

VALUE DISTRIBUTION OF A CLASS OF COMPLEX q -DIFFERENCE-DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, by employing Nevanlinna theory, we investigate the Borel exceptional values of a class of q -difference-differential polynomials of meromorphic functions. Our results generalize the results obtained by X. M. Zheng and M. L. Zhan.

1. Introduction and results

In this paper, we shall assume that the reader is familiar with fundamental results and the standard notation of the value distribution theory of meromorphic functions or Nevanlinna theory (see [4, 6]). A meromorphic function $a(z) (\neq 0)$ is called a small function of f if $T(r, a) = S(r, f)$, where $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ outside a possible exceptional set of finite linear measure.

The zero distribution of complex differential-difference polynomials of meromorphic functions is one of the core research directions in complex analysis. The origin of this problem can be traced back to a conjecture proposed by Hayman in 1959.

Hayman [5] posed the following famous conjecture.

CONJECTURE. *If f is a transcendental meromorphic function and $n \geq 1$, then $f^n f'$ takes every finite nonzero value $b \in \mathbb{C}$ infinitely often.*

This conjecture has been solved by Hayman [4] for $n \geq 3$, by Mues [8] for $n = 2$, by Bergweiler and Eremenko [1] for $n = 1$.

Hayman [5] also proved the following famous result.

THEOREM 1. *If $f(z)$ is a transcendental meromorphic function, $n \geq 5$ is an integer, and $a (\neq 0)$ is a constant, then $f'(z) - af^n(z)$ assumes all finite values b infinitely often.*

He also conjectured in [4] that the same result holds for $n = 3$ and 4. However, Mues [8] proved that the result is not true by providing a counterexample (see [8, pp. 240–241]).

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DEFINITION 1. Let $f(z)$ be a meromorphic function in the complex plane. Then the order $\sigma(f)$ of $f(z)$ and the exponent of convergence of zeros of $f(z)$ are defined by

$$\sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}, \quad \lambda(f) = \limsup_{r \rightarrow \infty} \frac{\log N(r, 1/f)}{\log r}.$$

In recent years, significant progress has been made in Nevanlinna theory involving q -differences, which provides important tools for the study of q -difference equations and q -difference polynomials. Consequently, a large number of research works have focused on complex difference problems, enriching the theory of value distribution of meromorphic functions (see [2, 3]).

In 2011, Liu and Qi proved that the q -difference polynomial of the form $f^n(z)f(qz + c)$ assumes every nonzero complex value b infinitely often, where $f(z)$ is a zero-order transcendental meromorphic function and q is a nonzero complex constant (see [7]). In 2014, Xu and Cao studied the value distribution of the q -difference-differential polynomial of the form $(f^n(z)f(qz + c))^{(k)}$, where $f(z)$ is a transcendental entire function of finite positive order (see [9]).

In 2021, Zheng and Zhan further generalized the results of Xu and Cao by investigating the value distribution of q -difference-differential polynomials in a more general form, where $f(z)$ is extended to a polynomial in $f(z)$, and obtained the following results (see [11]).

Let a nonzero polynomial $P(z)$ be

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad n \in \mathbb{N}_+,$$

where $a_0, a_1, \dots, a_n (\neq 0)$ are complex constants. Let $m \in \mathbb{N}_+$ denote the number of distinct zeros of $P(z)$. Then $P(z)$ can be written as

$$P(z) = a_n (z - \alpha_1)^{n_1} \cdots (z - \alpha_m)^{n_m},$$

where $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{C}$ are the distinct zeros of $P(z)$, and $n_1, n_2, \dots, n_m \in \mathbb{N}_+$ satisfy

$$\sum_{i=1}^m n_i = n.$$

THEOREM 2. Let $f(z)$ be a transcendental entire function with $0 < \sigma(f) < \infty$, let a be a finite Borel exceptional value of $f(z)$, let $\eta \in \mathbb{C}$, and let $q \in \mathbb{C} \setminus \{0, 1\}$ satisfy $q^{\sigma(f)} \neq -1, -2, \dots, -n$. Define

$$H_1(z) = [P(f)f(qz + \eta)]^{(k)}, \quad k \in \mathbb{N}_+.$$

Then $H_1(z)$ has no nonzero finite Borel exceptional values, i.e., $H_1(z)$ takes every value $c \in \mathbb{C} \setminus \{0\}$ infinitely often, and satisfies $\lambda(H_1 - c) = \sigma(f)$.

THEOREM 3. Let $f(z)$ be a transcendental entire function with $0 < \sigma(f) < \infty$, let a be a finite Borel exceptional value of $f(z)$, let $\eta \in \mathbb{C}$, and let $q \in \mathbb{C} \setminus \{0, 1\}$ satisfy $q^{\sigma(f)} \neq -1, -2, \dots, -n$ and $f(qz + \eta) \not\equiv f(z)$. Define

$$H_2(z) = [P(f)(f(qz + \eta) - f(z))]^{(k)}, \quad k \in \mathbb{N}.$$

Then the following conclusions hold:

(1) If $q^s = 1$, $m = 1$ and $a = \alpha_1$, then 0 is also a Borel exceptional value of $H_2(z)$, i.e., $H_2(z)$ has no nonzero finite Borel exceptional values.

(2) If $q^s \neq 1$, or $q^s = 1$ and $m \neq 1$, or $q^s = 1$, $m = 1$ and $a \neq \alpha_1$, then $H_2(z)$ has no finite Borel exceptional values.

(3) $H_2(z)$ takes every value $c \in \mathbb{C} \setminus \{0\}$ infinitely often, and satisfies $\lambda(H_2 - c) = \sigma(f)$.

In this paper, based on the work of Zheng and Zhan, we further extend the function $f(z)$ to the class of transcendental meromorphic functions and obtain the following results.

THEOREM 4. Let $f(z)$ be a transcendental meromorphic function with $0 < \sigma(f) < \infty$, and suppose that $f(z)$ has two Borel exceptional values $a \in \mathbb{C}$ and ∞ . Let $\eta \in \mathbb{C}$ and $q \in \mathbb{C} \setminus \{0, 1\}$ satisfy $q^{\sigma(f(z))} \neq -1, -2, \dots, -n$. Define

$$H_1(z) = (P(f)f(qz + \eta))^{(k)}, \quad k \in \mathbb{N}_+.$$

Then $H_1(z)$ has no nonzero finite Borel exceptional values, i.e., $H_1(z)$ takes every value $c \in \mathbb{C} \setminus \{0\}$ infinitely often, and satisfies $\lambda(H_1(z) - c) = \sigma(f)$.

The following example illustrates that the condition $q^{\sigma(f)} \neq -1, -2, \dots, -n$ in Theorem 4 cannot be omitted.

