

AN OPERATIONAL AND ASYMPTOTIC THEORY FOR
DEGENERATE LAURICELLA HYPERGEOMETRIC FUNCTIONS

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Abstract. This paper develops an operational and coefficientwise small-parameter theory for degenerate Lauricella hypergeometric functions of types A , B , C , and D . The construction is based on degenerate Pochhammer symbols associated with the degenerate gamma function. We prove first-order, higher-order, and mixed derivative formulas, and show that the deformed families preserve the parameter-shift structure of the classical Lauricella theory. We also obtain generating relations, diagonal reductions, and Euler-type integral representations based on a degenerate binomial kernel. The fixed-deformation case gives entire functions of the variables, while the small-parameter regime connects the deformed coefficients with their classical counterparts. In particular, we derive explicit first-order coefficient expansions as $\lambda \rightarrow 0^+$ and identify the corresponding correction series. Numerical examples in two variables illustrate how the deformation parameter changes the corresponding profiles in numerically stable compact regions.

1. Introduction

The classical Lauricella functions are standard multivariable extensions of the Gauss and Appell hypergeometric functions. Their theory connects multiple power series, parameter-shift identities, systems of partial differential equations, Euler-type integral representations, and reductions to lower-dimensional hypergeometric functions; see [3, 8, 19, 23, 24]. These functions appear in several branches of analysis and mathematical physics, and recent work has also emphasized analytic continuation and numerical evaluation of multivariable hypergeometric functions [2, 5, 6, 7].

Degenerate hypergeometric analysis does not use a single universal deformation convention. In one well-established line of work, the basic objects are the λ -falling and λ -rising factorials. Kim, Kim, Lee, and Kwon [13] used the λ -rising factorials to define degenerate Gauss and generalized hypergeometric series, while Kim, Kim, Jang, Lee, and Kim [14] combined the classical Gauss hypergeometric function with the degenerate exponential and degenerate Stirling structures in their construction of generalized degenerate Bernoulli numbers and polynomials. A second analytic line starts from the degenerate gamma function introduced by Kim and Kim [11] and subsequently developed in greater detail in [12]. The present paper follows the gamma-ratio Pochhammer model of Yağcı and Şahin [27]: selected classical Pochhammer factors are replaced by Γ_λ -ratios, and each fixed deformed coefficient reduces to its classical

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counterpart as $\lambda \rightarrow 0$. Thus, the degenerate Lauricella families considered here are multivariable extensions of this gamma-ratio model and should not be identified with the λ -step hypergeometric functions of [13].

The λ -factorial framework has also been used extensively in recent work on degenerate Bell, Dowling, and Stirling structures, including Spivey-type recurrence relations [15], degenerate r -Bell numbers and coherent states [17], combinatorial identities for degenerate Stirling numbers of the second kind [18], and boson-operator derivations of recurrences for degenerate Bell and Dowling polynomials [16]. These papers provide important notational and methodological context for degenerate factorial techniques. The precise distinction between those λ -step symbols and the gamma-ratio Pochhammer symbol used in the present paper is stated explicitly in Section 2. Recent developments in degenerate operational methods, degenerate special functions, degenerate polynomial sequences, and transform techniques also include modified-degenerate $W_{\alpha,\beta,v}$ -type functions, degenerate Mittag-Leffler functions, degenerate two-variable q -Legendre polynomials, and degenerate q -derangement polynomials [9, 10, 25, 26]. A natural question is whether the gamma-ratio deformation can be made coherent in the Lauricella setting, where the coefficient structure involves both total degrees and separate variables. This paper answers that question by developing the operational, integral, and coefficientwise small-parameter theory of the corresponding degenerate Lauricella families.

The definitions are included for completeness, but the main point is not merely to introduce new notation. The goal is to show that the degenerate coefficient structure behaves systematically under differentiation, parameter shifts, generating operations, diagonal specialization, Euler-type integration, and small-parameter expansion. In particular, for fixed admissible $\lambda \neq 0$ the resulting functions have a convergence behavior different from the classical Lauricella functions: the deformed families are entire in the variables. As $\lambda \rightarrow 0^+$, each fixed Taylor coefficient recovers its classical Lauricella counterpart, and the first-order coefficient correction can be written explicitly. A function-level limit of the infinite series requires additional uniform estimates and is therefore not inferred from coefficientwise convergence alone.

The first group of results consists of derivative and operational formulas. We prove first-order, higher-order, and mixed derivative identities for the four degenerate families. These identities retain the classical parameter-shift pattern, but the shift coefficients are modified by explicit λ -dependent factors. We then derive generating relations and diagonal reductions. The diagonal type D reduction is especially useful because it converts the multivariable expression into a one-variable degenerate Gauss-type series.

The second group of results is analytic. We introduce a degenerate binomial kernel and use it to obtain Euler-type integral representations for the type A family and for the diagonal type D family. The hypotheses are stated so that the beta-integrals and the interchange of summation and integration are justified. We then use the explicit gamma-ratio form of the degenerate gamma function to derive first-order small- λ expansions for each fixed Taylor coefficient. These expansions identify convergent correction series in the classical domains, while keeping distinct the separate question of a uniform function-level asymptotic expansion.

The numerical section is intended as a reproducible illustration of the analytic

formulas rather than as an independent numerical algorithm. Two-variable examples show the effect of λ on diagonal profiles and surface plots in compact regions where the summation is stable.

The main contributions of the paper may be summarized as follows.

- Four degenerate Lauricella families are treated in a unified framework based on the degenerate gamma and Pochhammer symbols.
- Fixed- λ convergence is proved, showing that the degenerate families are entire functions of the variables for admissible nonzero deformation parameters.
- First-order, higher-order, and mixed derivative formulas are established with explicit shifted parameters and deformation factors.
- Generating relations, diagonal reductions, and Euler-type integral representations are obtained.
- First-order small- λ coefficient expansions are derived, with explicit correction series measuring the formal first-order departure from the classical Lauricella coefficients.

The paper is organized as follows. Section 2 recalls the classical and degenerate background. Section 3 defines the degenerate Lauricella functions and records their basic convergence properties. Section 4 derives first- and higher-order derivative formulas. Section 5 establishes generating functions and diagonal reductions. Section 6 proves Euler-type integral representations. Section 7 studies the coefficientwise small- λ expansion, and Section 8 presents numerical and graphical examples.

2. Preliminaries

In this section, we recall the basic notation and the main classical objects that will be deformed later. For the one-variable classical background on the gamma function, the Pochhammer symbol, and generalized hypergeometric series, we rely on the standard treatments of Bailey [4], Slater [22], Andrews, Askey, and Roy [1], and the NIST Digital Library of Mathematical Functions [20]. For the multivariable theory, especially the Lauricella series, their convergence properties, differential equations, and integral representations, we use Lauricella's original paper [19] together with the systematic expositions of Appell and Kampé de Fériet [3], Exton [8], Srivastava and Karlsson [23], and Srivastava and Manocha [24]. For the degenerate framework, Kim and Kim [11, 12] provide the analytic background for the degenerate exponential and gamma functions. We also distinguish this gamma-based framework from the λ -step factorial and hypergeometric constructions of [13, 14] and from recent applications of λ -factorial techniques to Bell, Dowling, Stirling, boson-operator, and coherent-state problems [15, 16, 17, 18]. Yağcı and Şahin [27] supply the particular gamma-ratio Pochhammer symbol and the one-variable degenerate hypergeometric model that we extend in the present paper. We also draw on related recent contributions to degenerate operational reformulations, fractional viscoelastic applications, degenerate q -polynomial theory, and degenerate q -derangement polynomials [9, 10, 25, 26].

The classical gamma function Γ is defined, for $\Re(\alpha) > 0$, by [20, §5.2(i)]:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt. \quad (1)$$

It satisfies the well-known functional equation

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha),$$

and the duplication and reflection formulas (see, for example, [4, 22, 23]).

The rising factorial or Pochhammer symbol is given, for $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, by [20, §15.2(i)] and [1, Ch. 1]:

$$(a)_0 := 1, \quad (a)_n := a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \quad (n \geq 1), \quad (2)$$

whenever the right-hand side is defined. The Pochhammer symbol is the basic building block for the hypergeometric and multiple hypergeometric series [4, 22, 23, 24].

2.1. Generalized hypergeometric series

For complex parameters $a_1, \dots, a_p, b_1, \dots, b_q$ and a complex variable z , the generalized hypergeometric series is defined by [20, §16.2(i)], [4], and [22]:

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \quad (3)$$

with the usual convention that no denominator parameter b_j is a non-positive integer [4, 22, 23, 24]. The series (3) converges for all $z \in \mathbb{C}$ when $p \leq q$, for $|z| < 1$ when $p = q + 1$, and diverges for $z \neq 0$ when $p > q + 1$.

In the special case $p = 2, q = 1$, we obtain the Gauss hypergeometric function

$${}_2F_1(a, b; c; z),$$

which plays a central role in the theory of special functions and will appear as a one-variable limit of the Lauricella functions described below.

Throughout the paper, we work with $r \in \mathbb{N}$ complex variables

$$x = (x_1, \dots, x_r) \in \mathbb{C}^r,$$

and with multi-indices

$$m = (m_1, \dots, m_r) \in \mathbb{N}_0^r.$$

We use the standard multi-index notation from multivariable hypergeometric analysis [8, 23]:

$$|m| := m_1 + \cdots + m_r, \quad x^m := x_1^{m_1} \cdots x_r^{m_r}, \quad m! := m_1! \cdots m_r!.$$

For differential operators we adopt the Euler operators, standard in the operator formulation of Lauricella systems [8, 23],

$$\theta_j := x_j \frac{\partial}{\partial x_j} \quad (j = 1, \dots, r), \quad \theta := \theta_1 + \dots + \theta_r. \quad (4)$$

The classical Lauricella hypergeometric functions in r variables were introduced by Lauricella [19]. Their foundational structure was subsequently incorporated into the broader theory of multiple hypergeometric functions by Appell and Kampé de Fériet [3]. Later expositions by Exton [8] and by Srivastava and Karlsson [23] remain standard sources for their convergence domains, differential equations, transformation formulas, and integral representations, while Srivastava and Manocha [24] is particularly useful for generating-function methods related to this class. Recent work has focused on analytic continuation and numerical computation: Antonova, Dmytryshyn, and Goran [2] treated continuation questions for a Lauricella–Saran function, Blümlein, Saragnese, and Schneider [7] studied algorithmic hypergeometric structures including Lauricella-type systems in Feynman-integral problems, Bera and Pathak [5] developed continuation and numerical evaluation methods for Appell and Lauricella functions, and Bezuglov et al. [6] proposed a high-precision evaluation method for Lauricella functions. We now recall the classical series definitions.

Let $a, b_1, \dots, b_r, c_1, \dots, c_r \in \mathbb{C}$ be such that no c_j is a non-positive integer. The Lauricella function $F_A^{(r)}$ is defined by [3, 8, 19, 23]

$$F_A^{(r)}(a; b_1, \dots, b_r; c_1, \dots, c_r; x_1, \dots, x_r) := \sum_{m \in \mathbb{N}_0^r} \frac{(a)_{|m|} \prod_{j=1}^r (b_j)_{m_j} x_j^{m_j}}{\prod_{j=1}^r (c_j)_{m_j} m!}, \quad (5)$$

which converges absolutely in the domain $|x_1| + \dots + |x_r| < 1$ [8, 23].

