

CERTAIN GEOMETRIC PROPERTIES OF THE QUASI-CESÀRO MATRIX

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Abstract. The purpose of this research is to examine some geometric properties of the Quasi-Cesàro matrix. It is shown that these spaces have interesting properties such as uniform rotundity, uniform smoothness, and the Radon-Riesz property. More specifically, we investigate some of the further properties defined with respect to the bounded linear operators on them. The Approximation property, the Dunford-Pettis property and Hahn-Banach property are other important properties that belong to these spaces.

1. Introduction and notations

Throughout the text, let us use the notation $\mathbb{N} = \{0, 1, 2, \dots\}$. Let ω indicate the space of all complex sequences. For any sequence space Φ and infinite matrix A , the matrix domain Φ_A of A is defined as $\Phi_A = \{t = (t_v) \in \omega : At \in \Phi\}$. The classical sequence spaces ℓ_p for $1 \leq p < \infty$ and ℓ_∞ are the fundamental objects in functional analysis with well-known geometric properties. The Banach spaces ℓ_p for $1 \leq p < \infty$ and ℓ_∞ are the set of all sequences $t = (t_v)$ such that

$$\|t\|_{\ell_p} = \left(\sum_{v=0}^{\infty} |t_v|^p \right)^{1/p} < \infty \quad \text{and} \quad \|t\|_{\ell_\infty} = \sup_{v \in \mathbb{N}} |t_v| < \infty, \quad (1)$$

respectively. A Banach space X is called a *BK* space if it has continuous coordinates. The space ℓ_p be a *BK* space with respect to the ℓ_p -norm described in (1). An infinite matrix can be observed as the linear operator from a sequence space to another sequence space. For this, let X and Y be arbitrary subsets of ω . Let $A = (a_{nv})$ be an infinite matrix with complex entries a_{nv} . By $At = ((At)_n)$, we write the A -transform of a sequence $t = (t_v)$, if the series $(At)_n = \sum_v a_{nv} t_v$ converges for $n \in \mathbb{N}$.

If $At \in Y$ with $t \in X$, then A defines a matrix mapping from X to Y . In addition, (X, Y) indicates the family of all infinite matrices that map X to Y . Thus, A is in (X, Y) if and only if $At = ((At)_n) \in Y$, $\forall t \in X$, that is, $A \in (X, Y)$ if and only if $A_n \in X^\beta$, $\forall n$. For more information on summability theory and infinite matrix mappings (see [2], [12]).

For every sequence $t \in \ell_p$, the relation $\|At\|_{\ell_p} \leq M \|t\|_{\ell_p}$ holds, in this case the operator A is called bounded in ℓ_p . The constant M is called an upper bound for the operator A and the smallest choice of M that satisfies the inequality $\|At\|_{\ell_p} \leq M \|t\|_{\ell_p}$

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is called the norm of A . The upper bound and the norm of an operator play a crucial role in obtaining various inequalities in matrix domains. The famous Hardy's inequality

$$\sum_{n=0}^{\infty} \left(\sum_{v=0}^n \frac{|t_v|}{n+1} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{v=0}^{\infty} |t_v|^p$$

is the consequence of the boundedness of the Cesàro operator.

2. Preliminaries

Let X be a normed linear space and S_X indicate the unit sphere of X . Then, X is called rotund (strictly convex) if $\|\lambda t + (1 - \lambda)r\| < 1$, whenever $\lambda \in (0, 1)$, and t and r are distinct points of S_X . In other words, a normed linear space is rotund when its unit sphere contains no nontrivial line segments. In addition, the normed linear space X is rotund if and only if $\|\frac{1}{2}(t + r)\| < 1$, for distinct points t and r of S_X . For example, the space ℓ_p ($1 < p < \infty$) is rotund, whereas the spaces ℓ_∞, ℓ_1 and c_0 lack this property. For a normed linear space X , we denote the closed unit ball by $B_X = \{t \in X : \|t\| \leq 1\}$, and the unit sphere by $S_X = \{t \in X : \|t\| = 1\}$. Furthermore, the weak convergence in X is denoted as $t_n \xrightarrow{w} t$.