EXAMPLE 1. Let

$$f(z) = e^z \frac{z-1}{z+1}$$

be a meromorphic function. Take the polynomial $P(z) = z$ of the degree $n = 1$, the constants $q = -1$, $\eta = 0$, and set $k = 1$. Define

$$H_1(z) = (P(f)f(qz + \eta))^{(k)} = (f(z)f(-z))'.$$

Since $T(r, f) = T(r, e^z) + O(\log r)$, we have $\sigma(f) = 1$ satisfying $0 < \sigma(f) < \infty$. As $e^z \neq 0$, $f(z) = 0$ holds only at $z = 1$, and the zero counting function satisfies $N(r, 1/f) = O(\log r)$, so 0 is a Borel exceptional value. Furthermore, the unique pole of f is $z = -1$ with the pole counting function $N(r, f) = O(\log r)$; thus ∞ is also a Borel exceptional value. That is, f has two Borel exceptional values 0 and ∞ . Since $q = -1$, we obtain $q^{\sigma(f)} = (-1)^1 = -1$, which does not satisfy the condition $q^{\sigma(f)} \neq -1, -2, \dots, -n$ (here $n = 1$, the corresponding set is $\{-1\}$).

Since

$$f(-z) = e^{-z} \frac{-z-1}{-z+1} = e^{-z} \frac{z+1}{z-1},$$

we get

$$f(z)f(-z) = e^z e^{-z} \cdot \frac{z-1}{z+1} \cdot \frac{z+1}{z-1} = 1,$$

and therefore

$$H_1(z) = (f(z)f(-z))' = (1)' = 0.$$

Hence, for any nonzero constant $c \in \mathbb{C} \setminus \{0\}$, the equation $H_1(z) = c$ has no solutions, so every nonzero constant c is a Borel exceptional value. In particular, the exponent of convergence of zeros $\lambda(H_1 - c) = 0$, which contradicts the assertion of Theorem 4 that “ H_1 takes every nonzero value infinitely often and $\lambda(H_1 - c) = \sigma(f)$ ”.

THEOREM 5. *Let $f(z)$ be a transcendental meromorphic function with $0 < \sigma(f) < \infty$, and suppose that $f(z)$ has two Borel exceptional values $a \in \mathbb{C}$ and ∞ . Let $\eta \in \mathbb{C}$ and $q \in \mathbb{C} \setminus \{0, 1\}$ satisfy $q^{\sigma(f(z))} \neq -1, -2, \dots, -n$ and $f(qz + \eta) \not\equiv f(z)$. Define*

$$H_2(z) = (P(f)(f(qz + \eta) - f(z)))^{(k)}, \quad k \in \mathbb{N}_+.$$

Then the following conclusions hold:

(1) *If $q^s = 1$, $m = 1$ and $a = \alpha_1$, then 0 is also a Borel exceptional value of $H_2(z)$, i.e., $H_2(z)$ has no nonzero finite Borel exceptional values.*

(2) *If $q^s \neq 1$, or $q^s = 1$ and $m \neq 1$, or $q^s = 1$, $m = 1$ but $a \neq \alpha_1$, then $H_2(z)$ has no finite Borel exceptional values.*

(3) *$H_2(z)$ has no nonzero finite Borel exceptional values, i.e., $H_2(z)$ takes every value $c \in \mathbb{C} \setminus \{0\}$ infinitely often, and satisfies $\lambda(H_2(z) - c) = \sigma(f)$.*

The following example shows that the condition $q^{\sigma(f)} \neq -1, -2, \dots, -n$ in Theorem 5 cannot be removed.

EXAMPLE 2. Let the meromorphic function

$$f(z) = e^{z \frac{z-1}{z+1}},$$

the polynomial $P(z) = z$ of the degree $n = 1$, the constants $q = -1$, $\eta = 0$, and take $k = 1$. Define

$$H_2(z) = (P(f)(f(qz + \eta) - f(z)))^{(k)}.$$

Since $e^z \neq 0$, $f(z) = 0$ holds only at $z = 1$, so the zero counting function satisfies $N(r, 1/f) = O(\log r)$. The only pole of $f(z)$ is $z = -1$, hence $N(r, f) = O(\log r)$. Then both 0 and ∞ are Borel exceptional values of $f(z)$. From $T(r, f) \sim \frac{1}{\pi}r$, we obtain the order $\sigma(f) = 1$. Since $q = -1$, we have $q^{\sigma(f)} = (-1)^1 = -1$, which does not satisfy the condition $q^{\sigma(f)} \neq -1$. Note that

$$f(-z) = e^{-z \frac{-z-1}{-z+1}} = e^{-z \frac{z+1}{z-1}},$$

$$f(z)f(-z) = e^z e^{-z} \cdot \frac{z-1}{z+1} \cdot \frac{z+1}{z-1} = 1,$$

which yields

$$f(-z) = \frac{1}{f(z)}.$$

Since $P(f) = f$, we have

$$P(f)(f(qz + \eta) - f(z)) = f(z)(f(-z) - f(z)) = f(z) \left(\frac{1}{f(z)} - f(z) \right) = 1 - f(z)^2.$$

Taking $k = 1$, we get

$$H_2(z) = (1 - f(z)^2)' = -2f(z)f'(z).$$

Set $R(z) = \frac{z-1}{z+1}$, then $f(z) = e^z R(z)$. Differentiating gives

$$f'(z) = e^z (R(z) + R'(z)),$$

and substitution leads to

$$H_2(z) = -2e^{2z}R(z)(R(z) + R'(z)).$$

Since $R(z)$ and $R'(z)$ are both rational functions, $R(z)(R(z) + R'(z))$ is a non-zero rational function. Thus $H_2(z)$ has only finitely many zeros, and

$$\lambda(H_2) = 0 < 1 = \sigma(f).$$

Therefore, 0 is a finite Borel exceptional value of $H_2(z)$.

Conclusion (2) of Theorem 5 asserts that if $q^s \neq 1$ (here $s = 1$, $q^s = -1 \neq 1$), then H_2 possesses no finite Borel exceptional values. Nevertheless, 0 is exactly a Borel exceptional value of H_2 in this example, so Conclusion (2) is invalid. This counterexample shows that the condition $q^{\sigma(f)} \neq -1, -2, \dots, -n$ is necessary for the validity of the statement “ H_2 has no finite Borel exceptional values” in Theorem 5.

2. Some lemmas

To prove our theorems, we require the following lemmas.