Let $a_1, \dots, a_r, b_1, \dots, b_r, c \in \mathbb{C}$ with $c \notin \{0, -1, -2, \dots\}$. The Lauricella function $F_B^{(r)}$ is given by [3, 8, 19, 23]

$$F_B^{(r)}(a_1, \dots, a_r; b_1, \dots, b_r; c; x_1, \dots, x_r) := \sum_{m \in \mathbb{N}_0^r} \frac{\prod_{j=1}^r (a_j)_{m_j} (b_j)_{m_j} x_j^{m_j}}{(c)_{|m|} m!}. \quad (6)$$

The series converges absolutely in the polydisc $|x_j| < 1$ ($j = 1, \dots, r$) [8, 23].

Let $a, b, c_1, \dots, c_r \in \mathbb{C}$ with $c_j \notin \{0, -1, -2, \dots\}$. The function $F_C^{(r)}$ is defined by [3, 8, 19, 23]

$$F_C^{(r)}(a, b; c_1, \dots, c_r; x_1, \dots, x_r) := \sum_{m \in \mathbb{N}_0^r} \frac{(a)_{|m|} (b)_{|m|} x^{m_j}}{\prod_{j=1}^r (c_j)_{m_j} m!}, \quad (7)$$

which converges absolutely in the domain $\sqrt{|x_1|} + \dots + \sqrt{|x_r|} < 1$ [8, 23].

Finally, for $a, b_1, \dots, b_r, c \in \mathbb{C}$ with $c \notin \{0, -1, -2, \dots\}$, the Lauricella function $F_D^{(r)}$ is defined by [3, 8, 19, 23]

$$F_D^{(r)}(a; b_1, \dots, b_r; c; x_1, \dots, x_r) := \sum_{m \in \mathbb{N}_0^r} \frac{(a)_{|m|} \prod_{j=1}^r (b_j)_{m_j} x_j^{m_j}}{(c)_{|m|} m!}. \quad (8)$$

The series (8) converges for $|x_j| < 1$ ($j = 1, \dots, r$).

For $r = 1$ each Lauricella function reduces to the Gauss hypergeometric function ${}_2F_1$, while for $r = 2$ they give the Appell functions. Detailed discussions of their analytic properties, differential equations, and integral representations can be found in [3, 8, 23, 24].

We now recall the degenerate analogues of the exponential and gamma functions introduced by Kim and Kim [11]. Their later note [12] studies the same degenerate gamma function further, including its meromorphic continuation, difference formulas, values at positive integers, and additional representations. Thus, the citation [12] is complementary to, rather than a competing definition with, [11]. For the elementary degenerate exponential, let $\lambda \in \mathbb{R}$ be fixed. The *degenerate exponential function* is defined by [11, 12]

$$e_\lambda(t) := (1 + \lambda t)^{1/\lambda}, \quad t \in \mathbb{C}, \quad (9)$$

for $1 + \lambda t \neq 0$. In the limit $\lambda \rightarrow 0$ we recover the classical exponential function:

$$\lim_{\lambda \rightarrow 0} e_\lambda(t) = e^t.$$

The *degenerate gamma function* Γ_λ is then defined, for $0 < \Re(\alpha) < 1/\lambda$ and $\lambda > 0$, by [11, 12]:

$$\Gamma_\lambda(\alpha) := \int_0^\infty t^{\alpha-1} (1 + \lambda t)^{-1/\lambda} dt. \quad (10)$$

Equivalently, the kernel may be written as $e_\lambda(t)^{-1}$, since $e_\lambda(t) = (1 + \lambda t)^{1/\lambda}$. The function Γ_λ admits the classical limit

$$\lim_{\lambda \rightarrow 0^+} \Gamma_\lambda(\alpha) = \Gamma(\alpha)$$

for every fixed α in the domain of analyticity [11, 12].

A beta-integral substitution gives the explicit form

$$\Gamma_\lambda(\alpha) = \lambda^{-\alpha} \frac{\Gamma(\alpha)\Gamma(1/\lambda - \alpha)}{\Gamma(1/\lambda)}, \quad \lambda > 0, \quad 0 < \Re(\alpha) < 1/\lambda. \quad (11)$$

Formula (11) is also used below as the meromorphic continuation in the parameters wherever no pole is encountered; this use is consistent with the analytic continuation developed in [12].

2.2. Standing assumptions and branch conventions

In the series identities we allow fixed $\lambda \in \mathbb{C} \setminus \{0\}$, but every degenerate upper parameter u is assumed to satisfy

$$u + 1 - \frac{1}{\lambda} \notin \{0, -1, -2, \dots\}. \quad (12)$$

All ordinary denominator parameters are also assumed to avoid the usual non-positive integers. These non-pole assumptions will not be repeated in every formula unless a stronger hypothesis is needed.

The exceptional values in (12) accumulate at $\lambda = 0$. Nevertheless, for every fixed index n , the recurrence (15) shows directly that $(u)_{\lambda;n} \rightarrow (u)_n$ as $\lambda \rightarrow 0$ through admissible values. This fixed-index statement does not by itself justify interchanging the limit $\lambda \rightarrow 0$ with an infinite Lauricella sum: such an interchange requires a separate estimate uniform in the summation indices. Accordingly, all small- λ statements below are formulated coefficientwise unless an independent uniform argument is explicitly supplied.

The case $\lambda = 0$ is reserved for the classical Lauricella functions. Whenever the integral definition (10) or a fixed-parameter gamma asymptotic is used, we take $\lambda > 0$ and let $\lambda \rightarrow 0^+$. Classical binomial factors such as $(1-z)^{-a}$ are taken with the principal branch; inside the unit disc, this is exactly the branch determined by the binomial power series.

Following Yağcı and Şahin [27], we define the *degenerate Pochhammer symbol* associated with Γ_λ by

$$(a)_{\lambda;0} := 1, \quad (a)_{\lambda;n} := \frac{\Gamma_\lambda(a+n)}{\Gamma_\lambda(a)} \quad (n \in \mathbb{N}, a \in \mathbb{C}), \quad (13)$$

whenever the right-hand side is well defined. For each fixed λ , this introduces a deformation of the rising factorial, and we have

$$\lim_{\lambda \rightarrow 0} (a)_{\lambda;n} = (a)_n,$$

in agreement with (2).

LEMMA 1. (Degenerate Pochhammer) *Let $\lambda \in \mathbb{C} \setminus \{0\}$ and suppose that (12) holds for the parameter a . Then, for $n \in \mathbb{N}_0$,*

$$(a)_{\lambda;n} = \frac{(a)_n}{(-\lambda)^n (a+1-1/\lambda)_n}. \quad (14)$$

Consequently

$$(a)_{\lambda;n+1} = \phi_\lambda(a+n)(a)_{\lambda;n}, \quad \phi_\lambda(u) := \frac{\Gamma_\lambda(u+1)}{\Gamma_\lambda(u)} = \frac{u}{1-\lambda(u+1)}, \quad (15)$$

provided the denominator is nonzero. In particular,

$$\phi_\lambda(u) \longrightarrow u \quad (\lambda \rightarrow 0^+). \quad (16)$$

Proof. For $\lambda > 0$ and in the strip of validity of (11), the ratio $\Gamma_\lambda(a+n)/\Gamma_\lambda(a)$ gives

$$(a)_{\lambda;n} = \lambda^{-n} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(1/\lambda - a - n)}{\Gamma(1/\lambda - a)}.$$

Using $\Gamma(z-n)/\Gamma(z) = (-1)^n/(1-z)_n$ with $z = 1/\lambda - a$ yields (14). The same identity then extends meromorphically to the stated admissible parameter set. Dividing the

formula with $n + 1$ by the formula with n gives the recurrence and the expression for ϕ_λ . $\square$

Using (13), it is natural in the present context to consider the *degenerate generalized hypergeometric function* introduced in [27]:

$${}_pF_q^{(\lambda)} \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right) := \sum_{n=0}^{\infty} \frac{(a_1)_{\lambda;n} (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \quad (17)$$

under suitable assumptions on the parameters and on z . For every fixed summation index, the coefficient in (17) tends to the corresponding coefficient of the classical hypergeometric series (3) as $\lambda \rightarrow 0$ through admissible values. Thus the degenerate series has the classical ${}_pF_q$ series as its coefficientwise limit. A function-level limit requires an additional uniform-in- n argument and is not assumed here; see [27] for related one-variable constructions.

Comparison with other degenerate factorial and hypergeometric conventions

A notational distinction is essential because several inequivalent objects are called “degenerate Pochhammer symbols” or “degenerate hypergeometric functions” in the literature. In the λ -step framework used in [13, 14] and in many papers on degenerate Stirling, Bell, and Dowling structures, one starts from the λ -falling factorial

$$(x)_{n,\lambda}^{\text{fall}} := x(x-\lambda) \cdots (x-(n-1)\lambda), \quad (x)_{0,\lambda}^{\text{fall}} := 1,$$

and the corresponding λ -rising factorial

$$\langle x \rangle_{n,\lambda} := x(x+\lambda) \cdots (x+(n-1)\lambda), \quad \langle x \rangle_{0,\lambda} := 1.$$

Kim, Kim, Lee, and Kwon [13], for example, define a degenerate Gauss-type series of the form

$$F_\lambda \left(\begin{matrix} a, b \\ c \end{matrix}; z \right) = \sum_{n=0}^{\infty} \frac{\langle a \rangle_{n,\lambda} \langle b \rangle_{n,\lambda}}{\langle c \rangle_{n,\lambda}} \frac{z^n}{n!},$$

and similarly introduce a generalized degenerate hypergeometric series by deforming the parameter factorials with the same λ -step rule. In a different use of the Gauss function, Kim, Kim, Jang, Lee, and Kim [14] define generalized degenerate Bernoulli numbers through the classical Gauss hypergeometric function evaluated at a degenerate-exponential argument; in particular their generating function has the form

$$\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(p)} \frac{t^n}{n!} = {}_2F_1(1-\lambda, 1; p+2; 1-e_\lambda(t)).$$

Thus, in [14] the deformation is carried by the generating kernel and its argument rather than by the gamma-ratio coefficient used below.

By contrast, the symbol used in the present paper and in [27] is

$$(a)_{\lambda;n} = \frac{\Gamma_\lambda(a+n)}{\Gamma_\lambda(a)} = \prod_{k=0}^{n-1} \frac{a+k}{1-\lambda(a+k+1)}.$$

It therefore advances the parameter by unit increments through a ratio of degenerate gamma functions; it is not the λ -rising factorial $\langle a \rangle_{n,\lambda}$. For $\lambda \neq 0$ the two coefficient systems are generally different, even though both have classical limiting interpretations. Correspondingly, the one-variable function (17) deforms a selected upper Pochhammer factor by the gamma-ratio symbol while retaining the remaining ordinary Pochhammer factors. This is the convention extended to the Lauricella families in the next section.