2.1. Uniform rotundity

The normed linear space X is uniformly rotund if for each $\varepsilon > 0$, there is a $\delta > 0$ that depends on $\varepsilon > 0$ such that $\|\frac{t+r}{2}\| < 1 - \delta$ whenever $t, r \in S_X$ and $\|t - r\| \geq \varepsilon$. For any positive measure μ , the space $L_p(\mu)$ for $1 < p < \infty$ is uniformly rotund [6], while $L_1(\mu)$, $L_\infty(\mu)$ and $C[0, 1]$ are not. Evidently, every uniformly rotund space is rotund; however, there are rotund spaces that are not uniformly rotund. Indeed, rotundity as well as uniform rotundity is an isometric property and is usually not preserved by isomorphisms.

2.2. Uniform smoothness

The modulus of smoothness of the normed linear space X is defined as

$$\rho_X(m) = \sup \left\{ \frac{\|t + mr\| + \|t - mr\|}{2} - 1 : t, r \in S_X \right\},$$

X is uniformly smooth if $\lim_{m \rightarrow 0^+} \rho_X(m)/m = 0$.

It is noted that any subspace of a uniformly smooth normed space is uniformly smooth, and any uniformly smooth complete normed space is reflexive. For instance, c_0 is not uniformly smooth as it is not reflexive. Furthermore, $L_p(\mu)$ for $1 < p < \infty$ is uniformly smooth; however, $L_1(\mu)$ and $L_\infty(\mu)$ are not uniformly smooth [11].

2.3. Radon-Riesz property

A normed linear space X has the Radon-Riesz property whenever $t_n \xrightarrow{w} t$ in X and $\|t_n\| \rightarrow \|t\|$ implies $\|t_n - t\| \rightarrow 0$. For example, $L_p(\mu)$ for $1 < p < \infty$ possesses the Radon-Riesz property. Moreover, every uniformly rotund space has the Radon-Riesz property [11]. For relevant literature on the geometric structure of normed linear spaces; see the studies [1, 3, 4, 7, 10, 13, 14, 17, 18].

In this work, we are greatly benefiting from [11]. Suppose X and Y are the Banach spaces. A linear operator $\mathcal{T}$ from X to Y is compact if $\mathcal{T}(B)$ is a relatively compact subset of Y whenever B is a bounded subset of X . Let us denote the family of all compact linear operators from X to Y by $\kappa(X, Y)$ or $\kappa(X)$ if $X = Y$. Further, the range of compact operator from the Banach space into the Banach space is closed iff the operator has finite rank, i.e., the range of the operator is finite dimensional.

DEFINITION 1. [8] A Banach space X possesses the approximation property if, for every Banach space Y , the set of finite-rank members of $B(Y, X)$ is dense in $\kappa(Y, X)$.

PROPOSITION 1. [11] *Sequence spaces c_0 and ℓ_p , $1 \leq p < \infty$ have the approximation property.*

Suppose X and Y are the Banach spaces. A linear operator $\mathcal{T}$ from X to Y is weakly compact if $\mathcal{T}(B)$ is a relatively compact subset of Y whenever B is a bounded subset of X . Here, $\kappa^w(X, Y)$ denotes the family of all weakly compact linear operators. Furthermore, it is known from the Eberlein-Smulian theorem [11], a subset L of X is relatively compact implies $\bar{L}$ is a weakly compact subset of Y . $\bar{L}$ is a weakly compact subset of Y whenever $\bar{L}$ is a compact subset of Y in its weak topology.

PROPOSITION 2. *Let X and Y be any two Banach spaces and $\mathcal{T}$ be a linear operator from X into Y . Then $\mathcal{T}$ is weakly compact iff any bounded sequence (t_n) in X has a subsequence (t_{n_i}) with $(\mathcal{T}t_{n_i})$ converges weakly.*

DEFINITION 2. [9] Suppose X and Y are two Banach spaces. A linear operator $\mathcal{T}$ from X into Y is completely continuous or a Dunford-Pettis operator if $\mathcal{T}(B)$ is a compact subset of Y whenever B is a weakly compact subset of X .