LEMMA 1. ([12]) *Let $p_1, \dots, p_m$ be polynomials in z with leading terms $\alpha_1 z^{q_1}, \dots, \alpha_m z^{q_m}$, respectively, where $q_1, \dots, q_m$ are positive integers. Let $q = \min\{q_1, \dots, q_m\}$ and $q_0 = \max\{q_1, \dots, q_m\}$, where $\alpha_1, \dots, \alpha_m$ are nonzero complex numbers satisfying $\alpha_h \neq \alpha_k$ whenever $q_h = q_k$ for $1 \leq h < k \leq m$. Let*

$$G(z) = D_0(z) + D_1(z)e^{p_1(z)} + \dots + D_m(z)e^{p_m(z)},$$

where $D_0, D_1, \dots, D_m$ are meromorphic functions of growth order less than q , and $D_j \not\equiv 0$ for $j = 1, \dots, m$. Then:

(1) *There exist two positive constants $a_1 < a_2$ such that*

$$a_1 r^{q_0} \leq T(r, G) \leq a_2 r^{q_0}, \quad r \rightarrow \infty.$$

(2) If $D_0 \neq 0$, then

$$m\left(r, \frac{1}{G}\right) = o(r^{q_0}), \quad r \rightarrow \infty.$$

(3) If $D_0 \equiv 0$ and $m \geq 2$, then there exist two positive constants $b_1 < b_2$ such that

$$b_1 r^{q_0} \leq N\left(r, \frac{1}{G}\right) \leq b_2 r^{q_0}, \quad r \rightarrow \infty.$$

LEMMA 2. Let $n \in \mathbb{N}_+$, and let $P_1(z), P_2(z), \dots, P_n(z)$ be non-constant polynomials with degrees $d_1, d_2, \dots, d_n$, respectively, satisfying $\deg(P_i - P_j) = \max\{d_i, d_j\}$ for all $i \neq j$. Let

$$A(z) = \sum_{j=1}^n B_j(z) e^{P_j(z)},$$

where $B_j(z) (\neq 0)$, $j = 1, 2, \dots, n$, are meromorphic functions with $\sigma(B_j) < d_j$ for $j = 1, 2, \dots, n$. Then $A(z)$ satisfies

$$\sigma(A) = \max_{1 \leq j \leq n} \{d_j\}.$$

Proof. Let the leading terms of $P_1(z), P_2(z), \dots, P_n(z)$ be $\alpha_1 z^{d_1}, \alpha_2 z^{d_2}, \dots, \alpha_n z^{d_n}$, respectively, where $d_1, d_2, \dots, d_n$ are positive integers. Let

$$d = \max_{1 \leq j \leq n} \{d_j\}.$$

From the given condition that $\deg(P_i - P_j) = \max\{d_i, d_j\}$ for $i \neq j$, it follows that $\alpha_i \neq \alpha_j$ whenever $i \neq j$. By Lemma 1, there exist positive constants $a_1 < a_2$ such that

$$a_1 r^d \leq T(r, A) \leq a_2 r^d \quad \text{as } r \rightarrow \infty.$$

By the definition of the order, we conclude that $\sigma(A) = d$. $\square$

LEMMA 3. ([10]) Let $n \geq 2$, let $f_1(z), f_2(z), \dots, f_n(z)$ be meromorphic functions, and $g_1(z), g_2(z), \dots, g_n(z)$ be entire functions satisfying the following conditions:

(1) $\sum_{j=1}^n f_j(z) e^{g_j(z)} \equiv 0$;

(2) For $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is not constant;

(3) For $1 \leq j \leq n$ and $1 \leq h < k \leq n$, $T(r, f_j) = o(T(r, e^{g_h - g_k}))$ as $r \rightarrow \infty$, $r \notin E$, where $E \subset (1, +\infty)$ is a set of finite linear measure or finite logarithmic measure. Then $f_j(z) \equiv 0$ for $j = 1, 2, \dots, n$.

3. Proof of Theorem 4

Proof. Since $f(z)$ has two Borel exceptional values a and ∞ , by the Hadamard factorization theorem, there exist $s \geq 1$ with $\sigma(f(z)) = s$ and a nonzero complex constant α such that

$$f(z) = a + \frac{g(z)}{p(z)} e^{\alpha z^s}, \quad (3.1)$$

where $p(z)$ is the canonical product of the poles of $f(z)$ with $\rho(p(z)) < s$, and $g(z)$ is the canonical product of the zeros of $f(z)$ with $\rho(g(z)) < s$.

For q -difference of $f(z)$, we have

$$f(qz + \eta) = a + \frac{g(qz + \eta)}{p(qz + \eta)} e^{\alpha(qz + \eta)^s}.$$

By the binomial theorem,

$$(qz + \eta)^s = q^s z^s + s q^{s-1} \eta z^{s-1} + \binom{s}{2} q^{s-2} \eta^2 z^{s-2} + \cdots + \eta^s = q^s z^s + R(z),$$

where $R(z) = s q^{s-1} \eta z^{s-1} + \binom{s}{2} q^{s-2} \eta^2 z^{s-2} + \cdots + \eta^s$. Clearly, $R(z)$ is a polynomial of degree at most $s-1$.

Then

$$e^{\alpha(qz + \eta)^s} = e^{\alpha q^s z^s} e^{\alpha R(z)} = e^{\alpha q^s z^s} h(z),$$

where $h(z) = e^{\alpha R(z)}$.

Thus,

$$f(qz + \eta) = a + \frac{g(qz + \eta)}{p(qz + \eta)} e^{\alpha(qz + \eta)^s} = a + \frac{g(qz + \eta)}{p(qz + \eta)} h(z) e^{\alpha q^s z^s}. \quad (3.2)$$

Substituting (3.1) into $P(f)$, we obtain

$$P(f) = a_n \prod_{i=1}^m \left(a - \alpha_i + \frac{g(z)}{p(z)} e^{\alpha z^s} \right)^{n_i}.$$

By the binomial theorem,

$$\left(a - \alpha_i + \frac{g(z)}{p(z)} e^{\alpha z^s} \right)^{n_i} = \sum_{j_i=0}^{n_i} \binom{n_i}{j_i} (a - \alpha_i)^{n_i - j_i} \left(\frac{g(z)}{p(z)} \right)^{j_i} e^{j_i \alpha z^s}.$$

Then

$$P(f) = \sum_{j=0}^n \left(a_n \sum_{\substack{j_1 + \cdots + j_m = j \\ 0 \leq j_i \leq n_i}} \prod_{i=1}^m \binom{n_i}{j_i} (a - \alpha_i)^{n_i - j_i} \left(\frac{g(z)}{p(z)} \right)^{j_i} \right) e^{j_i \alpha z^s} = \sum_{j=0}^n C_j(z) e^{j \alpha z^s}, \quad (3.3)$$

where

$$C_j(z) = a_n \sum_{\substack{j_1 + \dots + j_m = j \\ 0 \leq j_i \leq n_i}} \prod_{i=1}^m \binom{n_i}{j_i} (a - \alpha_i)^{n_i - j_i} \left(\frac{g(z)}{p(z)} \right)^{j_i}.$$

Multiplying (3.2) and (3.3), we get

$$P(f)f(qz + \eta) = a \sum_{j=0}^n C_j e^{j\alpha z^s} + \sum_{j=0}^n C_j \frac{g(qz + \eta)}{p(qz + \eta)} h(z) e^{(j+q^s)\alpha z^s}. \quad (3.4)$$

When $m = 1$ and $a = \alpha_1 = 0$, the equation (3.4) reduces to

$$P(f)f(qz + \eta) = a_n \left(\frac{g(z)}{p(z)} \right)^n \frac{g(qz + \eta)}{p(qz + \eta)} h(z) e^{(n+q^s)\alpha z^s}. \quad (3.5)$$