The recent papers [15, 16, 17, 18] illustrate how the λ -factorial convention is used in combinatorial recurrence relations, degenerate Stirling-number identities, Bell and Dowling polynomial theory, boson normal ordering, and coherent-state formulations. They are therefore relevant precedents for the broader degenerate-factorial methodology, whereas the present work has a different analytic object and objective: it develops multivariable Lauricella series from the Γ_λ -ratio Pochhammer symbol and studies their differential, integral, convergence, and coefficientwise small- λ structure. This explicit separation of conventions is intended to prevent the two usages from being conflated.

3. Degenerate Lauricella hypergeometric functions

In this section, we introduce the degenerate Lauricella hypergeometric functions of types A , B , C , and D in r variables. The construction follows the classical Lauricella functions of Appell–Lauricella type [3, 8, 19, 23], but we replace one (or several) of the rising factorials by the gamma-ratio degenerate Pochhammer symbol defined in Section 2. Accordingly, these functions are the natural multivariable analogues of the one-variable model ${}_pF_q^{(\lambda)}$ studied in [27]; they are not obtained by replacing every parameter factorial with the λ -rising factorial of [13].

Throughout this section, we fix $r \in \mathbb{N}$ and use the standard multi-index notation from Section 2; compare [8, Ch. 5] and [23, Chs. 5–7]. For $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ we write

$$|\mathbf{n}| := n_1 + \dots + n_r, \quad \mathbf{x}^{\mathbf{n}} := x_1^{n_1} \dots x_r^{n_r}, \quad \mathbf{n}! := n_1! \dots n_r!.$$

We also use the boldface abbreviations $\mathbf{a} = (a_1, \dots, a_r)$, $\mathbf{b} = (b_1, \dots, b_r)$ and $\mathbf{c} = (c_1, \dots, c_r)$.

For later comparison, we recall the classical Lauricella functions in this notation; for these standard vector-valued forms, see [3, 8, 23]. Let $a, b_1, \dots, b_r, c_1, \dots, c_r \in \mathbb{C}$ and $\mathbf{x} = (x_1, \dots, x_r) \in \mathbb{C}^r$. The Lauricella function of type A is

$$F_A^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) := \sum_{\mathbf{n} \in \mathbb{N}_0^r} \frac{(a)_{|\mathbf{n}|} \prod_{j=1}^r (b_j)_{n_j}}{\prod_{j=1}^r (c_j)_{n_j}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}, \quad (18)$$

where $(\cdot)_n$ denotes the classical Pochhammer symbol [4, 20, 22].

Similarly, for $\mathbf{a} = (a_1, \dots, a_r)$ and $\mathbf{b} = (b_1, \dots, b_r)$ we set

$$F_B^{(r)}(\mathbf{a}; \mathbf{b}; \mathbf{c}; \mathbf{x}) := \sum_{\mathbf{n} \in \mathbb{N}_0^r} \frac{\prod_{j=1}^r (a_j)_{n_j} (b_j)_{n_j}}{(c)_{|\mathbf{n}|}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}, \quad (19)$$

$$F_C^{(r)}(a, b; \mathbf{c}; \mathbf{x}) := \sum_{\mathbf{n} \in \mathbb{N}_0^r} \frac{(a)_{|\mathbf{n}|} (b)_{|\mathbf{n}|}}{\prod_{j=1}^r (c_j)_{n_j}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}, \quad (20)$$

$$F_D^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) := \sum_{\mathbf{n} \in \mathbb{N}_0^r} \frac{(a)_{|\mathbf{n}|} \prod_{j=1}^r (b_j)_{n_j}}{(c)_{|\mathbf{n}|}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}. \quad (21)$$

The regions of absolute convergence of these series, and the systems of partial differential equations they satisfy are classical; see, for example, [8, Ch. 5] and [23, Chs. 5–7].

We now define the *degenerate* Lauricella functions by replacing, in the series (5)–(8), selected classical Pochhammer symbols by the degenerate Pochhammer symbol $(a)_{\lambda; n}$ introduced in (13); compare with the one-variable construction in [27].

DEFINITION 1. (Type A) Let $a, b_1, \dots, b_r, c_1, \dots, c_r \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^r$. The *degenerate Lauricella function of type A in r variables* is defined by

$$F_{A, \lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) := \sum_{\mathbf{n} \in \mathbb{N}_0^r} \frac{(a)_{\lambda; |\mathbf{n}|} \prod_{j=1}^r (b_j)_{n_j}}{\prod_{j=1}^r (c_j)_{n_j}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}, \quad (22)$$

where $(a)_{\lambda; |\mathbf{n}|}$ is the degenerate Pochhammer symbol defined in (13).

DEFINITION 2. (Types B, C, and D) Let $\mathbf{a} = (a_1, \dots, a_r)$ and $\mathbf{b} = (b_1, \dots, b_r) \in \mathbb{C}^r$, and let $a, b, c, c_1, \dots, c_r \in \mathbb{C}$. We put

$$F_{B, \lambda}^{(r)}(\mathbf{a}; \mathbf{b}; \mathbf{c}; \mathbf{x}) := \sum_{\mathbf{n} \in \mathbb{N}_0^r} \frac{\prod_{j=1}^r (a_j)_{\lambda; n_j} (b_j)_{n_j}}{(c)_{|\mathbf{n}|}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}, \quad (23)$$

$$F_{C, \lambda}^{(r)}(a, b; \mathbf{c}; \mathbf{x}) := \sum_{\mathbf{n} \in \mathbb{N}_0^r} \frac{(a)_{\lambda; |\mathbf{n}|} (b)_{|\mathbf{n}|}}{\prod_{j=1}^r (c_j)_{n_j}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}, \quad (24)$$

$$F_{D, \lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) := \sum_{\mathbf{n} \in \mathbb{N}_0^r} \frac{(a)_{\lambda; |\mathbf{n}|} \prod_{j=1}^r (b_j)_{n_j}}{(c)_{|\mathbf{n}|}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}. \quad (25)$$

The series in (22)–(25) will be called the degenerate Lauricella functions of types A, B, C and D, respectively. By construction they depend on the additional parameter λ , and each fixed Taylor coefficient reduces to the corresponding classical Lauricella coefficient when $\lambda \rightarrow 0$.

PROPOSITION 1. (Coefficientwise classical limit) Assume the stated non-pole conditions. Then, for every fixed multi-index $\mathbf{n} \in \mathbb{N}_0^r$, the coefficient of $\mathbf{x}^{\mathbf{n}}$ in each degenerate Lauricella series (22)–(25) converges, as $\lambda \rightarrow 0$ through admissible values, to the

corresponding coefficient of the classical series (5)–(8). Equivalently, the four degenerate Lauricella series converge coefficientwise, as formal power series, to their classical counterparts.

Proof. For every fixed integer $N \geq 0$, repeated use of the recurrence (15) gives

$$(u)_{\lambda;N} = (u)_N \prod_{k=0}^{N-1} \frac{1}{1 - \lambda(u+k+1)} \longrightarrow (u)_N \quad (\lambda \rightarrow 0),$$

with the empty product interpreted as 1. This identity remains valid when $(u)_N = 0$. Substituting the fixed-index limit into each coefficient in (22)–(25) proves the claim. Notice that the argument is coefficientwise; no interchange of the λ -limit with an infinite sum is used. $\square$

We next record the fixed- λ convergence result needed in the later termwise differentiations and integrations.

THEOREM 1. (Entire convergence) *Let $\lambda \in \mathbb{C} \setminus \{0\}$. Assume that every degenerate upper parameter satisfies (12), and assume that the ordinary denominator parameters satisfy the usual non-pole conditions: $c_j \notin \{0, -1, -2, \dots\}$ for every denominator parameter c_j , and $c \notin \{0, -1, -2, \dots\}$ whenever a total denominator $(c)_{|\mathbf{n}|}$ occurs. Then the four series (22)–(25) converge absolutely and locally uniformly for every $\mathbf{x} \in \mathbb{C}^r$. Hence, for fixed admissible $\lambda \neq 0$, the functions $F_{A,\lambda}^{(r)}$, $F_{B,\lambda}^{(r)}$, $F_{C,\lambda}^{(r)}$ and $F_{D,\lambda}^{(r)}$ are entire functions of the variables $x_1, \dots, x_r$. For $\lambda = 0$ one recovers the classical Lauricella convergence domains.*

Proof. We use (14). For each admissible degenerate upper parameter u there are constants $C_u, M_u > 0$ and $R_u > |\lambda|^{-1}$ such that

$$|(u)_{\lambda;N}| \leq C_u R_u^N (1+N)^{M_u}, \quad N \in \mathbb{N}_0. \quad (26)$$

Indeed, the quotient of two consecutive terms is $|\phi_\lambda(u+N)|$, and by (15) this quotient tends to $|\lambda|^{-1}$ as $N \rightarrow \infty$. Ordinary Pochhammer quotients satisfy only polynomial growth after division by the factorials present in the Lauricella coefficients.

For $F_{A,\lambda}^{(r)}$, (26) gives a majorant of the form

$$CR^{|\mathbf{n}|} (1 + |\mathbf{n}|)^M \prod_{j=1}^r (1 + n_j)^M \frac{|x_j|^{n_j}}{n_j!},$$

which is summable for all $\mathbf{x} \in \mathbb{C}^r$ and locally uniformly on compact subsets. For $F_{B,\lambda}^{(r)}$ and $F_{D,\lambda}^{(r)}$, the total denominator $(c)_{|\mathbf{n}|}$ supplies at least a factorial decay in the total degree, while the remaining ordinary Pochhammer factors contribute only polynomial factors after division by $\mathbf{n}!$. For $F_{C,\lambda}^{(r)}$, the factor $(b)_{|\mathbf{n}|}$ is controlled by $C(1 + |\mathbf{n}|)^M |\mathbf{n}|!$, and the quotient $|\mathbf{n}|!/\mathbf{n}!$ is bounded by $r^{|\mathbf{n}|}$ times a polynomial factor; the remaining

denominator Pochhammer symbols again give factorial decay in the separate variables. In each case the Weierstrass test gives absolute and locally uniform convergence on every compact subset of $\mathbb{C}^r$. $\square$

The degenerate Lauricella functions inherit many qualitative features of the classical Lauricella functions [3, 8, 23]. Directly from the series definitions we obtain the following symmetry properties:

- $F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x})$ is invariant under any simultaneous permutation of the triples (b_j, c_j, x_j) , $1 \leq j \leq r$.
- $F_{B,\lambda}^{(r)}(\mathbf{a}; \mathbf{b}; \mathbf{c}; \mathbf{x})$ is invariant under any permutation of the triples (a_j, b_j, x_j) , $1 \leq j \leq r$.
- $F_{C,\lambda}^{(r)}(a, b; \mathbf{c}; \mathbf{x})$ is invariant under any simultaneous permutation of the pairs (c_j, x_j) , $1 \leq j \leq r$.
- $F_{D,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x})$ is invariant under any simultaneous permutation of the pairs (b_j, x_j) , $1 \leq j \leq r$.