PROPOSITION 3. [11] *The space ℓ_1 has the Dunford-Pettis property.*

DEFINITION 3. [11] Let X be a normed space. Define a function $\rho_X : (0, \infty) \rightarrow [0, \infty)$ as follows

$$\rho_X(m) = \sup \left\{ \frac{1}{2} (\|t + mr\| + \|t - mr\|) - 1 : t, r \in S_X \right\}$$

if $X \neq 0$, then

$$\rho_X(m) = \begin{cases} 0 & \text{if } 0 < m < 1, \\ m - 1, & \text{if } m \geq 1, \end{cases}$$

if $X = \{0\}$. Then ρ_X be the modulus of smoothness of X . In addition, the space X is uniformly smooth if $\lim_{m \rightarrow 0^+} \frac{\rho_X(m)}{m} = 0$.

3. The quasi-Cesàro sequence spaces

Cesàro matrix. The infinite Cesàro operator $C = (c_{nv})$ is defined by

$$c_{nv} = \begin{cases} \frac{1}{n+1} & 0 \leq v \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

for all $n, v \in \mathbb{N}$. Moreover, $\|C\|_{\ell_p} = p^*$, where $\frac{1}{p} + \frac{1}{p^*} = 1$.

Generalized Cesàro matrix. Let $K \geq 1$ be a real number. Then the generalized Cesàro matrix $C^K = (c_{nv}^K)$ is defined as

$$c_{nv}^K = \begin{cases} \frac{1}{n+K} & 0 \leq v \leq n, \\ 0, & \text{otherwise} \end{cases}$$

has the ℓ_p -norm $\|C^K\|_{\ell_p} = p^*$ (see [5]).

Quasi-Cesàro matrix. Let $(k_v)_{v=0}^\infty$ be a sequence with positive elements satisfying

$$\sup_v \frac{k_v}{v+1} = M < \infty \quad \text{and} \quad \sup_v \frac{v+1}{k_v} = L < \infty,$$

where the supremum runs over the set $\mathbb{N}$. Then the quasi-Cesàro operator $C^k = (c_{nv}^k)$ generated by $(k_v)_{v=0}^\infty$ is defined as

$$c_{nv}^k = \begin{cases} \frac{1}{k_v} & 0 \leq v \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Explicitly

$$C^k = \begin{pmatrix} \frac{1}{k_0} & 0 & 0 & \cdots \\ \frac{1}{k_1} & \frac{1}{k_1} & 0 & \cdots \\ \frac{1}{k_2} & \frac{1}{k_2} & \frac{1}{k_2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (2)$$

Furthermore; quasi-Cesàro matrix C^k is invertible and its inverse $C^{-k} = (c_{nv}^{-k})$ is defined as

$$c_{nv}^{-k} = \begin{cases} k_v & n = v, \\ -k_v, & n = v+1, \\ 0 & \text{otherwise.} \end{cases}$$

Define the diagonal matrix $D = (d_{nv})$ with entries $d_{nv} = \frac{k_v}{v+1}$ for all $v \in \mathbb{N}$. As D is a diagonal matrix and so $\|D\|_{\ell_p} = \sup_v |d_{vv}| = \sup_v \frac{k_v}{v+1} = M$. Clearly, $C = DC^k$, where C is the Cesàro matrix. Thus, we conclude that $\|C^k\|_{\ell_p} \geq \frac{1}{M} p^*$. In the same way, $C^k = D^{-1}C$ which deduces that $\|C^k\|_{\ell_p} \leq Lp^*$. Hence, the quasi-Cesàro matrix is a bounded operator and

$$\frac{1}{M} p^* \leq \|C^k\|_{\ell_p} \leq Lp^*.$$

In addition, if we take $k_v = v+1$ and $k_v = v+K$, then C^k reduces to Cesàro matrix C and the generalized Cesàro matrix C^K .

