarrow LDf(x)$	$LDf(x) \not\Rightarrow C_r Df(x)$	Example 5.7

Table 1. Examples showing that the implications in Figure 1 are not reversible.

Mukhopadhyay gives several strictness examples in Chapter 2. In the Laplace–Peano comparison, he considers

$$f(x) = \begin{cases} x^n, & x = 0 \text{ or } x \text{ is irrational,} \\ 1, & x \text{ is rational and } x \neq 0, \end{cases}$$

and shows that $LD_n f(0)$ exists with value $n!$, although the ordinary derivative at 0 does not exist; for $n = 1$ this example is not continuous at 0 [12, Example after Theorem 2.18.5, p. 169]. In the Borel– L^p comparison, a singular function

$$f(x) = \begin{cases} x^{-1} \cos(x^{-3}), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is presented to show that $BD_1 f(0) = 0$, although $f \notin L^p$ in any neighborhood of 0; hence no first-order L^p derivative can exist there [12, Theorem 2.13.1 and the following example, pp. 142–143]. For Cesàro derivatives, the examples in Section 1.8.2 show that the inclusions in the Cesàro scale can be strict [12, Examples after Theorem 1.8.10, pp. 38–40].

5.1. Borel and Cesàro differentiability without an L^p derivative

LEMMA 5.1. *Let $w \in L^1([a, \infty))$, and let ϕ be a continuous 2π -periodic function. Then*

$$\lim_{\lambda \rightarrow \infty} \int_a^\infty w(v) \phi(\lambda v) dv = \left(\frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) d\theta \right) \int_a^\infty w(v) dv.$$

Proof. Put $\phi_{av} := \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) d\theta$ and define $\phi_0(\theta) := \phi(\theta) - \phi_{av}$. Then ϕ_0 is continuous and 2π -periodic, and $\int_0^{2\pi} \phi_0(\theta) d\theta = 0$. Also,

$$\int_a^\infty w(v) \phi(\lambda v) dv = \phi_{av} \int_a^\infty w(v) dv + \int_a^\infty w(v) \phi_0(\lambda v) dv.$$

It is enough to prove that the second term tends to 0.

Let $\varepsilon > 0$. By the Weierstrass approximation theorem, there is a trigonometric polynomial $Q(\theta) = \sum_{|n| \leq N} d_n e^{in\theta}$ such that $\|\phi_0 - Q\|_\infty < \frac{\varepsilon}{2}$. Since ϕ_0 has mean zero,

$$|d_0| = \left| \frac{1}{2\pi} \int_0^{2\pi} \{Q(\theta) - \phi_0(\theta)\} d\theta \right| \leq \|\phi_0 - Q\|_\infty < \frac{\varepsilon}{2}.$$

Therefore the trigonometric polynomial

$$P(\theta) := Q(\theta) - d_0 = \sum_{0 < |n| \leq N} d_n e^{in\theta}$$

has no constant term and satisfies

$$\|\phi_0 - P\|_\infty < \varepsilon.$$

Hence

$$\left| \int_a^\infty w(v) \{ \phi_0(\lambda v) - P(\lambda v) \} dv \right| \leq \varepsilon \|w\|_{L^1([a, \infty))}.$$

Next,

$$\int_a^\infty w(v) P(\lambda v) dv = \sum_{0 < |n| \leq N} d_n \int_a^\infty w(v) e^{in\lambda v} dv.$$

Extend w to a function $\tilde{w}$ on $\mathbb{R}$ by setting

$$\tilde{w}(v) := \begin{cases} w(v), & v \geq a, \\ 0, & v < a. \end{cases}$$

Then $\tilde{w} \in L^1(\mathbb{R})$. By the Riemann–Lebesgue lemma [8, Theorem 8.22(f), p. 249],

$$\int_{\mathbb{R}} \tilde{w}(v) e^{-i\xi v} dv \rightarrow 0 \quad (|\xi| \rightarrow \infty).$$

For each fixed integer $n \neq 0$, taking $\xi = -n\lambda$ gives

$$\int_a^\infty w(v) e^{in\lambda v} dv = \int_{\mathbb{R}} \tilde{w}(v) e^{in\lambda v} dv \rightarrow 0 \quad (\lambda \rightarrow \infty).$$