If we set all but one of the variables equal to zero, the degenerate Lauricella functions reduce to the one-variable degenerate hypergeometric functions ${}_pF_q^{(\lambda)}$ introduced in [27]. For example,

$$F_{D,\lambda}^{(r)}(a; b_1, \dots, b_r; c; x, 0, \dots, 0) = {}_2F_1^{(\lambda)}(a, b_1; c; x), \quad (27)$$

and analogous reductions hold for $F_{A,\lambda}^{(r)}$, $F_{B,\lambda}^{(r)}$ and $F_{C,\lambda}^{(r)}$. More generally, by restricting the variables to suitable diagonals or to other linear relations, one obtains further reductions to degenerate ${}_pF_q$ -type functions. Several explicit examples of this kind will appear later for $F_{D,\lambda}^{(r)}$, especially in the discussion of diagonal reductions and coefficientwise small-parameter behaviour.

These observations show that the functions defined in Definitions 1–2 are natural multivariable extensions of the degenerate hypergeometric functions of [27], while at the same time they are deformations of the classical Lauricella functions of Appell–Lauricella type [3, 8, 19, 23].

4. Differential and mixed-derivative formulas

In this section, we use a direct coefficient approach, in the spirit of the classical Lauricella theory [8, 23, 24], and then comment on the coefficientwise classical specialization $\lambda \rightarrow 0$. Our results may be viewed as degenerate analogues of the differential formulas for the classical Lauricella functions and of the one-variable degenerate hypergeometric functions studied in [27].

Throughout this section, $\theta_j = x_j \partial / \partial x_j$ and $\theta = \theta_1 + \dots + \theta_r$ denote the Euler operators, and ϕ_λ is the function defined in (15). We also use the multi-index notation from Section 2.

By Theorem 1, all differentiations below are justified termwise for fixed admissible $\lambda \neq 0$, locally uniformly on compact subsets of $\mathbb{C}^r$. In the classical case $\lambda = 0$, the same statements hold on compact subsets of the usual Lauricella domains of absolute convergence.

We begin with the degenerate Lauricella functions of types A and D , which will be the main objects in later sections.

PROPOSITION 2. (Type A derivative) *Let $a, b_1, \dots, b_r, c_1, \dots, c_r \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^r$ be such that the series $F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x})$ converges absolutely. Then, for each $j = 1, \dots, r$, we have*

$$\frac{\partial}{\partial x_j} F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) = \frac{\phi_\lambda(a) b_j}{c_j} F_{A,\lambda}^{(r)}(a+1; b_1, \dots, b_j+1, \dots, b_r; c_1, \dots, c_j+1, \dots, c_r; \mathbf{x}). \quad (28)$$

Equivalently,

$$\theta_j F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) = x_j \frac{\phi_\lambda(a) b_j}{c_j} F_{A,\lambda}^{(r)}(a+1; \mathbf{b} + \mathbf{e}_j; \mathbf{c} + \mathbf{e}_j; \mathbf{x}),$$

where $\mathbf{e}_j$ is the j th standard unit vector in $\mathbb{R}^r$. At the coefficient level, $\phi_\lambda(a)$ tends to a as $\lambda \rightarrow 0$ (see (16)), and the shifted-parameter identity (28) reduces formally to the familiar formula for $F_A^{(r)}$, cf. [8, 23].

Proof. We write

$$F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}_0^r} A_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}, \quad A_{\mathbf{n}} = \frac{(a)_{\lambda; |\mathbf{n}|} \prod_{k=1}^r (b_k)_{n_k}}{\prod_{k=1}^r (c_k)_{n_k}} \frac{1}{\mathbf{n}!}.$$

Differentiating termwise with respect to x_j , and reindexing $n_j \mapsto n_j + 1$, we obtain

$$\frac{\partial}{\partial x_j} F_{A,\lambda}^{(r)} = \sum_{\mathbf{n} \in \mathbb{N}_0^r} (n_j + 1) A_{\mathbf{n} + \mathbf{e}_j} \mathbf{x}^{\mathbf{n}}.$$

Using the recurrence relations for the degenerate and classical Pochhammer symbols, together with the identity

$$\phi_\lambda(a + |\mathbf{n}|) (a)_{\lambda; |\mathbf{n}|} = \phi_\lambda(a) (a+1)_{\lambda; |\mathbf{n}|},$$

one checks that

$$(n_j + 1) A_{\mathbf{n} + \mathbf{e}_j} = \frac{\phi_\lambda(a) b_j}{c_j} \frac{(a+1)_{\lambda; |\mathbf{n}|} (b_j+1)_{n_j} \prod_{k \neq j} (b_k)_{n_k}}{(c_j+1)_{n_j} \prod_{k \neq j} (c_k)_{n_k}} \frac{1}{\mathbf{n}!}.$$

Substituting this into the differentiated series yields (28), which is precisely the degenerate Lauricella series with shifted parameters. The coefficientwise classical specialization is immediate from (16). $\square$

A completely analogous argument gives the corresponding formula for $F_{D,\lambda}^{(r)}$.

PROPOSITION 3. (Type D derivative) Let $a, b_1, \dots, b_r, c \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^r$ be such that $F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; \mathbf{x})$ converges absolutely. Then for each $j = 1, \dots, r$,

$$\frac{\partial}{\partial x_j} F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; \mathbf{x}) = \frac{\phi_\lambda(a) b_j}{c} F_{D,\lambda}^{(r)}(a+1; b_1, \dots, b_j+1, \dots, b_r; c+1; \mathbf{x}). \quad (29)$$

In particular,

$$\theta_j F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; \mathbf{x}) = x_j \frac{\phi_\lambda(a) b_j}{c} F_{D,\lambda}^{(r)}(a+1; \mathbf{b} + \mathbf{e}_j; c+1; \mathbf{x}).$$

Coefficientwise as $\lambda \rightarrow 0$, (29) reduces formally to the standard first-order differential formula for the classical Lauricella function $F_D^{(r)}$; see [8, Ch. 5] or [23, Ch. 7].

For the remaining two types, we obtain similar formulas, with the degenerate symbol attached to the appropriate parameter.

PROPOSITION 4. (Types B and C derivatives)

(a) Let $\mathbf{a} = (a_1, \dots, a_r)$, $\mathbf{b} = (b_1, \dots, b_r)$, $c \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^r$ be such that $F_{B,\lambda}^{(r)}(\mathbf{a}; \mathbf{b}; c; \mathbf{x})$ converges absolutely. Then for each $j = 1, \dots, r$,

$$\frac{\partial}{\partial x_j} F_{B,\lambda}^{(r)}(\mathbf{a}; \mathbf{b}; c; \mathbf{x}) = \frac{\phi_\lambda(a_j) b_j}{c} F_{B,\lambda}^{(r)}(a_1, \dots, a_j+1, \dots, a_r; b_1, \dots, b_j+1, \dots, b_r; c+1; \mathbf{x}). \quad (30)$$

(b) Let $a, b, c_1, \dots, c_r \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^r$ be such that $F_{C,\lambda}^{(r)}(a, b; \mathbf{c}; \mathbf{x})$ converges absolutely. Then for each $j = 1, \dots, r$,

$$\frac{\partial}{\partial x_j} F_{C,\lambda}^{(r)}(a, b; \mathbf{c}; \mathbf{x}) = \frac{\phi_\lambda(a) b}{c_j} F_{C,\lambda}^{(r)}(a+1, b+1; c_1, \dots, c_j+1, \dots, c_r; \mathbf{x}). \quad (31)$$

In both cases, the coefficientwise specialization $\lambda \rightarrow 0$ reproduces the well-known first-order formulas for the classical Lauricella functions $F_B^{(r)}$ and $F_C^{(r)}$, respectively [8, 23, 24].

By iterating the first-order formulas, we obtain compact expressions for higher-order and mixed derivatives. The structure is very close to the classical one; the only difference is that the products of successive factors in the parameter a are now expressed in terms of the degenerate Pochhammer symbol.

We first treat the type A case.

THEOREM 2. (Type A mixed derivatives) Let $a, b_1, \dots, b_r, c_1, \dots, c_r \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^r$ be such that $F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x})$ converges absolutely. For a multi-index $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in$

$\mathbb{N}_0^r$ we have

$$\begin{aligned} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_r^{\alpha_r}} F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) &= (a)_{\lambda;|\alpha|} \prod_{j=1}^r \frac{(b_j)_{\alpha_j}}{(c_j)_{\alpha_j}} \\ &\times F_{A,\lambda}^{(r)}(a + |\alpha|; b_1 + \alpha_1, \dots, b_r + \alpha_r; \\ &\quad c_1 + \alpha_1, \dots, c_r + \alpha_r; \mathbf{x}). \end{aligned} \quad (32)$$

Proof. The case $|\alpha| = 0$ is trivial. For $|\alpha| = 1$ the result reduces to (28), since $(a)_{\lambda;1} = \phi_\lambda(a)$ and $(b_j)_1 = b_j$, $(c_j)_1 = c_j$. Assume that (32) holds for some $\alpha \in \mathbb{N}_0^r$ and consider $\alpha + \mathbf{e}_k$, where $k \in \{1, \dots, r\}$. By differentiating (32) with respect to x_k , and then applying (28) to the inner $F_{A,\lambda}^{(r)}$ with parameters shifted by $|\alpha|$, we obtain an additional factor

$$\phi_\lambda(a + |\alpha|) \frac{b_k + \alpha_k}{c_k + \alpha_k},$$

and the parameters are incremented by 1 in a , b_k and c_k . Using the identity

$$(a)_{\lambda;|\alpha|+1} = (a)_{\lambda;|\alpha|} \phi_\lambda(a + |\alpha|)$$

and the classical relations $(b_k)_{\alpha_k+1} = (b_k + \alpha_k)(b_k)_{\alpha_k}$ and $(c_k)_{\alpha_k+1} = (c_k + \alpha_k)(c_k)_{\alpha_k}$, we obtain (32) with α replaced by $\alpha + \mathbf{e}_k$. The result follows by induction on $|\alpha|$. $\square$

A similar argument yields the higher-order formulas for the type D function.

THEOREM 3. (Type D mixed derivatives) *Let $a, b_1, \dots, b_r, c \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^r$ be such that $F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; \mathbf{x})$ converges absolutely. Then, for every multi-index $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}_0^r$,*

$$\begin{aligned} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_r^{\alpha_r}} F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; \mathbf{x}) &= (a)_{\lambda;|\alpha|} \frac{\prod_{j=1}^r (b_j)_{\alpha_j}}{(c)_{|\alpha|}} \\ &\times F_{D,\lambda}^{(r)}(a + |\alpha|; b_1 + \alpha_1, \dots, b_r + \alpha_r; \\ &\quad c + |\alpha|; \mathbf{x}). \end{aligned} \quad (33)$$

For the remaining two types we simply state the corresponding results, which again follow by induction based on the first-order formulas (30) and (31).