Following [16], the sequence spaces C_p^k ($1 \leq p < \infty$) and C_∞^k are defined as

$$C_p^k = \left\{ t = (t_v) \in \omega : C^k t \in \ell_p \right\},$$

$$C_\infty^k = \left\{ t = (t_v) \in \omega : C^k t \in \ell_\infty \right\}.$$

In fact, C_p^k is the set of all sequences such that C^k -transforms belong to the space ℓ_p . The sequence spaces C_p^k and C_∞^k are BK spaces equipped with the norm

$$\|t\|_{C_p^k} = \|C^k t\|_{\ell_p} \quad \text{and} \quad \|t\|_{C_\infty^k} = \|C^k t\|_{\ell_\infty}.$$

Moreover, from the inclusion $\ell_p \subset \ell_q$ it follows that the inclusion $C_p^k \subset C_q^k$ holds for $1 \leq p < q < \infty$. The study of geometric properties of Banach spaces has deep implications in fixed point theory, approximation theory, and optimization. However, the spaces ℓ_p for $(1 \leq p < \infty)$ act as paradigmatic examples of spaces with well geometric properties, viz uniform rotundity, uniform smoothness, and the Radon-Riesz property. The construction of novel sequence spaces has further certain fascinating properties, while being structurally distinct from spaces ℓ_p remains a functional research area.

4. Our results

In this segment, we examine some geometric properties of the quasi-Cesàro sequence spaces C_p^k ($1 \leq p \leq \infty$). We address the properties like uniform rotundity, uniform smoothness, Radon-Riesz property, approximation property, Dunford-Pettis property and the Hahn-Banach extension property.

THEOREM 1. *The space C_p^k is uniformly smooth for $1 \leq p < \infty$.*

Proof. Let $t, r \in S_X$. Then

$$\|t + mr\|_{C_p^k}^p = \sum_{n=0}^{\infty} \left| \sum_{v=0}^n \frac{1}{k_v} (t_v + mr_v) \right|^p,$$

and

$$\|t - mr\|_{C_p^k}^p = \sum_{n=0}^{\infty} \left| \sum_{v=0}^n \frac{1}{k_v} (t_v - mr_v) \right|^p.$$

Since $\lim_{m \rightarrow 0^+} \frac{\rho_X(m)}{m} = \frac{0}{0}$. Using the L'Hospital's rule, we have

$$\lim_{m \rightarrow 0^+} \frac{\rho_X(m)}{m} = \lim_{m \rightarrow 0^+} \frac{d}{dm} \rho_X(m).$$

Applying the linearity of derivatives and the properties of the supremum, we get

$$\frac{d}{dm} \rho_X(m) = \sup_{t, r \in S_X} \left\{ \frac{1}{2} \frac{d}{dm} \|t + mr\|_{C_p^k} + \frac{1}{2} \frac{d}{dm} \|t - mr\|_{C_p^k} \right\}.$$

Furthermore,

$$\begin{aligned} \frac{d}{dm} \|t + mr\|_{C_p^k}^p &= \frac{d}{dm} \left\| C^k(t + mr) \right\|_{C_p^k}^p \\ &= \frac{d}{dm} \sum_{n=0}^{\infty} \left| (C^k(t + mr))_n \right|^p \\ &= \sum_{n=0}^{\infty} \frac{d}{dm} \left| (C^k(t + mr))_n \right|^p \\ &= \sum_{n=0}^{\infty} \left(p \left| (C^k(t + mr))_n \right|^{p-1} \frac{d}{dm} \left| (C^k(t + mr))_n \right| \right), \end{aligned}$$

and similarly

$$\frac{d}{dm} \|t - mr\|_{C_p^k}^p = \sum_{n=0}^{\infty} \left(p \left| (C^k(t - mr))_n \right|^{p-1} \frac{d}{dm} \left| (C^k(t - mr))_n \right| \right).$$

In particular,

$$\frac{d}{dm} \left| (C^k(t + mr))_n \right| = \begin{cases} (C^k r)_n & (C^k(t + mr))_n \geq 0, \\ -(C^k r)_n & (C^k(t + mr))_n < 0, \end{cases}$$

and

$$\frac{d}{dm} \left| (C^k(t - mr))_n \right| = \begin{cases} -(C^k r)_n & (C^k(t - mr))_n \geq 0, \\ (C^k r)_n & (C^k(t - mr))_n < 0. \end{cases}$$