Since P is a finite trigonometric polynomial, it follows that

$$\int_a^\infty w(v) P(\lambda v) dv \rightarrow 0.$$

Consequently,

$$\limsup_{\lambda \rightarrow \infty} \left| \int_a^\infty w(v) \phi_0(\lambda v) dv \right| \leq \varepsilon \|w\|_{L^1([a, \infty))}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\int_a^\infty w(v) \phi_0(\lambda v) dv \rightarrow 0.$$

Therefore

$$\lim_{\lambda \rightarrow \infty} \int_a^\infty w(v) \phi(\lambda v) dv = \phi_{av} \int_a^\infty w(v) dv. \quad \square$$

EXAMPLE 5.2. Fix $1 \leq p < \infty$ and define

$$f(x) = \begin{cases} x \sin(x^{-2}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is continuous at 0, $\text{BD}f(0) = 0$, and $C_r Df(0) = 0$ for every $r \geq 1$, but $\mathcal{D}_p f(0)$ does not exist.

Proof. We first compute the Borel derivative at 0. Putting $t = hv^{-1/2}$, we get

$$\frac{1}{h} \int_0^h \frac{f(t) - f(0)}{t} dt = \frac{1}{h} \int_0^h \sin(t^{-2}) dt = \frac{1}{2} \int_1^\infty v^{-3/2} \sin(h^{-2}v) dv.$$

Now $v^{-3/2} \in L^1[1, \infty)$, and $h^{-2} \rightarrow \infty$ as $h \downarrow 0$. By Lemma 5.1, applied with

$$w(v) = \frac{1}{2}v^{-3/2}, \quad \phi(\theta) = \sin \theta, \quad \lambda = h^{-2},$$

we obtain

$$\frac{1}{2} \int_1^\infty v^{-3/2} \sin(h^{-2}v) dv \longrightarrow \left(\frac{1}{2\pi} \int_0^{2\pi} \sin \theta d\theta \right) \int_1^\infty \frac{1}{2}v^{-3/2} dv = 0.$$

Thus $\text{BD}^+ f(0) = 0$. The left-hand calculation is the same. Consequently, $\text{BD}f(0) = 0$. Lemma 4.5 and Mukhopadhyay's Cesàro scale theorem [12, Corollary 1.8.5] therefore give

$$C_r Df(0) = 0 \quad (r \geq 1).$$

Suppose, for a contradiction, that $\mathcal{D}_p f(0)$ exists with value A . Then

$$\frac{1}{h^{p+1}} \int_0^h |t \sin(t^{-2}) - At|^p dt \rightarrow 0. \quad (5)$$

Putting $t = hv^{-1/2}$ and $B = h^{-2}$ the left-hand side equals

$$\frac{1}{2} \int_1^\infty v^{-(p+3)/2} |\sin(Bv) - A|^p dv.$$

By Lemma 5.1, applied with

$$w(v) = \frac{1}{2}v^{-(p+3)/2}, \quad \phi(\theta) = |\sin \theta - A|^p,$$

the integral tends to

$$\frac{1}{p+1} \cdot \frac{1}{2\pi} \int_0^{2\pi} |\sin \theta - A|^p d\theta.$$

The last number is strictly positive for every real A , because $|\sin \theta - A|^p$ cannot vanish identically on $[0, 2\pi]$. This contradicts (5). Hence $\mathcal{D}_p f(0)$ does not exist. $\square$

5.2. Cesàro differentiability (C_rD) without Borel differentiability

EXAMPLE 5.3. Define

$$f(x) = \begin{cases} |x|^{1/2} \sin(|x|^{-1/2}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is continuous at 0 and $C_rDf(0) = 0$ for every $r \geq 2$, but $\text{BD}f(0)$ does not exist. Consequently, neither $C_1Df(0)$ nor $\mathcal{D}_pf(0)$ exists.

Proof. We first compute the second right Cesàro derivative. For $q > -1$, put

$$J_q(h) := \int_0^h t^q \sin(t^{-1/2}) dt.$$

With the substitution $t = u^{-2}$ and $X = h^{-1/2}$, we get

$$J_q(h) = 2 \int_X^\infty u^{-2q-3} \sin u du.$$

We shall use the following elementary Dirichlet estimate: if $\beta > 0$, then

$$\int_X^\infty u^{-\beta} \cos u du = O(X^{-\beta}) \quad (X \rightarrow \infty).$$

Indeed, for $R > X$, set

$$C_R(y) = \int_X^y \cos u du.$$

Then $|C_R(y)| \leq 2$. Abel's integration formula gives

$$\int_X^R u^{-\beta} \cos u du = R^{-\beta} C_R(R) + \beta \int_X^R C_R(u) u^{-\beta-1} du,$$

and hence

$$\left| \int_X^R u^{-\beta} \cos u du \right| \leq 2R^{-\beta} + 2\beta \int_X^R u^{-\beta-1} du \leq 2X^{-\beta}.$$

Letting $R \rightarrow \infty$ gives the estimate.