THEOREM 4. (Types B and C mixed derivatives)

(a) *Let $\mathbf{a} = (a_1, \dots, a_r)$, $\mathbf{b} = (b_1, \dots, b_r)$, $c \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^r$ be such that $F_{B,\lambda}^{(r)}(\mathbf{a}; \mathbf{b}; c; \mathbf{x})$ converges absolutely. Then, for each $\alpha \in \mathbb{N}_0^r$,*

$$\begin{aligned} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_r^{\alpha_r}} F_{B,\lambda}^{(r)}(\mathbf{a}; \mathbf{b}; c; \mathbf{x}) &= \frac{\prod_{j=1}^r (a_j)_{\lambda; \alpha_j} (b_j)_{\alpha_j}}{(c)_{|\alpha|}} \\ &\times F_{B,\lambda}^{(r)}(a_1 + \alpha_1, \dots, a_r + \alpha_r; \\ &\quad b_1 + \alpha_1, \dots, b_r + \alpha_r; c + |\alpha|; \mathbf{x}). \end{aligned} \quad (34)$$

(b) Let $a, b, c_1, \dots, c_r \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^r$ be such that $F_{C,\lambda}^{(r)}(a, b; \mathbf{c}; \mathbf{x})$ converges absolutely. Then, for each $\boldsymbol{\alpha} \in \mathbb{N}_0^r$,

$$\begin{aligned} \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \dots \partial x_r^{\alpha_r}} F_{C,\lambda}^{(r)}(a, b; \mathbf{c}; \mathbf{x}) &= (a)_{\lambda; |\boldsymbol{\alpha}|} (b)_{|\boldsymbol{\alpha}|} \prod_{j=1}^r \frac{1}{(c_j)_{\alpha_j}} \\ &\times F_{C,\lambda}^{(r)}(a + |\boldsymbol{\alpha}|, b + |\boldsymbol{\alpha}|; \\ &\quad c_1 + \alpha_1, \dots, c_r + \alpha_r; \mathbf{x}). \end{aligned} \quad (35)$$

Coefficientwise as $\lambda \rightarrow 0$, the degenerate Pochhammer symbols reduce to the classical ones, and (34)–(35) recover the standard higher-order differential formulas for the classical Lauricella functions $F_B^{(r)}$ and $F_C^{(r)}$, respectively [8, 23].

The differential formulas derived above can be used to obtain *contiguous relations*, that is, linear relations between degenerate Lauricella functions whose parameters differ by integers. Such relations are the multivariable analogues of the contiguous relations for the one-variable degenerate hypergeometric functions of [27] and deform the classical contiguous relations for the Lauricella functions [8, 23, 24].

As a simple example, we rewrite the first-order formula (28) for $F_{A,\lambda}^{(r)}$ in the form

$$F_{A,\lambda}^{(r)}(a + 1; \mathbf{b} + \mathbf{e}_j; \mathbf{c} + \mathbf{e}_j; \mathbf{x}) = \frac{c_j}{\phi_\lambda(a) b_j} \frac{\partial}{\partial x_j} F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}).$$

Together with the corresponding differential systems, such identities yield three-term relations between $F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x})$, $F_{A,\lambda}^{(r)}(a + 1; \mathbf{b} + \mathbf{e}_j; \mathbf{c} + \mathbf{e}_j; \mathbf{x})$ and a suitable third function obtained by shifting a , b_j or c_j by ± 1 . Analogous relations hold for the other degenerate Lauricella functions and can be derived systematically from their higher-order derivative formulas. We omit the detailed list of such relations, since in the subsequent sections we shall mainly use the differential forms (32)–(35) to derive generating functions and integral representations.

5. Generating functions and diagonal reductions

In this section, we construct generating functions for the degenerate Lauricella families introduced in Section 3 and studied through the derivative formulas of Section 4. Our main focus is on the types *A* and *D*, which will be used later in the integral representations and in the asymptotic analysis. We also discuss diagonal specializations which connect the multivariable theory to the one-variable degenerate hypergeometric functions considered in [27]. For comparison with the classical case, see [8, 21, 23].

We first consider generating functions for the sequence of repeated derivatives of the degenerate Lauricella functions with respect to one or several variables. These generating functions are essentially operator identities, but they will be convenient later when we act with differential operators under the integral sign.

PROPOSITION 5. (One-variable generating function) *Let $F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x})$ be defined by (22) and assume that it is analytic in a neighbourhood of $\mathbf{x} \in \mathbb{C}^r$. For a fixed index $j \in \{1, \dots, r\}$, consider the exponential generating function*

$$G_{A,\lambda}^{(j)}(t; \mathbf{x}) := \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial^n}{\partial x_j^n} F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}). \quad (36)$$

Then, for t in a neighbourhood of 0,

$$G_{A,\lambda}^{(j)}(t; \mathbf{x}) = F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; x_1, \dots, x_j + t, \dots, x_r). \quad (37)$$

An analogous statement holds for $F_{B,\lambda}^{(r)}$, $F_{C,\lambda}^{(r)}$ and $F_{D,\lambda}^{(r)}$.

Proof. By absolute convergence in a small polydisc, we may differentiate the defining series (22) termwise with respect to x_j . The series in (36) is then the exponential generating function of the n th derivatives of an analytic function, so (37) follows from the Taylor expansion of $F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; x_1, \dots, x_j + t, \dots, x_r)$ in the variable t around $t = 0$. $\square$

Similarly, we may consider simultaneous derivatives with respect to all variables. In the standard multivariable Taylor notation [8, 23], let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}_0^r$ and write

$$\partial^{\boldsymbol{\alpha}} := \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \dots \partial x_r^{\alpha_r}}, \quad \boldsymbol{\alpha}! := \alpha_1! \dots \alpha_r!.$$

PROPOSITION 6. (Multivariable generating function) *Let $F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x})$ be analytic in a neighbourhood of $\mathbf{x} \in \mathbb{C}^r$ and define*

$$G_{A,\lambda}(\mathbf{t}; \mathbf{x}) := \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^r} \frac{\mathbf{t}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \partial^{\boldsymbol{\alpha}} F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}), \quad (38)$$

where $\mathbf{t} = (t_1, \dots, t_r)$ and $\mathbf{t}^{\boldsymbol{\alpha}} = t_1^{\alpha_1} \dots t_r^{\alpha_r}$. Then, for $\mathbf{t}$ small enough,

$$G_{A,\lambda}(\mathbf{t}; \mathbf{x}) = F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; x_1 + t_1, \dots, x_r + t_r). \quad (39)$$

Proof. Formula (39) is simply the multivariable Taylor expansion of $F_{A,\lambda}^{(r)}$ at the point $\mathbf{x}$, with increment $\mathbf{t}$. $\square$

We now use the higher-order formulas of Section 4 to rewrite the generating functions (36) and (38) as generating functions for families of degenerate Lauricella functions with shifted parameters. This is the degenerate analogue of well-known constructions for the classical Lauricella functions (see, for example, [8, 23]).

We begin with the multivariable case for $F_{A,\lambda}^{(r)}$. Combining Proposition 6 with Theorem 2 we obtain the following identity.

THEOREM 5. (Type A generating relation) *Let $a, b_1, \dots, b_r, c_1, \dots, c_r \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^r$ be such that $F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x})$ converges absolutely. For $\mathbf{t}$ in a neighbourhood of the origin we have*

$$F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; x_1 + t_1, \dots, x_r + t_r) = \sum_{\alpha \in \mathbb{N}_0^r} (a)_{\lambda; |\alpha|} \prod_{j=1}^r \frac{(b_j)_{\alpha_j}}{(c_j)_{\alpha_j}} \frac{\mathbf{t}^\alpha}{\alpha!} \quad (40)$$

$$F_{A,\lambda}^{(r)}(a + |\alpha|; b_1 + \alpha_1, \dots, b_r + \alpha_r; c_1 + \alpha_1, \dots, c_r + \alpha_r; \mathbf{x}).$$

Proof. Starting from (38) and applying the higher-order formula (32), we obtain

$$G_{A,\lambda}(\mathbf{t}; \mathbf{x}) = \sum_{\alpha \in \mathbb{N}_0^r} (a)_{\lambda; |\alpha|} \prod_{j=1}^r \frac{(b_j)_{\alpha_j}}{(c_j)_{\alpha_j}} \frac{\mathbf{t}^\alpha}{\alpha!}$$

$$F_{A,\lambda}^{(r)}(a + |\alpha|; b_1 + \alpha_1, \dots, b_r + \alpha_r; c_1 + \alpha_1, \dots, c_r + \alpha_r; \mathbf{x}).$$

Using (39) to identify $G_{A,\lambda}(\mathbf{t}; \mathbf{x})$ with the left-hand side of (40), we obtain (40). $\square$

A similar result holds for $F_{D,\lambda}^{(r)}$.

THEOREM 6. (Type D generating relation) *Let $a, b_1, \dots, b_r, c \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^r$ be such that $F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; \mathbf{x})$ converges absolutely. Then, for $\mathbf{t}$ in a neighbourhood of the origin,*

$$F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; x_1 + t_1, \dots, x_r + t_r) = \sum_{\alpha \in \mathbb{N}_0^r} (a)_{\lambda; |\alpha|} \frac{\prod_{j=1}^r (b_j)_{\alpha_j}}{(c)_{|\alpha|}} \frac{\mathbf{t}^\alpha}{\alpha!} \quad (41)$$

$$F_{D,\lambda}^{(r)}(a + |\alpha|; b_1 + \alpha_1, \dots, b_r + \alpha_r; c + |\alpha|; \mathbf{x}).$$

Proof. This follows from the multivariable Taylor expansion as in Proposition 6, together with the higher-order formula (33). $\square$

Analogous generating functions for $F_{B,\lambda}^{(r)}$ and $F_{C,\lambda}^{(r)}$ can be obtained from Theorem 4. We omit the explicit formulas, since in the subsequent sections, we will mainly use (40) and (41).

Coefficientwise in the relevant Taylor variables, the limit $\lambda \rightarrow 0$ replaces each degenerate Pochhammer symbol $(a)_{\lambda; n}$ by the ordinary Pochhammer symbol $(a)_n$. In this formal coefficientwise sense, (40)–(41) recover the classical generating functions for the Lauricella functions $F_A^{(r)}$ and $F_D^{(r)}$; compare with [8, 21, 23].

We now consider diagonal specializations of the variables $x_1, \dots, x_r$ and of the auxiliary variables $t_1, \dots, t_r$. Such specializations lead to generating functions for one-variable degenerate hypergeometric functions of the type introduced in [27] and provide a bridge between the multivariable degenerate Lauricella theory and the one-variable case.

For simplicity, we concentrate on the degenerate Lauricella function of type D .

Let

$$\mathbf{x} = (x, \dots, x), \quad \mathbf{t} = (t, \dots, t),$$

and write

$$F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; x) := F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; x, \dots, x),$$

for the diagonal restriction.

PROPOSITION 7. (Type D diagonal relation) *Under the above notation and assumptions, we have*

$$F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; x+t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} \left(\sum_{|\boldsymbol{\alpha}|=m} (a)_{\lambda:m} \frac{\prod_{j=1}^r (b_j)_{\alpha_j}}{(c)_m} \frac{m!}{\boldsymbol{\alpha}!} F_{D,\lambda}^{(r)}(a+m; b_1+\alpha_1, \dots, b_r+\alpha_r; c+m; x) \right), \quad (42)$$

where the inner sum runs over all multi-indices $\boldsymbol{\alpha} \in \mathbb{N}_0^r$ with $|\boldsymbol{\alpha}| = m$.