Therefore, we have the following conditions

$$\lim_{m \rightarrow 0^+} \frac{d}{dm} \|t + mr\|_{C_p^k}^p = \begin{cases} p \sum_{n=0}^{\infty} \left| (C^k t)_n \right|^{p-1} (C^k r)_n & (C^k t)_n \geq 0, \\ -p \sum_{n=0}^{\infty} \left| (C^k t)_n \right|^{p-1} (C^k r)_n & (C^k t)_n < 0, \end{cases}$$

and

$$\lim_{m \rightarrow 0^+} \frac{d}{dm} \|t - mr\|_{C_p^k}^p = \begin{cases} -p \sum_{n=0}^{\infty} |(C^k t)_n|^{p-1} (C^k r)_n & (C^k t)_n \geq 0, \\ p \sum_{n=0}^{\infty} |(C^k t)_n|^{p-1} (C^k r)_n & (C^k t)_n < 0. \end{cases}$$

Thus, we have

$$\lim_{m \rightarrow 0^+} \frac{d}{dm} \|t + mr\|_{C_p^k}^p + \lim_{m \rightarrow 0^+} \frac{d}{dm} \|t - mr\|_{C_p^k}^p = 0.$$

This gives

$$\lim_{m \rightarrow 0^+} \frac{d}{dm} \|t + mr\|_{C_p^k} + \lim_{m \rightarrow 0^+} \frac{d}{dm} \|t - mr\|_{C_p^k} = 0.$$

Hence, $\lim_{m \rightarrow 0^+} \frac{d}{dm} (\rho_X(m)) = 0$. $\square$

THEOREM 2. *The space C_p^k is uniformly rotund for $1 < p < \infty$.*

Proof. Let $\varepsilon > 0$ and take $t, r \in S_{C_p^k}$ with $\|t - r\|_{C_p^k} \geq \varepsilon$. Then $\|C^k t - C^k r\|_{\ell_p} = \|t - r\|_{C_p^k} \geq \varepsilon$. As ℓ_p is uniformly rotund, $\exists \delta = \delta_{\ell_p}(\varepsilon) > 0$ such that $\left\| \frac{C^k t + C^k r}{2} \right\|_{\ell_p} < 1 - \delta$. But $\left\| \frac{C^k t + C^k r}{2} \right\|_{\ell_p} = \left\| \frac{t+r}{2} \right\|_{C_p^k}$, showing that C_p^k is uniformly rotund. $\square$

THEOREM 3. *The spaces C_1^k and C_∞^k are not rotund.*

Proof. Let $t = (1, 0, 0, \dots)$, $r = (0, 1, 0, \dots)$, $u_1 = t + r$ and $u_2 = t - r$. Then, it is easy to see that $\|t\|_{C_1^k} = \|r\|_{C_1^k} = \left\| \frac{t+r}{2} \right\|_{C_1^k} = \|u_1\|_{C_\infty^k} = \|u_2\|_{C_\infty^k} = \left\| \frac{u_1+u_2}{2} \right\|_{C_\infty^k} = 1$, i.e., the line segment connecting the points lies on the unit sphere. Hence C_1^k and C_∞^k are not rotund. $\square$

THEOREM 4. *The space C_2^k possesses the Radon-Riesz property.*

Proof. Assume that the sequence (t_n) and the element t in C_2^k are such that $t_n \xrightarrow{w} t$ and $\|t_n\|_{C_2^k} \rightarrow \|t\|_{C_2^k}$. In this case, we will show that $\|t_n - t\|_{C_2^k} \rightarrow 0$ as $n \rightarrow \infty$. Consider the equality

$$\begin{aligned} \|t_n - t\|_{C_2^k}^2 &= \|C^k t_n - C^k t\|_{\ell_2}^2 \\ &= \langle C^k t_n - C^k t, C^k t_n - C^k t \rangle_{\ell_2} \\ &= \|C^k t_n\|_{\ell_2}^2 + \|C^k t\|_{\ell_2}^2 - \langle C^k t_n, C^k t \rangle_{\ell_2} - \langle C^k t, C^k t_n \rangle_{\ell_2}. \end{aligned}$$