Since the coefficients in (3.5) are not identically zero, $q^{\sigma(f(z))} \neq -1, -2, \dots, -n$, $\alpha \neq 2\alpha \neq \dots \neq n\alpha$, $q^s \alpha \neq (1+q^s)\alpha \neq \dots \neq (n+q^s)\alpha$, $P(f)f(qz + \eta) \neq 0$, $\sigma\left(\frac{g(z)}{p(z)}\right) < s$, $\sigma\left(\frac{g(qz + \eta)}{p(qz + \eta)}\right) < s$ and $\sigma(h(z)) \leq s-1$, it follows from (3.4), (3.5) and Lemma 2 that $P(f)f(qz + \eta)$ is a transcendental meromorphic function satisfying

$$\sigma(H_1) = \sigma(P(f)f(qz + \eta)) = s.$$

Therefore, $H_1(z)$ can be expressed as

$$H_1(z) = (P(f)f(qz + \eta))^{(k)} = \sum_{l=1}^{2n+1} g_l(z) e^{\beta_l z^s}. \quad (3.6)$$

where

$$\beta_l = \begin{cases} l\alpha, & l = 1, 2, \dots, n, \\ (l - n - 1 + q^s)\alpha, & l = n+1, n+2, \dots, 2n+1, \end{cases}$$

and $g_l(z)$ for $l = 1, 2, \dots, 2n+1$ are differential polynomials in $g(z)$, $p(z)$, $g(qz + \eta)$, $p(qz + \eta)$, and $h(z)$ with $\rho(g_l(z)) < s$ for all $l = 1, 2, \dots, 2n+1$.

Define the set

$$B = \{\alpha, 2\alpha, \dots, n\alpha\} \cup \{q^s \alpha, (1+q^s)\alpha, \dots, (n+q^s)\alpha\}.$$

Since q^s may be an integer, the two sets above may have a non-empty intersection, so that $|B| \leq 2n+1$.

By combining like terms in (3.6), the proof is divided into the following two cases.

Case 1. There exists exactly one $\beta_{l_0} \in \{\alpha, 2\alpha, \dots, n\alpha\} \cup \{q^s \alpha, (1+q^s)\alpha, \dots, (n+q^s)\alpha\}$ such that

$$H_1(z) = g_{l_0}(z) e^{\beta_{l_0} z^s},$$

where $g_{l_0}(z)$ is the sum of the coefficients corresponding to the term $e^{\beta_{l_0}z^s}$ in (3.6). Clearly, $\beta_{l_0} \neq 0$, $g_{l_0}(z) \not\equiv 0$, and $\sigma(g_{l_0}(z)) < s$. It follows that 0 is a Borel exceptional value of $H_1(z)$, i.e., $H_1(z)$ has no non-zero finite Borel exceptional values.

Case 2. For the remaining cases, without loss of generality, let

$$H_1(z) = (P(f)f(qz + \eta))^{(k)} = \sum_{j=1}^r g_{l_j}(z)e^{\beta_{l_j}z^s}, \quad (3.7)$$

where $r \in \{2, \dots, 2n+1\}$, and $\beta_{l_1}, \beta_{l_2}, \dots, \beta_{l_r} \in \{\alpha, 2\alpha, \dots, n\alpha\} \cup \{q^s\alpha, (1+q^s)\alpha, \dots, (n+q^s)\alpha\}$ satisfy $\beta_{l_i} \neq \beta_{l_j}$ for $i \neq j$ and $g_{l_i}(z) \not\equiv 0$ for $i = 1, 2, \dots, r$.

Let a^* be a finite Borel exceptional value of $H_1(z)$. Then $H_1(z)$ can be expressed as

$$H_1(z) = a^* + g^*(z)e^{\beta z^s}, \quad (3.8)$$

where $\beta \neq 0$ is a complex constant, and $g^*(z) \not\equiv 0$ is a meromorphic function with $\sigma(g^*(z)) < s$.

From (3.7) and (3.8), we obtain

$$\sum_{j=1}^r g_{l_j}(z)e^{\beta_{l_j}z^s} - a^* - g^*(z)e^{\beta z^s} \equiv 0. \quad (3.9)$$

If $\beta \neq \beta_{l_j}$ for all $j = 1, 2, \dots, r$, then by Lemma 3 and (3.9), we have

$$g_{l_j}(z) \equiv 0, \quad j = 1, 2, \dots, r, \quad g^*(z) \equiv 0, \quad a^* \equiv 0. \quad (3.10)$$

Since $g^*(z) \not\equiv 0$, this contradicts equation (3.10).

If $\beta = \beta_{l_j}$ for some $j = 1, 2, \dots, r$, then the equation (3.9) can be rewritten as

$$(g_{l_j}(z) - g^*(z))e^{\beta z^s} - \sum_{\substack{i=1 \\ i \neq j}}^r g_{l_i}(z)e^{\beta_{l_i}z^s} - a^* \equiv 0. \quad (3.11)$$

By Lemma 3 and (3.11), we obtain

$$g_{l_j}(z) - g^*(z) \equiv 0, \quad g_{l_i}(z) \equiv 0, \quad i \neq j, \quad a^* \equiv 0. \quad (3.12)$$

Since $g^*(z) \not\equiv 0$, this contradicts equation (3.12).

Therefore, $H_1(z)$ has no finite Borel exceptional values.

From Case 1 and Case 2, $H_1(z)$ takes every value $c \in \mathbb{C} \setminus \{0\}$ infinitely often, and satisfies $\lambda(H_1(z) - c) = \sigma(f)$.

This completes the proof of Theorem 4. $\square$

4. Proof of Theorem 5

Proof. Since $f(z)$ has two Borel exceptional values a and ∞ , by the Hadamard factorization theorem, there exist $s \geq 1$ with $\sigma(f(z)) = s$ and a nonzero complex constant α such that

$$f(z) = a + \frac{g(z)}{p(z)}e^{\alpha z^s}, \quad (4.1)$$

where $p(z)$ is the canonical product of the poles of $f(z)$ with $\rho(p(z)) < s$, and $g(z)$ is the canonical product of the zeros of $f(z)$ with $\rho(g(z)) < s$.

For the shift, we have

$$f(qz + \eta) = a + \frac{g(qz + \eta)}{p(qz + \eta)} e^{\alpha(qz + \eta)^s}.$$

By the binomial theorem,

$$(qz + \eta)^s = q^s z^s + s q^{s-1} \eta z^{s-1} + \binom{s}{2} q^{s-2} \eta^2 z^{s-2} + \cdots + \eta^s = q^s z^s + R(z),$$

where $R(z) = s q^{s-1} \eta z^{s-1} + \binom{s}{2} q^{s-2} \eta^2 z^{s-2} + \cdots + \eta^s$. Clearly, $R(z)$ is a polynomial of degree at most $s - 1$.

Then

$$e^{\alpha(qz + \eta)^s} = e^{\alpha q^s z^s} e^{\alpha R(z)} = e^{\alpha q^s z^s} h(z),$$

where $h(z) = e^{\alpha R(z)}$.