Proof. Substitute $\mathbf{x} = (x, \dots, x)$ and $\mathbf{t} = (t, \dots, t)$ into (41). On the left-hand side, we obtain the diagonal shift $x_j \mapsto x_j + t$, which gives $F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; x+t)$. On the right-hand side we have $\mathbf{t}^{\boldsymbol{\alpha}} = t^{|\boldsymbol{\alpha}|}$, so the sum over all multi-indices can be grouped according to $m = |\boldsymbol{\alpha}|$. This yields (42). $\square$

The diagonal type D reduction can also be written explicitly as an ordinary generalized hypergeometric function. Put $B = b_1 + \dots + b_r$ and assume $\lambda \neq 0$ and $a+1 - 1/\lambda \notin \{0, -1, -2, \dots\}$. Then, by (14),

$$F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; x, \dots, x) = {}_2F_2 \left(\begin{matrix} a, B \\ c, a+1-1/\lambda \end{matrix}; -\frac{x}{\lambda} \right). \quad (43)$$

Thus the diagonal degenerate F_D is an entire one-variable function for every fixed admissible $\lambda \neq 0$. Coefficientwise as $\lambda \rightarrow 0$, the diagonal series reduces to the classical identity $F_D^{(r)}(a; \mathbf{b}; c; x, \dots, x) = {}_2F_1(a, B; c; x)$ on $|x| < 1$.

6. Integral representations

In this section, we derive Euler-type integral representations for the degenerate Lauricella functions introduced in Section 3 and studied in Sections 4–5. We focus on the function $F_{A,\lambda}^{(r)}$ and on the diagonal restriction of $F_{D,\lambda}^{(r)}$, in accordance with the discussion in the Introduction.

The classical Euler integrals for the Lauricella functions are obtained by inserting the binomial expansion

$$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n$$

into suitable beta integrals, and then rearranging the resulting series [8, 23]. In the degenerate setting it is natural to introduce an analogous kernel based on the degenerate Pochhammer symbol $(a)_{\lambda;n}$ defined in (13).

DEFINITION 3. (Binomial kernel) Let $a \in \mathbb{C}$ and let $\lambda \in \mathbb{C} \setminus \{0\}$ be admissible in the sense that $a + 1 - 1/\lambda \notin \{0, -1, -2, \dots\}$. We define the *degenerate binomial kernel* by the entire power series

$$(1-z)_{\lambda}^{-a} := \sum_{n=0}^{\infty} \frac{(a)_{\lambda;n}}{n!} z^n. \quad (44)$$

Using (14), this kernel may also be written as the confluent hypergeometric function

$$(1-z)_{\lambda}^{-a} = {}_1F_1 \left(a; a + 1 - 1/\lambda; -\frac{z}{\lambda} \right). \quad (45)$$

For $\lambda = 0$ we use the classical binomial kernel $(1-z)^{-a}$, taken with the principal branch, and in particular with its power-series meaning in $|z| < 1$. For every fixed n , the coefficient of z^n in (44) tends to $(a)_n/n!$ as $\lambda \rightarrow 0$ through admissible values. Hence the kernel has $(1-z)^{-a}$ as its coefficientwise formal limit. We do not infer a locally uniform function-level limit from this coefficientwise statement.

Series of the form (44) appear naturally in the study of degenerate hypergeometric functions and degenerate integral transforms; see [11, 27] for related constructions in the one-variable case. Here we use (44) as a kernel in multiple integrals over the unit cube, in direct analogy with the one-variable degenerate constructions in [27].

We first derive an Euler-type integral representation for the degenerate Lauricella function of type A, defined by (22). The formula below should be compared with the classical representation of $F_A^{(r)}$; see, for example, [8, Ch. 5] and [23, Ch. 7].

THEOREM 7. (Type A integral representation) Let $a, b_1, \dots, b_r, c_1, \dots, c_r \in \mathbb{C}$ satisfy

$$\Re(c_j) > \Re(b_j) > 0 \quad (j = 1, \dots, r),$$

let $\lambda \in \mathbb{C} \setminus \{0\}$ be admissible for a , and let $\mathbf{x} = (x_1, \dots, x_r) \in \mathbb{C}^r$. Then the degenerate Lauricella function $F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x})$ admits the Euler-type integral representation

$$F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) = \prod_{j=1}^r \frac{\Gamma(c_j)}{\Gamma(b_j)\Gamma(c_j - b_j)} \int_{(0,1)^r} \left[\prod_{j=1}^r t_j^{b_j-1} (1-t_j)^{c_j-b_j-1} \right] \times (1-x_1 t_1 - \dots - x_r t_r)_{\lambda}^{-a} d\mathbf{t}, \quad (46)$$

where $d\mathbf{t} = dt_1 \cdots dt_r$ and $(1-z)_{\lambda}^{-a}$ is given by (44).

Proof. We start from the right-hand side of (46) and insert the series expansion (44) with $z = x_1 t_1 + \dots + x_r t_r$:

$$(1-x_1 t_1 - \dots - x_r t_r)_{\lambda}^{-a} = \sum_{n=0}^{\infty} \frac{(a)_{\lambda;n}}{n!} (x_1 t_1 + \dots + x_r t_r)^n.$$

Using the multinomial expansion,

$$(x_1 t_1 + \cdots + x_r t_r)^n = \sum_{|\mathbf{n}|=n} \frac{n!}{\mathbf{n}!} (x_1 t_1)^{n_1} \cdots (x_r t_r)^{n_r},$$

where $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ and $|\mathbf{n}| = n_1 + \cdots + n_r$, we obtain

$$(1 - x_1 t_1 - \cdots - x_r t_r)^{-a} = \sum_{\mathbf{n} \in \mathbb{N}_0^r} (a)_{\lambda; |\mathbf{n}|} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} \mathbf{t}^{\mathbf{n}},$$

where $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_r^{n_r}$ and $\mathbf{t}^{\mathbf{n}} = t_1^{n_1} \cdots t_r^{n_r}$. Substituting this into the integral and interchanging sum and integral (justified by the local uniform convergence of the entire kernel (44) on the compact set $\{x_1 t_1 + \cdots + x_r t_r : 0 \leq t_j \leq 1\}$ and by the beta-integrability assumptions) yields

$$\begin{aligned} \text{RHS of (46)} &= \sum_{\mathbf{n} \in \mathbb{N}_0^r} (a)_{\lambda; |\mathbf{n}|} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} \prod_{j=1}^r \frac{\Gamma(c_j)}{\Gamma(b_j) \Gamma(c_j - b_j)} \int_0^1 t_j^{b_j + n_j - 1} (1 - t_j)^{c_j - b_j - 1} dt_j \\ &= \sum_{\mathbf{n} \in \mathbb{N}_0^r} (a)_{\lambda; |\mathbf{n}|} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} \prod_{j=1}^r \frac{\Gamma(c_j)}{\Gamma(b_j) \Gamma(c_j - b_j)} B(b_j + n_j, c_j - b_j), \end{aligned}$$

where $B(\cdot, \cdot)$ is the classical beta function. Using

$$B(b_j + n_j, c_j - b_j) = \frac{\Gamma(b_j + n_j) \Gamma(c_j - b_j)}{\Gamma(c_j + n_j)} = \frac{(b_j)_{n_j}}{(c_j)_{n_j}} \frac{\Gamma(b_j) \Gamma(c_j - b_j)}{\Gamma(c_j)},$$

we find that the prefactors cancel and we are left with

$$\text{RHS of (46)} = \sum_{\mathbf{n} \in \mathbb{N}_0^r} \frac{(a)_{\lambda; |\mathbf{n}|} \prod_{j=1}^r (b_j)_{n_j}}{\prod_{j=1}^r (c_j)_{n_j}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!},$$

which is exactly the defining series (22) of $F_{A, \lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x})$. This proves (46). $\square$

Coefficientwise in $\mathbf{x}$, replacing the degenerate kernel coefficients by their $\lambda \rightarrow 0$ limits in (46) gives the well-known classical Euler integral for $F_A^{(r)}$; see [3, 8, 23, 24].

We now derive an integral representation for the diagonal restriction of $F_{D, \lambda}^{(r)}$, that is, for

$$F_{D, \lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; x) := F_{D, \lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; x, \dots, x),$$

which will be particularly useful in the asymptotic and numerical sections. The key point is that, on the diagonal, the r -variable series collapses to a one-variable degenerate Gauss-type series with second upper parameter $b_1 + \cdots + b_r$.

THEOREM 8. (Type D diagonal integral) *Let $a, b_1, \dots, b_r, c \in \mathbb{C}$ satisfy*

$$\Re(c) > \Re(b_1 + \cdots + b_r) > 0,$$

let $\lambda \in \mathbb{C} \setminus \{0\}$ be admissible for a , and let $x \in \mathbb{C}$. Then the diagonal restriction of the degenerate Lauricella function of type D admits the one-dimensional integral representation

$$F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; x) = \frac{\Gamma(c)}{\Gamma(b_1 + \cdots + b_r) \Gamma(c - b_1 - \cdots - b_r)} \int_0^1 t^{b_1 + \cdots + b_r - 1} \times (1-t)^{c-b_1-\cdots-b_r-1} (1-xt)_\lambda^{-a} dt, \quad (47)$$

where $(1-z)_\lambda^{-a}$ is the degenerate binomial kernel (44).

Proof. Set $B := b_1 + \cdots + b_r$. From the defining series (25) with $x_1 = \cdots = x_r = x$ we obtain

$$F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; x) = \sum_{m=0}^{\infty} \frac{(a)_{\lambda;m}}{(c)_m} x^m \sum_{|\mathbf{n}|=m} \frac{\prod_{j=1}^r (b_j)_{n_j}}{\mathbf{n}!}.$$

By the multinomial identity coming from $\prod_{j=1}^r (1-z)^{-b_j} = (1-z)^{-B}$, we have

$$\sum_{|\mathbf{n}|=m} \frac{\prod_{j=1}^r (b_j)_{n_j}}{\mathbf{n}!} = \frac{(B)_m}{m!}.$$

Hence

$$F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; x) = \sum_{m=0}^{\infty} \frac{(a)_{\lambda;m} (B)_m}{(c)_m} \frac{x^m}{m!}.$$

Now expand the kernel in the proposed integral:

$$(1-xt)_\lambda^{-a} = \sum_{m=0}^{\infty} \frac{(a)_{\lambda;m}}{m!} (xt)^m.$$

The interchange of sum and integral is justified by uniform convergence of the entire kernel on the compact interval of integration. Thus

$$\begin{aligned} & \frac{\Gamma(c)}{\Gamma(B) \Gamma(c-B)} \int_0^1 t^{B-1} (1-t)^{c-B-1} (1-xt)_\lambda^{-a} dt \\ &= \sum_{m=0}^{\infty} \frac{(a)_{\lambda;m} x^m}{m!} \frac{\Gamma(c)}{\Gamma(B) \Gamma(c-B)} \int_0^1 t^{B+m-1} (1-t)^{c-B-1} dt \\ &= \sum_{m=0}^{\infty} \frac{(a)_{\lambda;m} x^m}{m!} \frac{\Gamma(c)}{\Gamma(B) \Gamma(c-B)} B(B+m, c-B) \\ &= \sum_{m=0}^{\infty} \frac{(a)_{\lambda;m} (B)_m}{(c)_m} \frac{x^m}{m!}, \end{aligned}$$

which is exactly the diagonal series above. $\square$

Coefficientwise in the variable x , replacing $(a)_{\lambda;m}$ by $(a)_m$ in (47) yields the classical Euler integral for the diagonal specialization of $F_D^{(r)}$; equivalently, the Euler integral for ${}_2F_1(a, b_1 + \cdots + b_r; c; x)$ [8, 23]. The same coefficientwise reduction applies to (46). A genuine function-level limit under the integral sign would require a uniform bound for the degenerate binomial kernel as $\lambda \rightarrow 0$; no such bound is assumed here. This distinction agrees with Proposition 1, which is deliberately coefficientwise.