Assume $y = C^k t \in \ell_2 = \ell_2^*$ and consider $y \circ C^k$ such that $(y \circ C^k)r = \langle C^k r, C^k t \rangle_{\ell_2}$. It is known from the Riesz representation theorem for ℓ_2 and using the properties of C^k , we conclude that $y \circ C^k$ is a continuous linear functional on C_2^k and so

$$(y \circ C^k)t_n = y(C^k t_n) = \langle C^k t_n, C^k t \rangle_{\ell_2}.$$

Using assumption $t_n \xrightarrow{w} t$, we obtain

$$\langle C^k t_n, C^k t \rangle_{\ell_2} = (y \circ C^k)t_n \rightarrow (y \circ C^k)t = \|C^k t\|_{\ell_2}^2 \quad (n \rightarrow \infty).$$

Similarly, take $y_n = C^k t_n \in \ell_2 = \ell_2^*$, for each n . Then $y_n \circ C^k$ is a continuous linear functional on C_2^k , and so $(y_n \circ C^k)r = \langle C^k r, C^k t_n \rangle_{\ell_2}$. Hence, as $t_n \xrightarrow{w} t$, we have

$$\langle C^k t, C^k t_n \rangle_{\ell_2} = (y_n \circ C^k)t = \overline{\langle C^k t_n, C^k t \rangle_{\ell_2}} \rightarrow \overline{\langle C^k t, C^k t \rangle_{\ell_2}} = \|C^k t\|_{\ell_2}^2,$$

as $n \rightarrow \infty$. Thus, as $\|t_n\|_{C_2^k} \rightarrow \|t\|_{C_2^k}$, it is seen that

$$\begin{aligned} \|t_n - t\|_{C_2^k}^2 &= \|C^k t_n\|_{\ell_2}^2 + \|C^k t\|_{\ell_2}^2 - \langle C^k t_n, C^k t \rangle_{\ell_2} - \langle C^k t, C^k t_n \rangle_{\ell_2} \\ &\rightarrow \|C^k t\|_{\ell_2}^2 + \|C^k t\|_{\ell_2}^2 - \|C^k t\|_{\ell_2}^2 - \|C^k t\|_{\ell_2}^2 \\ &= 0, \text{ as } n \rightarrow \infty. \quad \square \end{aligned}$$

THEOREM 5. For $1 \leq p < \infty$, the space C_p^k has the approximation property.

Proof. Suppose $\mathcal{T}$ is a compact linear operator from space Y into C_p^k . For this, let (z_n) be a bounded sequence in Y , $(\mathcal{T}z_n)$ has a convergent sub-sequence $(\mathcal{T}z_{n_i})_{i=0}^\infty$ in C_p^k . Hence

$$\begin{aligned} \|\mathcal{T}z_{nj} - \mathcal{T}z_{ni}\|_{C_p^k}^p &= \|\mathcal{T}(z_{ni} - z_{nj})\|_{C_p^k}^p \rightarrow 0 \text{ as } i, j \rightarrow \infty \\ &= \|(C^k \mathcal{T})(z_{nj} - z_{ni})\|_{\ell_p}^p \rightarrow 0 \text{ as } i, j \rightarrow \infty. \end{aligned}$$

Therefore, the operator $C^k \mathcal{T} : Y \rightarrow \ell_p$ is well defined and compact, and so clearly $C_p^k \mathcal{T}$ is bounded linear. As ℓ_p has the approximation property, it follows that the sequence $(\mu_m)_{m=0}^\infty$ of a finite rank bounded linear operator from Y to ℓ_p exists such that $\|C^k \mathcal{T} - \mu_m\| \rightarrow 0$ as $m \rightarrow \infty$. Consequently, it is seen that the bounded linear operators $((C^k)^{-1} \mu_m)_{m=0}^\infty$ are the required sequence of a finite rank from Y to C_p^k and

therefore each $(C^k)^{-1}\mu_m$ is a finite rank bounded linear operator. Further,