Now put

$$\alpha = 2q + 3.$$

Since $q > -1$, we have $\alpha > 1$. Integration by parts gives

$$\begin{aligned} \int_X^\infty u^{-\alpha} \sin u du &= \left[-u^{-\alpha} \cos u \right]_X^\infty - \alpha \int_X^\infty u^{-\alpha-1} \cos u du \\ &= X^{-\alpha} \cos X + O(X^{-\alpha-1}), \end{aligned}$$

Therefore

$$J_q(h) = 2h^{q+3/2} \cos(h^{-1/2}) + O(h^{q+2}).$$

In particular,

$$J_{1/2}(h) = 2h^2 \cos(h^{-1/2}) + O(h^{5/2}) \quad \text{and} \quad J_{3/2}(h) = 2h^3 \cos(h^{-1/2}) + O(h^{7/2}).$$

Hence

$$\begin{aligned} \int_0^h (h-t)t^{1/2} \sin(t^{-1/2}) dt &= hJ_{1/2}(h) - J_{3/2}(h) \\ &= h\{2h^2 \cos(h^{-1/2}) + O(h^{5/2})\} \\ &\quad - \{2h^3 \cos(h^{-1/2}) + O(h^{7/2})\} \\ &= O(h^{7/2}). \end{aligned}$$

Therefore

$$C_2 D^+ f(0) = \lim_{h \downarrow 0} \frac{6}{h^3} \int_0^h (h-t)t^{1/2} \sin(t^{-1/2}) dt = 0.$$

The left-hand calculation also gives $C_2 D^- f(0) = 0$, because the corresponding quantity has the same order estimate. Hence $C_2 Df(0) = 0$, and the Cesàro scale theorem gives

$$C_r Df(0) = 0 \quad (r \geq 2).$$

On the other hand,

$$\begin{aligned} \frac{1}{h} \int_0^h \frac{f(t) - f(0)}{t} dt &= \frac{1}{h} \int_0^h t^{-1/2} \sin(t^{-1/2}) dt \\ &= \frac{1}{h} J_{-1/2}(h) \\ &= 2 \cos(h^{-1/2}) + O(h^{1/2}). \end{aligned}$$

This has no limit as $h \downarrow 0$. Therefore $BD^+ f(0)$ does not exist, and so $BDf(0)$ does not exist. By Lemma 4.5, $C_1 Df(0)$ does not exist. Also $\mathcal{D}_p f(0)$ cannot exist, because $\mathcal{D}_p f(0)$ would imply $BDf(0)$. $\square$

5.3. Laplace differentiability without Borel differentiability

LEMMA 5.4. *Let P be a polynomial and let $\delta > 0$. Then*

$$\int_0^{s\delta} e^{-u} P(u) \sin((\log(s/u))^2) du \rightarrow 0 \quad (s \rightarrow \infty).$$

Proof. Put $T = \log s$ and write $u = e^v$. Then $du = e^v dv$, and

$$\log(s/u) = T - v.$$

Hence

$$\int_0^{s\delta} e^{-u} P(u) \sin((\log(s/u))^2) du = \int_{-\infty}^{T+\log \delta} W(v) \sin((T-v)^2) dv,$$

where

$$W(v) = e^{-e^v} P(e^v) e^v.$$

The function W belongs to $L^1(\mathbb{R})$. Indeed, as $v \rightarrow -\infty$, $P(e^v)$ is bounded and so $W(v) = O(e^v)$. As $v \rightarrow +\infty$, the factor e^{-e^v} dominates every power of e^v , and therefore gives rapid decay.