7. Coefficientwise expansions for small λ

In this section, we study the fixed-index expansion as $\lambda \rightarrow 0^+$. To make the scope precise, we write $G_\lambda = G_0 + \lambda G_1 + \mathcal{O}_{\text{coeff}}(\lambda^2)$ when every fixed Taylor coefficient has such an expansion, with a remainder constant that may depend on the coefficient index. This notation does not assert a uniform function-level remainder after summation. The correction series identified below nevertheless converge absolutely and locally uniformly on compact subsets of the corresponding classical Lauricella domains.

A convenient way to obtain the first correction is to use the beta-formula (11). Put $z = 1/\lambda$. The standard gamma-ratio expansion [20, §5.11(iii)]

$$\frac{\Gamma(z - \alpha)}{\Gamma(z)} = z^{-\alpha} \left[1 + \frac{\alpha(\alpha + 1)}{2z} + \mathcal{O}(z^{-2}) \right] \quad (z \rightarrow \infty)$$

holds uniformly for α in compact subsets avoiding the poles. Substitution into (11) gives, for fixed admissible α ,

$$\Gamma_\lambda(\alpha) = \Gamma(\alpha) \left[1 + \frac{\lambda}{2} \alpha(\alpha + 1) + \mathcal{O}(\lambda^2) \right] \quad (\lambda \rightarrow 0^+). \quad (48)$$

Taking the ratio of (48) at $\alpha = a + n$ and at $\alpha = a$ yields, for every fixed $n \in \mathbb{N}_0$,

$$(a)_{\lambda;n} = (a)_n \left[1 + \lambda \rho_n(a) + \mathcal{O}(\lambda^2) \right], \quad \rho_n(a) := \frac{n(2a + n + 1)}{2}. \quad (49)$$

Indeed,

$$\rho_n(a) = \frac{(a + n)(a + n + 1) - a(a + 1)}{2} = \frac{n(2a + n + 1)}{2}.$$

The remainder in (49) is uniform for $a + n$ in a fixed compact set, but its constant is not uniform as $n \rightarrow \infty$. Therefore (49) is used below coefficientwise; summing it as a function-level asymptotic expansion would require an additional estimate uniform in the summation index.

Collecting the fixed- n first-order coefficients gives the coefficientwise expansion

$$(1 - z)_\lambda^{-a} = (1 - z)^{-a} + \lambda K_a(z) + \mathcal{O}_{\text{coeff}}(\lambda^2), \quad (50)$$

where

$$K_a(z) := \sum_{n=0}^{\infty} \frac{(a)_n \rho_n(a)}{n!} z^n = \frac{1}{2} [\theta_z^2 + (2a + 1)\theta_z] (1 - z)^{-a}, \quad \theta_z = z \frac{d}{dz}. \quad (51)$$

Equivalently,

$$K_a(z) = a(a + 1)z \left(1 - \frac{z}{2} \right) (1 - z)^{-a-2}, \quad (52)$$

with the same branch convention as the classical binomial factor. The series defining K_a converges locally uniformly for $|z| < 1$, whereas (50) itself is asserted only coefficientwise in z .

We first consider the degenerate Lauricella function of type A . Define

$$F_{A,1}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) := \sum_{\mathbf{n} \in \mathbb{N}_0^r} \frac{|\mathbf{n}|(2a + |\mathbf{n}| + 1)}{2} \frac{(a)_{|\mathbf{n}|} \prod_{j=1}^r (b_j)_{n_j}}{\prod_{j=1}^r (c_j)_{n_j}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!}. \quad (53)$$

This correction series converges absolutely and locally uniformly on compact subsets of the classical F_A domain, because the additional factor is only quadratic in the total degree.

PROPOSITION 8. (Type A coefficientwise first-order expansion) *Assume the usual non-pole conditions on the denominator parameters. Then, as $\lambda \rightarrow 0^+$, the type A series has the coefficientwise expansion*

$$F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) = F_A^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) + \lambda F_{A,1}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) + \mathcal{O}_{\text{coeff}}(\lambda^2). \quad (54)$$

Moreover, the correction series $F_{A,1}^{(r)}$ converges absolutely and locally uniformly on compact subsets of $|x_1| + \cdots + |x_r| < 1$.

Proof. For a fixed multi-index $\mathbf{n}$, insert (49) with $n = |\mathbf{n}|$ into the coefficient of $\mathbf{x}^{\mathbf{n}}$ in (22). The zeroth-order coefficient is the corresponding classical Lauricella coefficient, and the first-order coefficient is precisely the coefficient appearing in (53). The convergence of $F_{A,1}^{(r)}$ on compact subsets of the classical type A domain follows because the additional factor is quadratic in the total degree. No uniform $\mathcal{O}(\lambda^2)$ estimate in $|\mathbf{n}|$ is claimed. $\square$

The same formal correction coefficients admit an Euler-type integral representation. Under the hypotheses of Theorem 7, integrating each fixed Taylor coefficient in (50) gives

$$\begin{aligned} F_{A,\lambda}^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) &= F_A^{(r)}(a; \mathbf{b}; \mathbf{c}; \mathbf{x}) \\ &+ \lambda \prod_{j=1}^r \frac{\Gamma(c_j)}{\Gamma(b_j)\Gamma(c_j - b_j)} \int_{(0,1)^r} \left[\prod_{j=1}^r t_j^{b_j-1} (1-t_j)^{c_j-b_j-1} \right] \\ &\times K_a(x_1 t_1 + \cdots + x_r t_r) d\mathbf{t} + \mathcal{O}_{\text{coeff}}(\lambda^2). \end{aligned} \quad (55)$$

We now turn to the diagonal type D case. Put $B = b_1 + \cdots + b_r$ and define

$$F_{D,1}^{(r)}(a; \mathbf{b}; c; x) := \sum_{n=0}^{\infty} \frac{n(2a + n + 1)}{2} \frac{(a)_n (B)_n}{(c)_n} \frac{x^n}{n!}. \quad (56)$$

Equivalently, under the beta-integral assumptions $\Re(c) > \Re(B) > 0$,

$$F_{D,1}^{(r)}(a; \mathbf{b}; c; x) = \frac{\Gamma(c)}{\Gamma(B)\Gamma(c-B)} \int_0^1 t^{B-1} (1-t)^{c-B-1} K_a(xt) dt. \quad (57)$$

PROPOSITION 9. (Type D diagonal coefficientwise expansion) *Assume the usual non-pole conditions. Then, as $\lambda \rightarrow 0^+$, the diagonal type D series has the coefficientwise expansion*

$$F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; x, \dots, x) = F_D^{(r)}(a; \mathbf{b}; c; x, \dots, x) + \lambda F_{D,1}^{(r)}(a; \mathbf{b}; c; x) + \mathcal{O}_{\text{coeff}}(\lambda^2). \quad (58)$$

The correction series $F_{D,1}^{(r)}$ converges absolutely and locally uniformly on compact subsets of $|x| < 1$.

Proof. On the diagonal, the defining series is

$$F_{D,\lambda}^{(r)}(a; \mathbf{b}; c; x, \dots, x) = \sum_{n=0}^{\infty} \frac{(a)_{\lambda:n}(B)_n}{(c)_n} \frac{x^n}{n!}.$$

For each fixed n , substitution of (49) gives the classical coefficient and the first-order coefficient in (56). The latter series converges locally uniformly in $|x| < 1$ because the extra factor is quadratic in n . Again, no uniform function-level remainder is asserted. $\square$

For example, when $r = 2$ and $b_1 = b_2 = b$, the diagonal integral becomes

$$F_{D,\lambda}^{(2)}(a; b, b; c; x, x) = \frac{\Gamma(c)}{\Gamma(2b)\Gamma(c-2b)} \int_0^1 t^{2b-1} (1-t)^{c-2b-1} (1-xt)_{\lambda}^{-a} dt,$$

and (58) gives its coefficientwise two-term expansion with $B = 2b$. This identifies the formal first-order effect of the deformation through the explicit correction coefficient $n(2a+n+1)/2$.

8. Numerical results and graphical illustrations

In this section, we recompute the numerical examples for the two-variable degenerate Lauricella functions of types D and A using the formulas derived in the previous sections. The main goals are

- to compare the degenerate functions with their classical counterparts, and
- to illustrate the effect of the deformation parameter λ in compact parameter ranges where direct series summation is numerically stable.

All values below were obtained with high-precision arithmetic by summing the defining series shell-by-shell with respect to the total degree $m+n$. More precisely, for each total degree N we summed all terms with $m+n=N$, added the resulting shell contribution to the partial sum, and stopped when the absolute value of the last shell was below $10^{-14}(1+|S_N|)$. The computations used 80-digit working precision and a safety cap $N_{\max} = 200$ on the total degree; in the reported tables the stopping tolerance was reached before the cap. For the type D examples with $\lambda > 0$, the sampling points

are chosen in a small compact region where the shell-by-shell truncation is stable and the comparison with the classical surface is visually clear.

We recall the two-variable degenerate Lauricella function of type D , defined by

$$F_{D,\lambda}^{(2)}(a; b_1, b_2; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{\lambda; m+n} (b_1)_m (b_2)_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad (59)$$

and the type A function

$$F_{A,\lambda}^{(2)}(a; b_1, b_2; c_1, c_2; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{\lambda; m+n} (b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!}. \quad (60)$$

On the diagonal $x = y$, the type D series reduces to

$$F_{D,\lambda}^{(2)}(a; b_1, b_2; c; x, x) = \sum_{n=0}^{\infty} \frac{(a)_{\lambda; n} (b_1 + b_2)_n}{(c)_n} \frac{x^n}{n!}, \quad (61)$$

and, using

$$(a)_{\lambda; n} = \frac{(a)_n}{(-\lambda)^n (a + 1 - 1/\lambda)_n},$$

we see that for fixed $\lambda > 0$ the diagonal series is of generalized-hypergeometric type in the scaled variable $-x/\lambda$. Accordingly, in the type D examples below we choose

$$0 \leq x \leq 0.08 \quad \text{and} \quad (x, y) \in [0, 0.08]^2,$$

so that the sampled points remain in a numerically stable compact region for the positive admissible values $\lambda \in \{0.2, 0.3, 0.5\}$ used in the tables and figures. The fixed- λ convergence theorem above shows that these choices are not convergence restrictions; they are made for stable and comparable numerical illustration.