Thus,

$$f(qz + \eta) = a + \frac{g(qz + \eta)}{p(qz + \eta)} e^{\alpha(qz + \eta)^s} = a + \frac{g(qz + \eta)}{p(qz + \eta)} h(z) e^{\alpha q^s z^s}. \quad (4.2)$$

From (4.1) and (4.2), we obtain

$$f(qz + \eta) - f(z) = \frac{g(qz + \eta)}{p(qz + \eta)} h(z) e^{\alpha q^s z^s} - \frac{g(z)}{p(z)} e^{\alpha z^s} = U(z) e^{\alpha q^s z^s} - V(z) e^{\alpha z^s}, \quad (4.3)$$

where $U(z) = \frac{g(qz + \eta)}{p(qz + \eta)} h(z)$, $V(z) = \frac{g(z)}{p(z)}$, and $\rho(U(z)) < s$, $\rho(V(z)) < s$.

Substituting (4.1) into $P(f)$, we obtain

$$P(f) = a_n \prod_{i=1}^m \left(a - \alpha_i + \frac{g(z)}{p(z)} e^{\alpha z^s} \right)^{n_i} = a_n \prod_{i=1}^m \left(a - \alpha_i + V(z) e^{\alpha z^s} \right)^{n_i}.$$

By the binomial theorem,

$$\left(a - \alpha_i + V(z) e^{\alpha z^s} \right)^{n_i} = \sum_{j_i=0}^{n_i} \binom{n_i}{j_i} (a - \alpha_i)^{n_i - j_i} (V(z))^{j_i} e^{j_i \alpha z^s},$$

and thus

$$P(f) = \sum_{j=0}^n \left(a_n \sum_{\substack{j_1 + \cdots + j_m = j \\ 0 \leq j_i \leq n_i}} \prod_{i=1}^m \binom{n_i}{j_i} (a - \alpha_i)^{n_i - j_i} (V(z))^{j_i} \right) e^{j \alpha z^s} = \sum_{j=0}^n C_j(z) e^{j \alpha z^s}, \quad (4.4)$$

where

$$C_j(z) = a_n \sum_{\substack{j_1 + \cdots + j_m = j \\ 0 \leq j_i \leq n_i}} \prod_{i=1}^m \binom{n_i}{j_i} (a - \alpha_i)^{n_i - j_i} (V(z))^{j_i}.$$

Multiplying (4.3) and (4.4) yields

$$\begin{aligned} P(f)(f(qz + \eta) - f(z)) &= \left(\sum_{j=0}^n C_j(z) e^{j\alpha z^s} \right) \left(U(z) e^{\alpha q^s z^s} - V(z) e^{\alpha z^s} \right) \\ &= \sum_{j=0}^n C_j(z) U(z) e^{(j+q^s)\alpha z^s} - \sum_{j=0}^n C_j(z) V(z) e^{(j+1)\alpha z^s}. \end{aligned} \quad (4.5)$$

Define the index sets:

$$\Gamma_1 = \{q^s \alpha, (1+q^s)\alpha, \dots, (n+q^s)\alpha\}, \quad \Gamma_2 = \{\alpha, 2\alpha, \dots, (n+1)\alpha\}.$$

We now classify the cases according to whether $q^s = 1$, whether $m = 1$, and whether $a = \alpha_1$. These classifications are mutually exclusive and cover all possible scenarios.

Case 1. When $q^s = 1$ and $m = 1$, we have $\Gamma_1 = \Gamma_2 = \{\alpha, 2\alpha, \dots, (n+1)\alpha\}$, i.e., the two index sets are identical.

At this point,

$$P(z) = a_n(z - \alpha_1)^n.$$

Since $m = 1$, we have $n_1 = n$, and $C_j(z)$ simplifies to

$$C_j(z) = a_n \sum_{\substack{j_1 + \dots + j_m = j \\ 0 \leq j_i \leq n_i}} \prod_{i=1}^m C_{n_i}^{j_i}(a - \alpha_i)^{n_i - j_i} \left(\frac{g(z)}{p(z)} \right)^{j_i} = a_n C_n^j(a - \alpha_1)^{n-j} V(z)^j.$$

Substituting the above into (4.5), we obtain

$$\begin{aligned} P(f)(f(qz + \eta) - f(z)) &= \sum_{j=0}^n C_j(z) U(z) e^{(j+q^s)\alpha z^s} - \sum_{j=0}^n C_j(z) V(z) e^{(j+1)\alpha z^s} \\ &= a_n \sum_{j=0}^n \binom{n}{j} (a - \alpha_1)^{n-j} V(z)^j (U(z) - V(z)) e^{(j+1)\alpha z^s}. \end{aligned} \quad (4.6)$$

Since $f(qz + \eta) \not\equiv f(z)$, we have $U(z) \not\equiv V(z)$. Moreover, since $\alpha \not\equiv 2\alpha \not\equiv \dots \not\equiv (n+1)\alpha$, $V(z) \not\equiv 0$, $U(z) \not\equiv 0$, $\sigma(V(z)) < s$, $\sigma(U(z)) < s$, and $\sigma(h(z)) \leq s-1$, it follows from (4.6) and Lemma 2 that $P(f)(f(qz + \eta) - f(z))$ is a transcendental meromorphic function satisfying

$$\sigma(P(f)(f(qz + \eta) - f(z))) = s,$$

and hence

$$\sigma(H_2(z)) = \sigma\left((P(f)(f(qz + \eta) - f(z)))^{(k)}\right) = s.$$

Therefore, $H_2(z)$ can be expressed as

$$H_2(z) = h_1(z) e^{\alpha z^s} + h_2(z) e^{2\alpha z^s} + \dots + h_{n+1}(z) e^{(n+1)\alpha z^s}, \quad (4.7)$$

where each $h_i(z)$ is a differential polynomial in $U(z)$ and $V(z)$ with $\sigma(h_i(z)) < s$, for $i = 1, 2, \dots, n+1$.

We now consider two subcases depending on whether a equals some α_i .

Subcase 1.1. When $a = \alpha_1$, it follows from (4.6) and (4.7) that $H_2(z) = h_{n+1}e^{(n+1)\alpha z^s}$. Since $P(f)(f(qz + \eta) - f(z)) \not\equiv 0$, we have $h_{n+1}(z) \not\equiv 0$. Thus, when $q^s = 1$, $m = 1$, and $a = \alpha_1$, 0 is a Borel exceptional value of $H_2(z)$.

Subcase 1.2. When $a \neq \alpha_1$, we have $P(z) = a_n(z - \alpha_1)^n$. Set

$$A_i(z) = a_n \binom{n}{i-1} (a - \alpha_1)^{n-i+1} (V(z))^{i-1} (U(z) - V(z)), \quad i = 1, 2, \dots, n+1.$$

Then (4.6) can be rewritten as

$$P(f)(f(qz + \eta) - f(z)) = A_1(z)e^{\alpha z^s} + A_2(z)e^{2\alpha z^s} + \dots + A_{n+1}(z)e^{(n+1)\alpha z^s}.$$

Clearly, $A_i(z) \not\equiv 0$ and $\sigma(A_i(z)) < s$.