Since $W \in L^1(\mathbb{R})$ and $T + \log \delta \rightarrow \infty$, we have

$$\int_{T+\log \delta}^{\infty} |W(v)| dv \rightarrow 0.$$

Thus the original integral differs by $o(1)$ from

$$\int_{-\infty}^{\infty} W(v) \sin((T-v)^2) dv.$$

Now

$$\sin((T-v)^2) = \Im \left(e^{iT^2} e^{-2iTv} e^{iv^2} \right).$$

Therefore

$$\int_{-\infty}^{\infty} W(v) \sin((T-v)^2) dv = \Im \left(e^{iT^2} \int_{-\infty}^{\infty} W(v) e^{iv^2} e^{-2iTv} dv \right).$$

The function

$$v \mapsto W(v) e^{iv^2}$$

belongs to $L^1(\mathbb{R})$. Hence, by the Riemann–Lebesgue lemma [8, Theorem 8.22(f), p. 249],

$$R(T) := \int_{-\infty}^{\infty} W(v) e^{iv^2} e^{-2iTv} dv \rightarrow 0 \quad (T \rightarrow \infty).$$

Since

$$\left| \Im \{ e^{iT^2} R(T) \} \right| \leq |e^{iT^2} R(T)| = |R(T)|,$$

we get

$$\Im \{ e^{iT^2} R(T) \} \rightarrow 0.$$

Thus the required limit follows. $\square$

EXAMPLE 5.5. Define a function in a neighborhood of 0 as follows. For $0 < x < e^{-1}$ set

$$A(x) := \sin((\log(1/x))^2), \quad Q(x) := xA(x),$$

and set

$$f(x) := \begin{cases} xQ'(x), & 0 < x < e^{-1}, \\ 0, & x \leq 0. \end{cases}$$

Equivalently, for $0 < x < e^{-1}$,

$$f(x) = x \sin((\log(1/x))^2) - 2x \log(1/x) \cos((\log(1/x))^2).$$

Then f is continuous at 0, $\text{LD}f(0) = 0$, but $\text{BD}f(0)$ does not exist. Consequently, neither $C_1 Df(0)$ nor $\mathcal{D}_p f(0)$ exists.

Proof. Continuity at 0 follows from

$$|f(x)| \leq x + 2x \log(1/x) \rightarrow 0.$$

For the right Borel average,

$$\frac{1}{h} \int_0^h \frac{f(t) - f(0)}{t} dt = \frac{1}{h} \int_0^h Q'(t) dt = \frac{Q(h)}{h} = \sin((\log(1/h))^2),$$

which has no limit as $h \downarrow 0$. Therefore $\text{BD}^+ f(0)$, and hence $\text{BD} f(0)$, does not exist.

It remains to compute the right Laplace derivative.

Since $tQ'(t) = (tQ(t))' - Q(t)$, we have using integration by parts,

$$\begin{aligned} s^2 \int_0^\delta e^{-st} tQ'(t) dt &= s^2 \int_0^\delta e^{-st} (tQ(t))' dt - s^2 \int_0^\delta e^{-st} Q(t) dt \\ &= s^2 \left[e^{-st} tQ(t) \right]_0^\delta + s^3 \int_0^\delta e^{-st} tQ(t) dt - s^2 \int_0^\delta e^{-st} Q(t) dt \end{aligned}$$

Now $tQ(t) \rightarrow 0$ as $t \downarrow 0$, and

$$s^2 e^{-s\delta} \delta Q(\delta) \rightarrow 0.$$

Hence the boundary term is $o(1)$. Since

$$Q(t) = t \sin((\log(1/t))^2),$$

we get

$$\begin{aligned} s^2 \int_0^\delta e^{-st} f(t) dt &= s^3 \int_0^\delta e^{-st} t^2 \sin((\log(1/t))^2) dt \\ &\quad - s^2 \int_0^\delta e^{-st} t \sin((\log(1/t))^2) dt + o(1). \end{aligned}$$

Putting $u = st$, this becomes

$$s^2 \int_0^\delta e^{-st} f(t) dt = \int_0^{s\delta} e^{-u} (u^2 - u) \sin((\log(s/u))^2) du + o(1).$$

By Lemma 5.4, applied to the polynomial $P(u) = u^2 - u$, the last integral tends to 0. Therefore $\text{LD}^+ f(0) = 0$. Since $f(-t) = 0$ for $t > 0$, we also have $\text{LD}^- f(0) = 0$. Thus $\text{LD} f(0) = 0$.