$$\begin{aligned}
 \left\| \mathcal{T} - (C^k)^{-1}\mu_m \right\| &= \sup_{\|z\|=1} \left\| (\mathcal{T} - (C^k)^{-1}\mu_m)z \right\|_{C_p^k}^p \\
 &= \sup_{\|z\|=1} \left\| \mathcal{T}z - ((C^k)^{-1}\mu_m)z \right\|_{C_p^k}^p \\
 &= \sup_{\|z\|=1} \left\| C^k \mathcal{T}z - C^k((C^k)^{-1}\mu_m)z \right\|_{\ell_p}^p \\
 &= \sup_{\|z\|=1} \left\| (C^k \mathcal{T} - \mu_m)z \right\|_{\ell_p}^p \\
 &\rightarrow 0 \text{ as } m \rightarrow \infty. \quad \square
 \end{aligned}$$

THEOREM 6. C_1^k has the Dunford-Pettis property.

Proof. Let $\mathcal{T} : C_1^k \rightarrow Y$ be a weakly compact linear operator. Then, compose $\mathcal{T}$ with $(C^k)^{-1}$ and so $\mathcal{T}(C_1^k)^{-1}$ is also the bounded linear operator from ℓ_1 into Y . Also, it is weakly compact. For this, suppose that M is a bounded in ℓ_1 . By the boundedness of the matrix $(C^k)^{-1}$, we conclude that $(C^k)^{-1}(M)$ is a bounded subset of C_1^k . Thus,

$$\mathcal{T}((C^k)^{-1}(M)) = (\mathcal{T}(C^k)^{-1}(M))$$

is relatively weakly compact in Y . Therefore, $\mathcal{T}(C^k)^{-1} : \ell_1 \rightarrow Y$ is a weakly compact operator. As ℓ_1 possesses the Dunford-Pettis property, it is reached that $\mathcal{T}(C^k)^{-1}$ is completely continuous. Let N be a weakly compact subset of C_1^k . Then by ([11] Exercise 3.50), $C^k(N)$ is a weakly compact subset of ℓ_1 and hence

$$(\mathcal{T}(C^k)^{-1})C^k(N) = \mathcal{T}(N)$$

is a compact subset of Y . $\square$

THEOREM 7. [15] Assume that $\mathcal{T} : Y \rightarrow \ell_\infty$ is a bounded linear operator and Y is any linear subspace of X . Then $\mathcal{T}$ can be extended to a bounded linear operator $\mathcal{S} : X \rightarrow \ell_\infty$ with the same norm as $\mathcal{T}$.

Let us show that C_∞^k has the Hahn-Banach extension property.

THEOREM 8. Let Y be a linear subspace of X and $\mathcal{T} : Y \rightarrow C_\infty^k$ be a bounded linear operator. Then $\mathcal{T}$ can be extended to a bounded linear operator $\mathcal{S} : Y \rightarrow C_\infty^k$ with the same norm as $\mathcal{T}$.

Proof. For any bounded linear operator $\mathcal{T} : Y \rightarrow C_\infty^k$, we have $C^k \mathcal{T} \in B(Y, \ell_\infty)$. As ℓ_∞ has the Hahn-Banach extension property according to Theorem 7. Therefore, $(C^k \mathcal{T})$ can be extended to a bounded linear operator $M : X \rightarrow \ell_\infty$ with the same norm as $C^k \mathcal{T}$. Consider the operator $(C^k)^{-1}M$. Then by the classical operator algebra

$(C^k)^{-1}U = \mathcal{S}$ is a bounded linear operator from X to C_∞^k . We only show that $\mathcal{S}$ is an extension of $\mathcal{T}$ and $\|\mathcal{T}\| = \|\mathcal{S}\|$. For any $r \in Y$,

$$\begin{aligned} Sr &= ((C^k)^{-1}M)r = (C^k)^{-1}(Mr) \\ &= (C^k)^{-1}(C^k \mathcal{T})r = \mathcal{T}r \end{aligned}$$

Now

$$\begin{aligned} \|\mathcal{S}\| &= \|(C^k)^{-1}M\| = \|(C^k)^{-1}(C^k \mathcal{T})\| \\ &= \|((C^k)^{-1}C^k)\mathcal{T}\| = \|\mathcal{T}\|. \quad \square \end{aligned}$$

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