Since $h_i(z)e^{\alpha z^s} = (A_i(z)e^{i\alpha z^s})^{(k)}$ for $i = 1, 2, \dots, n+1$, we have $h_i(z) \not\equiv 0$ for all $i = 1, 2, \dots, n+1$. By an argument similar to that in Case 2 of Theorem 4, it follows that when $q^s = 1$, $m = 1$, and $a \neq \alpha_1$, $H_2(z)$ has no finite Borel exceptional values.

Case 2. When $q^s = 1$ and $m \neq 1$, we have

$$P(z) = a_n(z - \alpha_1)^{n_1} \dots (z - \alpha_m)^{n_m},$$

with $m \geq 2$.

From (4.5), we obtain

$$P(f)(f(qz + \eta) - f(z)) = a_n \prod_{i=1}^m \left(a - \alpha_i + V(z)e^{\alpha z^s} \right)^{n_i} (U(z) - V(z))e^{\alpha z^s}. \quad (4.8)$$

We now consider two subcases according to whether a equals some α_i .

Subcase 2.1. If there exists $i \in \{1, 2, \dots, m\}$ such that $a = \alpha_i$, without loss of generality, let $a = \alpha_1$. From (4.8), we have

$$\begin{aligned} & P(f)(f(qz + \eta) - f(z)) \\ &= a_n (V(z))^{n_1} (U(z) - V(z)) e^{(n_1+1)\alpha z^s} \prod_{i=2}^m \left(a - \alpha_i + V(z)e^{\alpha z^s} \right)^{n_i} \\ &= a_n (V(z))^{n_1} (U(z) - V(z)) e^{(n_1+1)\alpha z^s} \prod_{i=2}^m \left(a - \alpha_i + V(z)e^{\alpha z^s} \right)^{n_i} \\ &= a_n (V(z))^{n_1} (U(z) - V(z)) e^{(n_1+1)\alpha z^s} \left(\prod_{i=2}^m (a - \alpha_i)^{n_i} + \dots + (V(z))^{n-n_1} e^{(n-n_1)\alpha z^s} \right) \\ &= B_1(z)e^{(n_1+1)\alpha z^s} + B_2(z)e^{(n_1+2)\alpha z^s} + \dots + B_{n-n_1+1}(z)e^{(n+1)\alpha z^s}, \end{aligned} \quad (4.9)$$

where

$$B_1(z) = a_n \prod_{i=2}^m (a - \alpha_i)^{n_i} (V(z))^{n_1} (U(z) - V(z)),$$

$$B_{n-n_1+1}(z) = a_n (V(z))^n (U(z) - V(z)).$$

Since $V(z) \not\equiv 0$, $U(z) \not\equiv 0$, $\sigma(V(z)) < s$, $\sigma(U(z)) < s$, $\sigma(h(z)) \leq s-1$, and $U(z) \not\equiv V(z)$, we have $B_1(z) \not\equiv 0$, $B_{n-n_1+1}(z) \not\equiv 0$, and $\sigma(B_i(z)) < s$ for $i = 1, 2, \dots, n - n_1 + 1$.

Moreover, since $(n_1 + 1)\alpha \neq (n_1 + 2)\alpha \neq \dots \neq (n + 1)\alpha$, it follows from (4.9) and Lemma 2 that $P(f)(f(qz + \eta) - f(z))$ is a transcendental meromorphic function with $\sigma(P(f)(f(qz + \eta) - f(z))) = s$.

Thus,

$$\sigma(H_2(z)) = \sigma\left((P(f)(f(qz + \eta) - f(z)))^{(k)}\right) = s.$$

Therefore, $H_2(z)$ can be expressed as

$$H_2(z) = h_{t_1}(z)e^{(n_1+1)\alpha z^s} + h_{t_2}(z)e^{(n_1+2)\alpha z^s} + \dots + h_{t_{n-n_1+1}}(z)e^{(n+1)\alpha z^s},$$

where $h_{t_i}(z)$ is a differential polynomial in $U(z)$ and $V(z)$ satisfying $\sigma(h_{t_i}(z)) < s$.

Moreover, since $B_1(z) \not\equiv 0$, $B_{n-n_1+1}(z) \not\equiv 0$, and $\sigma(B_i(z)) < s$ for $i = 1, 2, \dots, n - n_1 + 1$, we have $h_{t_1}(z) \not\equiv 0$ and $h_{t_{n-n_1+1}}(z) \not\equiv 0$.

By an argument similar to that in Case 2 of Theorem 4, it follows that when $q^s = 1$, $m \neq 1$, and there exists some $i \in \{1, 2, \dots, m\}$ such that $a = \alpha_i$, $H_2(z)$ has no finite Borel exceptional values.

Subcase 2.2. If $a \neq \alpha_i$ for all $i = 1, 2, \dots, m$, it follows from (4.8) that

$$\begin{aligned} & P(f)(f(qz + \eta) - f(z)) \\ &= a_n \prod_{i=1}^m (a - \alpha_i)^{n_i} \left(\frac{g(qz + \eta)}{p(qz + \eta)} h(z) - \frac{g(z)}{p(z)} \right) e^{\alpha z^s} \\ &+ a_n \left(\sum_{i=1}^m C_{n_j}^1 (a - \alpha_j)^{n_j-1} \prod_{\substack{i=1 \\ i \neq j}}^m (a - \alpha_i)^{n_i} \right) \frac{g(z)}{p(z)} \left(\frac{g(qz + \eta)}{p(qz + \eta)} h(z) - \frac{g(z)}{p(z)} \right) e^{2\alpha z^s} \\ &+ \dots + \left(\frac{g(z)}{p(z)} \right)^n \left(\frac{g(qz + \eta)}{p(qz + \eta)} h(z) - \frac{g(z)}{p(z)} \right) e^{(n+1)\alpha z^s} \\ &= C_1(z)e^{\alpha z^s} + C_2(z)e^{2\alpha z^s} + \dots + C_{n+1}(z)e^{(n+1)\alpha z^s}, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} C_1(z) &= a_n \prod_{i=1}^m (a - \alpha_i)^{n_i} \left(\frac{g(qz + \eta)}{p(qz + \eta)} h(z) - \frac{g(z)}{p(z)} \right), \\ C_{n+1}(z) &= \left(\frac{g(z)}{p(z)} \right)^n \left(\frac{g(qz + \eta)}{p(qz + \eta)} h(z) - \frac{g(z)}{p(z)} \right). \end{aligned}$$

Since $V(z) \not\equiv 0$, $U(z) \not\equiv 0$, $\sigma(V(z)) < s$, $\sigma(U(z)) < s$, $\sigma(h(z)) \leq s-1$, and $U(z) \not\equiv V(z)$, we have $C_1(z) \not\equiv 0$, $C_{n+1}(z) \not\equiv 0$, and $\sigma(C_i(z)) < s$ for $i = 1, 2, \dots, n+1$.