Finally, by Lemma 4.5, the non-existence of $\text{BD} f(0)$ implies the non-existence of $C_1 D f(0)$. Also, since the first L^p -derivative implies the Borel derivative, $\mathcal{D}_p f(0)$ cannot exist. $\square$

5.4. Laplace differentiability without Cesàro differentiability

LEMMA 5.6. *Let $r \geq 1$ be an integer, and let*

$$\psi(T) = \sin(T^2), \quad a_r(T) = \psi^{(r)}(T).$$

Then the following two assertions hold.

1. If

$$W(v) = e^{-e^{-v}} e^{-2v}, \quad v \in \mathbb{R},$$

then

$$\int_{-\infty}^{\infty} W(v) a_r(T+v) dv \rightarrow 0 \quad (T \rightarrow \infty).$$

2. If

$$W_r(v) = r(r+1)(1 - e^{-v})^{r-1} e^{-2v}, \quad v \geq 0,$$

then

$$\int_0^{\infty} W_r(v) a_r(T+v) dv = A_r \sin(T^2) + o(1) \quad (T \rightarrow \infty),$$

where

$$A_r = (-1)^r r(r+1)(r-1)! \neq 0.$$

Proof. For the first assertion, note that W and its derivatives up to order r belong to $L^1(\mathbb{R})$, and all boundary terms vanish under integration by parts. Since $a_r = \psi^{(r)}$, integrating by parts r times gives

$$\int_{-\infty}^{\infty} W(v) a_r(T+v) dv = (-1)^r \int_{-\infty}^{\infty} W^{(r)}(v) \psi(T+v) dv.$$

Now

$$\psi(T+v) = \sin((T+v)^2) = \Im \left(e^{iT^2} e^{2iTv} e^{iv^2} \right).$$

Since $W^{(r)}(v) e^{iv^2} \in L^1(\mathbb{R})$, the Riemann–Lebesgue lemma gives

$$\int_{-\infty}^{\infty} W^{(r)}(v) e^{iv^2} e^{2iTv} dv \rightarrow 0.$$

Therefore the first assertion follows.

For the second assertion, W_r and its derivatives have exponential decay at infinity. Also W_r vanishes at 0 to order $r-1$, and

$$W_r^{(r-1)}(0) = r(r+1)(r-1)!.$$

Integrating by parts r times on $[0, \infty)$, the only non-zero boundary term at 0 comes from $W_r^{(r-1)}(0)$. Thus

$$\int_0^{\infty} W_r(v) a_r(T+v) dv = (-1)^r W_r^{(r-1)}(0) \psi(T) + (-1)^r \int_0^{\infty} W_r^{(r)}(v) \psi(T+v) dv.$$

The last integral is $o(1)$ by the Riemann–Lebesgue lemma, as above. Hence

$$\int_0^{\infty} W_r(v) a_r(T+v) dv = (-1)^r r(r+1)(r-1)! \sin(T^2) + o(1),$$

which proves the second assertion. $\square$

EXAMPLE 5.7. Fix an integer $r \geq 1$. Put

$$\psi(T) = \sin(T^2), \quad a_r(T) = \psi^{(r)}(T),$$

and define, in a neighborhood of 0,

$$f_r(x) = \begin{cases} x a_r(\log(1/x)), & 0 < x < e^{-1}, \\ 0, & x \leq 0. \end{cases}$$

Then f_r is continuous at 0, $\text{LD}f_r(0) = 0$, but $C_r Df_r(0)$ does not exist.

Proof. The function $a_r(T)$ is a finite linear combination of terms of the form

$$P(T)\sin(T^2) \quad \text{or} \quad P(T)\cos(T^2),$$

where P is a polynomial. Hence

$$x a_r(\log(1/x)) \rightarrow 0 \quad (x \downarrow 0),$$

and so f_r is continuous at 0.

We first prove the Laplace differentiability. Choose $0 < \rho < e^{-1}$, and let $T = \log s$. Then

$$s^2 \int_0^\rho e^{-sx} f_r(x) dx = \int_0^{s\rho} e^{-u} u a_r(T - \log u) du.$$