As a first example, we take the symmetric parameter set

$$(a, b_1, b_2; c) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1 \right). \quad (62)$$

We evaluate the diagonal values

$$F_{D,\lambda}^{(2)}(a; b_1, b_2; c; x, x)$$

for $x \in \{0.02, 0.04, 0.06, 0.08\}$ and $\lambda \in \{0.2, 0.3, 0.5\}$.

Table 1 lists the recomputed values. Every entry is obtained from the diagonal series (61).

For this symmetric parameter set, the sampled diagonal values increase with x for every listed λ , and for each sampled x they also increase with the deformation parameter. Thus, in the displayed numerical window, the degenerate values lie above their classical counterparts.

As shown in Figure 1, the plotted curves are smooth and increasing on the displayed interval. The gap between the classical and degenerate curves becomes more visible as x approaches the right end of the interval.

Table 1: Diagonal values $F_{D,\lambda}^{(2)}(a; b_1, b_2; c; x, x)$ for the symmetric parameter set (62) and several values of λ .

x	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.5$
0.0200000	1.0101525	1.0147416	1.0189268	1.0376526
0.0400000	1.0206207	1.0305216	1.0380936	1.0708150
0.0600000	1.0314212	1.0475424	1.0559026	1.0997813
0.0800000	1.0425721	1.0659851	1.0710856	1.1248298

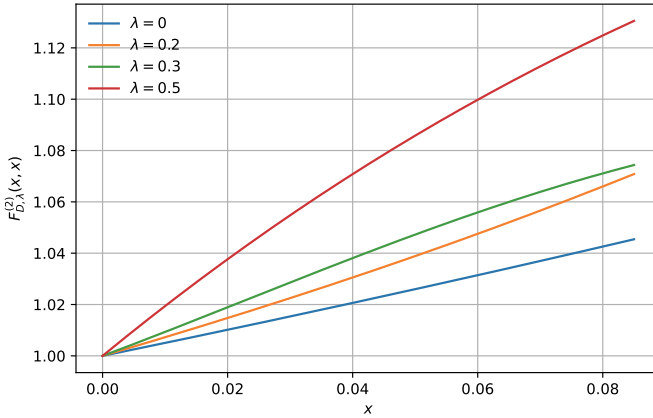


Figure 1: Diagonal profiles of the degenerate Lauricella function $F_{D,\lambda}^{(2)}(a; b_1, b_2; c; x, x)$ for the parameter set $(a, b_1, b_2; c) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1)$ and $\lambda \in \{0, 0.2, 0.3, 0.5\}$.

Next, we study the full two-variable behaviour of $F_{D,\lambda}^{(2)}(a; b_1, b_2; c; x, y)$ for the same symmetric parameter set (62). For a stable and visually comparable numerical window, we work on the square

$$(x, y) \in [0, 0.08] \times [0, 0.08]$$

and compare the classical surface $\lambda = 0$ with the degenerate surface $\lambda = 0.2$.

Figure 2 shows that the two sampled surfaces have the same qualitative shape: they are smooth, increase with both variables on the plotted grid, and remain close near the origin. On the displayed square, the degenerate surface for $\lambda = 0.2$ lies slightly above the classical one.

We next consider the asymmetric parameter set

$$(a, b_1, b_2; c) = \left(\frac{1}{2}, 1, \frac{3}{2}; 2\right), \quad (63)$$

and again study the diagonal values

$$F_{D,\lambda}^{(2)}(a; b_1, b_2; c; x, x)$$

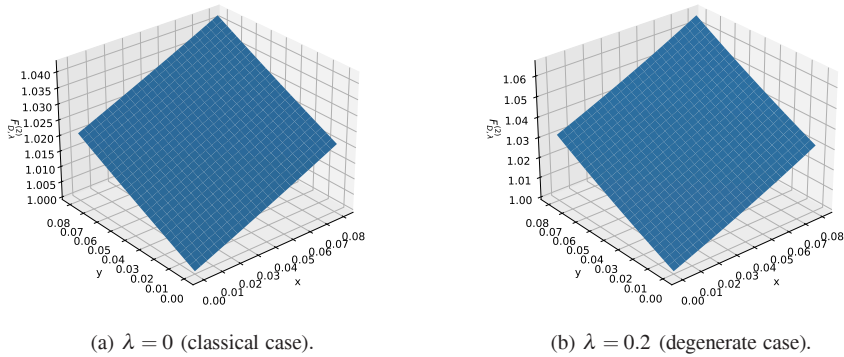


Figure 2: Three-dimensional surface plots of $F_{D,\lambda}^{(2)}(a;b_1,b_2;c;x,y)$ on the square $[0,0.08] \times [0,0.08]$ for the parameter set $(a,b_1,b_2;c) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1)$. Left: classical case; right: degenerate case.

for $x \in \{0.02, 0.04, 0.06, 0.08\}$ and $\lambda \in \{0, 0.2, 0.3, 0.5\}$.

Table 2 contains the recomputed values.

Table 2: Diagonal values $F_{D,\lambda}^{(2)}(a;b_1,b_2;c;x,x)$ for the asymmetric parameter set (63) and several values of λ .

x	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.5$
0.0200000	1.0127229	1.0185274	1.0237534	1.0465862
0.0400000	1.0259091	1.0386059	1.0475402	1.0866789
0.0600000	1.0395863	1.0605662	1.0688248	1.1207569
0.0800000	1.0537844	1.0846897	1.0856522	1.1492699

The same qualitative pattern is observed in the asymmetric sampled data: for each listed x , the degenerate values are larger than the classical ones, and the dependence on λ is increasing across the tabulated values. The shift is slightly more pronounced than in the symmetric example.

As shown in Figure 3, the four plotted curves are again smooth and increasing on the displayed interval, with a clear upward shift as λ grows.

We now turn to the two-variable degenerate Lauricella function of type A, with parameter set

$$(a,b_1,b_2;c_1,c_2) = \left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}; 1, 1\right), \tag{64}$$

and evaluate the diagonal values

$$F_{A,\lambda}^{(2)}(a;b_1,b_2;c_1,c_2;x,x)$$

for $x \in \{0.1, 0.2, 0.25, 0.3\}$ and $\lambda \in \{0, 0.1, 0.2\}$.

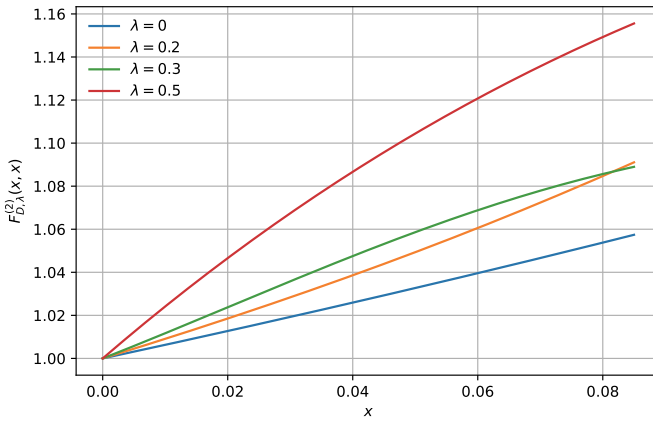


Figure 3: Diagonal profiles of $F_{D,\lambda}^{(2)}(a; b_1, b_2; c; x, x)$ for the asymmetric parameter set (63) and $\lambda \in \{0, 0.2, 0.3, 0.5\}$.

Table 3: Diagonal values $F_{A,\lambda}^{(2)}(a; b_1, b_2; c_1, c_2; x, x)$ for the parameter set (64) and several values of λ .

x	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.2$
0.1	1.0551825	1.0677255	1.0902401
0.2	1.1240100	1.1658201	1.2166441
0.25	1.1659064	1.2320283	1.2799098
0.3	1.2150468	1.3066504	1.3342332

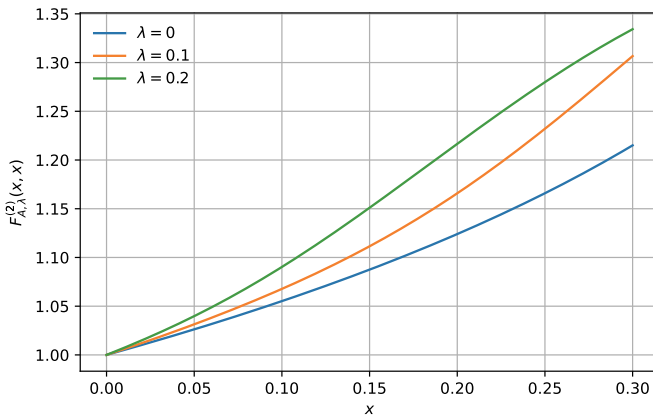


Figure 4: Diagonal profiles of $F_{A,\lambda}^{(2)}(a; b_1, b_2; c_1, c_2; x, x)$ for the parameter set (64) and $\lambda \in \{0, 0.1, 0.2\}$.

Table 3 gives the computed values.

On this interval the type A example shows a clear monotone response in the sampled values: for each listed x , the values increase with λ , and the difference between the three curves becomes more visible as x increases.

Figure 4 shows that the three plotted curves are smooth and increasing on the interval $0 \leq x \leq 0.3$, with the $\lambda = 0.2$ profile lying above the $\lambda = 0.1$ and classical profiles throughout the displayed range.

9. Concluding remarks and further directions

This paper developed the operational and analytic part of the degenerate Lauricella framework. Starting from the definitions based on the degenerate Pochhammer symbol, we established first-order, higher-order, and mixed-derivative formulas for the degenerate Lauricella functions. These identities show that the deformed families retain the parameter-shift structure characteristic of classical hypergeometric functions, with the deformation entering through explicit λ -dependent factors.

We also derived generating relations and diagonal reductions, especially for the types A and D . These reductions connect the multivariable theory to one-variable degenerate hypergeometric functions and provide useful forms for computation. The Euler-type integral representations obtained through the degenerate binomial kernel show that the integral side of Lauricella theory also admits a coherent degenerate analogue.

The coefficientwise small- λ expansions clarify the formal relationship with the classical Lauricella series: the classical coefficients are the leading terms, while the correction series record the first-order coefficient effect of the gamma-theoretic deformation. The numerical examples and plots separately show how the deformation parameter changes the corresponding two-variable profiles in numerically stable compact regions. Establishing a uniform function-level $\lambda \rightarrow 0$ asymptotic theory on complex classical domains remains a distinct problem.

Further work may address analytic continuation, monodromy, uniform function-level small- λ asymptotics (including behavior near exceptional parameter values), and concrete applications to deformed integral transforms or differential models involving degenerate kernels. The formulas developed here indicate that the degenerate Lauricella functions provide a workable setting for such investigations.

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