Moreover, since $\alpha \neq 2\alpha \neq \dots \neq (n+1)\alpha$, it follows from (4.10) and Lemma 2 that $P(f)(f(qz + \eta) - f(z))$ is a transcendental meromorphic function with

$$\sigma(P(f)(f(qz + \eta) - f(z))) = s.$$

Thus,

$$\sigma(H_2(z)) = \sigma\left((P(f)(f(qz + \eta) - f(z)))^{(k)}\right) = s.$$

Therefore, $H_2(z)$ can be expressed as

$$H_2(z) = h_{t_1}^*(z)e^{\alpha z^s} + h_{t_2}^*(z)e^{2\alpha z^s} + \dots + h_{t_{n+1}}^*(z)e^{(n+1)\alpha z^s},$$

where $h_{t_i}^*(z)$ is a differential polynomial in $U(z)$ and $V(z)$ satisfying $\sigma(h_{t_i}^*(z)) < s$.

Since $C_1(z) \not\equiv 0$, $C_{n+1}(z) \not\equiv 0$, and $\sigma(C_i(z)) < s$ for $i = 1, 2, \dots, n+1$, we have $h_{t_1}^*(z) \not\equiv 0$ and $h_{t_{n+1}}^*(z) \not\equiv 0$.

By an argument similar to that in Case 2 of Theorem 4, it follows that when $q^s = 1$, $m \neq 1$, and $a \neq \alpha_i$ for all i , $H_2(z)$ has no finite Borel exceptional values.

Case 3. When $q^s \neq 1$ and $m = 1$, we have $P(z) = a_n(z - \alpha_1)^n$. From (4.5), we obtain

$$P(f)(f(qz + \eta) - f(z)) = a_n \left(a - \alpha_1 + V(z)e^{\alpha z^s} \right)^n \left(U(z)e^{\alpha q^s z^s} - V(z)e^{\alpha z^s} \right). \quad (4.11)$$

We now consider two subcases according to whether a equals some α_i .

Subcase 3.1. When $a = \alpha_1$, from (4.11) we have

$$\begin{aligned} P(f)(f(qz + \eta) - f(z)) &= a_n(V(z))^n \left(U(z)e^{\alpha q^s z^s} - V(z)e^{\alpha z^s} \right) e^{n\alpha z^s} \\ &= a_n(V(z))^n U(z)e^{(n+q^s)\alpha z^s} - a_n(V(z))^{n+1} e^{(n+1)\alpha z^s}. \end{aligned} \quad (4.12)$$

Since $q^s \neq 1, -1, -2, \dots, -n$, $V(z)U(z) \not\equiv 0$, $\sigma(V(z)) < s$, and $\sigma(U(z)) < s$, it follows from (4.12) and Lemma 2 that $P(f)(f(qz + \eta) - f(z))$ is a transcendental meromorphic function satisfying

$$\sigma(P(f)(f(qz + \eta) - f(z))) = s.$$

Thus,

$$\sigma(H_2(z)) = \sigma\left((P(f)(f(qz + \eta) - f(z)))^{(k)}\right) = s.$$

Therefore, $H_2(z)$ can be expressed as

$$H_2(z) = h_{v_1}(z)e^{(n+q^s)\alpha z^s} + h_{v_2}(z)e^{(n+1)\alpha z^s},$$

where each $h_{v_i}(z)$ is a differential polynomial in $U(z)$ and $V(z)$ satisfying $\sigma(h_{v_i}(z)) < s$ for $i = 1, 2$.

Clearly, $h_{v_i}(z) \not\equiv 0$ for $i = 1, 2$. By an argument similar to that in Case 2 of Theorem 4, it follows that when $q^s \neq 1$, $m = 1$, and $a = \alpha_i$, $H_2(z)$ has no finite Borel exceptional values.

Subcase 3.2. If $a \neq \alpha_i$ for all $i = 1, 2, \dots, m$, it follows from (4.11) that

$$\begin{aligned} & P(f)(f(qz + \eta) - f(z)) \\ &= a_n \left((a - \alpha_1)^n + C_n^1(a - \alpha_1)^{n-1} V(z) e^{\alpha z^s} + \dots + (V(z))^n e^{n\alpha z^s} \right) \left(U(z) e^{\alpha q^s z^s} - V(z) e^{\alpha z^s} \right) \\ &= a_n(a - \alpha_1)^n U(z) e^{\alpha q^s z^s} + a_n C_n^1(a - \alpha_1)^{n-1} V(z) U(z) e^{(1+q^s)\alpha z^s} \\ &\quad + \dots + a_n(V(z))^n U(z) e^{(n+q^s)\alpha z^s} - a_n(a - \alpha_1)^n V(z) e^{\alpha z^s} \\ &\quad - a_n C_n^1(a - \alpha_1)^{n-1} (V(z))^2 e^{2\alpha z^s} - \dots - a_n(V(z))^{n+1} e^{(n+1)\alpha z^s}. \end{aligned} \quad (4.13)$$

We now discuss two cases depending on whether $q^s - 1$ is an integer.

Subcase 3.2.1. If $q^s - 1$ is not an integer, then $i\alpha \neq (j + q^s)\alpha$ for $i = 1, 2, \dots, n+1$ and $j = 0, 1, \dots, n$. Since $V(z) \not\equiv 0$, $U(z) \not\equiv 0$, $\sigma(V(z)) < s$, $\sigma(U(z)) < s$, and $\sigma(h(z)) \leq s - 1$, it follows from (4.13) and Lemma 2 that $P(f)(f(qz + \eta) - f(z))$ is a transcendental meromorphic function satisfying

$$\sigma(P(f)(f(qz + \eta) - f(z))) = s.$$

Thus,

$$\sigma(H_2(z)) = \sigma\left((P(f)(f(qz + \eta) - f(z)))^{(k)}\right) = s.$$

Subcase 3.2.2. If $q^s - 1$ is an integer, since $q^s - 1 \neq 0, -1$, we consider the following two situations.

(a) If $q^s - 1 \geq 1$, rearranging (4.13) gives

$$P(f)(f(qz + \eta) - f(z)) = -a_n(a - \alpha_1)^n V(z) e^{\alpha z^s} + \dots + a_n(V(z))^n U(z) e^{(n+q^s)\alpha z^s}.$$

Since $\alpha \neq (n + q^s)\alpha \neq i\alpha$ for $i = 2, 3, \dots, n+1$, $\alpha \neq (n + q^s)\alpha \neq (j + q^s)\alpha$ for $j = 0, 1, \dots, n-1$, $V(z) \not\equiv 0$, $U(z) \not\equiv 0$, $\sigma(V(z)) < s$, $\sigma(U(z)) < s$, and $\sigma(h(z)) \leq s - 1$, it follows from (4.13) and Lemma 2 that $P(f)(f(qz + \eta) - f(z))$ is a transcendental meromorphic function satisfying

$$\sigma(P(f)(f(qz + \eta) - f(z))) = s.$$

Thus,

$$\sigma(H_2(z)) = \sigma\left((P(f)(f(qz + \eta) - f(z)))^{(k)}\right) = s.$$

(b) If $q^s - 1 \leq -2$, rearranging (4.13) gives

$$P(f)(f(qz + \eta) - f(z)) = a_n(a - \alpha_1)^n U(z) e^{\alpha q^s z^s} + \dots - a_n(V(z))^{n+1} e^{(n+1)\alpha z^s}.$$