Putting $u = e^{-v}$, this becomes

$$\int_{-T-\log\rho}^\infty e^{-e^{-v}} e^{-2v} a_r(T+v) dv.$$

To apply Lemma 5.6 we show that

$$R_T := \int_{-\infty}^{-T-\log\rho} e^{-e^{-v}} e^{-2v} a_r(T+v) dv$$

tends to 0. Put $y = T + v$. Since $s = e^T$, this gives

$$R_T = s^2 \int_{-\infty}^{-\log\rho} e^{-se^{-y}} e^{-2y} a_r(y) dy.$$

Now put $z = e^{-y}$. Then $dy = -dz/z$, and the last integral becomes

$$R_T = s^2 \int_\rho^\infty e^{-sz} z a_r(\log(1/z)) dz.$$

Since a_r has at most polynomial growth, there are constants $C > 0$ and $N \in \mathbb{N}$ such that

$$|a_r(\log(1/z))| \leq C\{1 + |\log z|^N\}, \quad z \geq \rho.$$

Hence

$$|R_T| \leq C s^2 \int_{\rho}^{\infty} e^{-sz} z \{1 + |\log z|^N\} dz.$$

For $z \geq \rho$,

$$e^{-sz} \leq e^{-s\rho/2} e^{-sz/2}.$$

Thus, for large s ,

$$|R_T| \leq C s^2 e^{-s\rho/2} \int_{\rho}^{\infty} e^{-z/2} z \{1 + |\log z|^N\} dz \rightarrow 0.$$

Thus $R_T \rightarrow 0$. With $W(v) = e^{-e^{-v}} e^{-2v}$, we may write

$$s^2 \int_0^{\rho} e^{-sx} f_r(x) dx = \int_{-\infty}^{\infty} W(v) a_r(T+v) dv - R_T.$$

As $T \rightarrow \infty$, by Lemma 5.6,

$$\int_{-\infty}^{\infty} W(v) a_r(T+v) dv \rightarrow 0$$

Therefore

$$\text{LD}^+ f_r(0) = 0.$$

Since $f_r(-x) = 0$ for $x > 0$, the left Laplace integral is identically zero; hence

$$\text{LD}^- f_r(0) = 0.$$

Thus

$$\text{LD} f_r(0) = 0.$$

We now show that the right Cesàro derivative $C_r D^+ f_r(0)$ does not exist. Since $f_r(0) = 0$, the expression defining $C_r D^+ f_r(0)$ is

$$\frac{r(r+1)}{h^{r+1}} \int_0^h (h-t)^{r-1} f_r(t) dt.$$

Put $T = \log(1/h)$ and $t = he^{-v}$, so

$$\log(1/t) = \log(1/h) + v = T + v.$$

Then

$$\begin{aligned} & \frac{r(r+1)}{h^{r+1}} \int_0^h (h-t)^{r-1} f_r(t) dt \\ &= \frac{r(r+1)}{h^{r+1}} \int_{\infty}^0 h^{r-1} (1 - e^{-v})^{r-1} \cdot h e^{-v} a_r(T+v) \cdot (-h e^{-v}) dv \\ &= r(r+1) \int_0^{\infty} (1 - e^{-v})^{r-1} e^{-2v} a_r(T+v) dv \\ &= \int_0^{\infty} W_r(v) a_r(T+v) dv, \end{aligned}$$

where

$$W_r(v) = r(r+1)(1 - e^{-v})^{r-1}e^{-2v}.$$

By Lemma 5.6,

$$\int_0^\infty W_r(v)a_r(T+v)dv = A_r \sin(T^2) + o(1),$$

where $A_r \neq 0$. As $h \downarrow 0$, $T = \log(1/h) \rightarrow \infty$. Since $\sin(T^2)$ has no limit as $T \rightarrow \infty$, the right Cesàro derivative $C_r D^+ f_r(0)$ does not exist. Hence the bilateral Cesàro derivative $C_r D f_r(0)$ does not exist. $\square$

6. Conclusion

We proved a Flett-type mean value theorem for the first-order Laplace derivative. The result has the same form as the classical Flett theorem, but the ordinary derivative is replaced by the Laplace derivative and the endpoint condition is expressed in terms of the right and left Laplace derivatives.

We then used comparison results among first-order generalized derivatives to obtain corresponding Flett-type results for the Borel derivative, the Cesàro derivative $C_r D$, and the first-order L^p -derivative. In particular, the implication from the Borel derivative to the Laplace derivative, the equivalence between the Borel derivative and $C_1 D$, the implication from $C_r D$ to the Laplace derivative, and the implication from the first-order L^p -derivative to $C_r D$ allow these cases to be reduced to the Laplace case. Thus several Flett-type theorems follow from one common argument.

Finally, the examples show that the implication scheme is not reversible in general.

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