Since $q^s \alpha \neq (n+1)\alpha \neq i\alpha$ for $i = 2, 3, \dots, n$, $q^s \alpha \neq (n+1)\alpha \neq (j + q^s)\alpha$ for $j = 0, 1, \dots, n$, $V(z) \not\equiv 0$, $U(z) \not\equiv 0$, $\sigma(V(z)) < s$, $\sigma(U(z)) < s$, and $\sigma(h(z)) \leq s - 1$,

it follows from (4.13) and Lemma 2 that $P(f)(f(qz + \eta) - f(z))$ is a transcendental meromorphic function satisfying

$$\sigma(P(f)(f(qz + \eta) - f(z))) = s.$$

Thus,

$$\sigma(H_2(z)) = \sigma\left((P(f)(f(qz + \eta) - f(z)))^{(k)}\right) = s.$$

Therefore, combining Subcases 3.2.1 and 3.2.2, we see that $H_2(z)$ can be expressed as

$$H_2(z) = h_{w_1}(z)e^{\gamma_1 z^s} + h_{w_2}(z)e^{\gamma_2 z^s} + \cdots + h_{w_r}(z)e^{\gamma_r z^s},$$

where $r \in \{2, 3, \dots, 2n+2\}$, and $\gamma_1, \gamma_2, \dots, \gamma_r \in \{\alpha, 2\alpha, \dots, (n+1)\alpha\} \cup \{q^s\alpha, (1+q^s)\alpha, \dots, (n+q^s)\alpha\}$ are all distinct, $h_{w_j}^*(z) \not\equiv 0$, and $\sigma(h_{w_j}^*(z)) < s$. By an argument similar to that in Case 2 of Theorem 4, it follows that when $q^s \neq 1$, $m = 1$, and $a \neq \alpha_1$, $H_2(z)$ has no finite Borel exceptional values.

Case 4. When $q^s \neq 1$ and $m \neq 1$, we have $P(z) = a_n(z - \alpha_1)^{n_1} \cdots (z - \alpha_m)^{n_m}$ with $m \geq 2$. From (4.5), we obtain

$$P(f)(f(qz + \eta) - f(z)) = a_n \prod_{i=1}^m \left(a - \alpha_i + V(z)e^{\alpha_i z^s} \right)^{n_i} \left(U(z)e^{\alpha q^s z^s} - V(z)e^{\alpha z^s} \right). \quad (4.14)$$

We now consider two subcases according to whether a equals some α_i .

Case 4.1. If there exists some $i \in \{1, 2, \dots, m\}$ such that $a = \alpha_i$, without loss of generality, assume $a = \alpha_1$. From (4.14), we have

$$\begin{aligned} & P(f)(f(qz + \eta) - f(z)) \\ &= a_n (V(z))^{n_1} \prod_{i=2}^m \left(a - \alpha_i + V(z)e^{\alpha_i z^s} \right)^{n_i} e^{n_1 \alpha z^s} \left(U(z)e^{(q^s-1)\alpha z^s} - V(z) \right) \\ &= a_n \prod_{i=2}^m (a - \alpha_i)^{n_i} (V(z))^{n_1} U(z) e^{(n_1+q^s)\alpha z^s} \\ &\quad + a_n \left(\sum_{j=2}^m \left(\prod_{\substack{i=2 \\ i \neq j}}^m (a - \alpha_i)^{n_i} \right) C_{n_j}^1(a - \alpha_j)^{n_j-1} \right) (V(z))^{n_1+1} U(z) e^{(n_1+1+q^s)\alpha z^s} \\ &\quad + \cdots + a_n (V(z))^n U(z) e^{(n+q^s)\alpha z^s} - a_n \prod_{i=2}^m (a - \alpha_i)^{n_i} (V(z))^{n_1+1} e^{(n_1+1)\alpha z^s} \\ &\quad - a_n \left(\sum_{j=2}^m \left(\prod_{\substack{i=2 \\ i \neq j}}^m (a - \alpha_i)^{n_i} \right) C_{n_j}^1(a - \alpha_j)^{n_j-1} \right) (V(z))^{n_1+2} e^{(n_1+2)\alpha z^s} \\ &\quad - \cdots - a_n (V(z))^{n+1} e^{(n+1)\alpha z^s}. \end{aligned}$$

By an argument similar to that in Case 3.2 of Theorem 5, it follows that when $q^s \neq 1$, $m \neq 1$, and there exists some $i \in \{1, 2, \dots, m\}$ such that $a = \alpha_i$, $H_2(z)$ has no finite Borel exceptional values.

Case 4.2. If $a \neq \alpha_i$ for all $i = 1, 2, \dots, m$, from (4.14) we have

$$\begin{aligned} & P(f)(f(qz + \eta) - f(z)) \\ &= a_n \prod_{i=1}^m \left(a - \alpha_i + V(z) e^{\alpha_i z^s} \right)^{n_i} \left(U(z) e^{(q^s - 1)\alpha z^s} - V(z) \right) \\ &= a_n \prod_{i=1}^m (a - \alpha_i)^{n_i} U(z) e^{q^s \alpha z^s} \\ &\quad + a_n \left(\sum_{j=1}^m \left(\prod_{\substack{i=1 \\ i \neq j}}^m (a - \alpha_i)^{n_i} \right) C_{n_j}^1 (a - \alpha_j)^{n_j - 1} \right) V(z) U(z) e^{(1+q^s)\alpha z^s} \\ &\quad + \dots + a_n (V(z))^n U(z) e^{(n+q^s)\alpha z^s} - a_n \prod_{i=1}^m (a - \alpha_i)^{n_i} V(z) e^{\alpha z^s} \\ &\quad - a_n \left(\sum_{j=1}^m \left(\prod_{\substack{i=1 \\ i \neq j}}^m (a - \alpha_i)^{n_i} \right) C_{n_j}^1 (a - \alpha_j)^{n_j - 1} \right) (V(z))^2 U(z) e^{2\alpha z^s} \\ &\quad - \dots - a_n (V(z))^{n+1} e^{(n+1)\alpha z^s}. \end{aligned}$$

By an argument similar to that in Case 3.2 of Theorem 5, it follows that when $q^s \neq 1$, $m \neq 1$, and there exists some $i \in \{1, 2, \dots, m\}$ such that $a = \alpha_i$, $H_2(z)$ has no finite Borel exceptional values.

From Cases 1 to 4, it follows that $H_2(z)$ takes every value $c \in \mathbb{C} \setminus \{0\}$ infinitely often, and satisfies $\lambda(H_2(z) - c) = \sigma(f)$.

This completes the proof of Theorem 5. $\square$

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