

# INTEGRAL EXPRESSIONS OF EXTENDED APPELL'S DOUBLE $F_1$ AND SRIVASTAVA'S TRIPLE $H_A$ HYPERGEOMETRIC FUNCTIONS WITH APPLICATIONS

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*Abstract.* Motivated by recent extensions of the Euler's beta integral form  $B(s, t)$ , Gauss's hypergeometric function  ${}_2F_1$  and confluent hypergeometric function  ${}_1F_1$  with a Macdonald function  $K_\nu(z)$  in the integrand, we extend and systematically introduce and investigate Appell's first  $F_1$  and Srivastava's triple hypergeometric function  $H_A$  and its various properties, among others numerous integral representations of Euler and Laplace type, including integral representations involving Bessel  $J_\nu(z)$  and modified Bessel functions  $I_\nu(z)$  of the first kind. Several functional bounding inequalities are derived by virtue of bounds of Luke, von Lommel, Minakshisundaram and Szász, and Olenko upon the Bessel functions appearing in the kernels of the newly established integral representations for the extended Appell's double  $F_{1,p,q,\nu}^\lambda$  and Srivastava's triple hypergeometric  $H_{A,p,q,\nu}^\lambda$  functions. Further, certain contiguous relations and recursive formulae are also presented as an applications.

## 1. Introduction and motivation

The Beta function's integral form or Euler's integral of the first kind [4]

$$B(s, t) = \int_0^1 x^{s-1} (1-x)^{t-1} dx, \quad \min\{s, t\} > 0, \quad (1)$$

for generalizations by exponential, Kummer's hypergeometric function, Mittag-Leffler class functions, Macdonald function, and other functions of a suitably defined argument  $h(x)$ , the integral form of the beta function, or Euler's integral of the first kind  $B(s, t)$ , becomes the parent function. By incorporating the kernel  $h(x)$  into a parametric integral [20]

$$B_h(s, t) = B(s, t)[h] = \int_0^1 x^{s-1} (1-x)^{t-1} h(x) dx, \quad \min\{s, t\} > 0,$$

we arrive at the so-called Beta transform of the suitable  $h(x)$ .

In fact, this Beta function transform maps a suitable input function  $h$  into a multi-parameter function [7]

$$\varphi \mapsto \int_0^1 x^{s-1} (1-x)^{t-1} \varphi(x) dx.$$

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In defining integral (2) we have  $\varphi(x) = \sqrt{h_\theta(x)} K_{\nu+\frac{1}{2}}(h_\theta(x))$ , being  $\sqrt{h_\theta(x)}$  the necessarily implemented correcting factor function (up to the multiplicative constant  $\sqrt{2/\pi}$ ), compare with the relation (4) along with constraint  $\min\{s, t\} > \frac{\lambda}{2} > 0$  follows immediately by re-writing  $\sqrt{h_\theta(x)}$  in (2) into a convenient form.

The latest articles of several authors provide a comprehensive summary of the ancestry and presentation of the Beta-transforms obtaining  $B_p$ ,  $B_{p,q}$ ,  $B_{p,\nu}$ ,  $B_{p,q;m}$ , and others. Therefore, rather than needlessly restating the established facts and generalizations, we refer to these publications and the other pertinent references therein [1, 2, 3, 9, 13, 18, 19, 20, 21, 22].

The Macdonald function (modified Bessel function of the second kind) whose definition reads [16, p. 251, Eq. 10.27.4]

$$K_\nu(z) = \frac{\pi}{2} \csc(\pi\nu) (I_{-\nu}(z) - I_\nu(z)),$$

where [16, p. 249, Eq. 10.25.2]

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n \geq 0} \frac{\left(\frac{z}{2}\right)^{2n}}{\Gamma(\nu+1+n)n!},$$

is the modified Bessel function of the first kind. Next, the extended Beta function [19] is a Beta transform integral with the Macdonald function  $K_\nu$  for the building block containing the defining kernel  $h(x)$ . The resulting integral expression turns out to be [19, p. 2, Definition 1]

$$B_{p,q,\nu}^\lambda(s, t) := B(s, t, \Theta) = \sqrt{\frac{2}{\pi}} \int_0^1 x^{s-1} (1-x)^{t-1} \sqrt{h_\theta(x)} K_{\nu+\frac{1}{2}}(h_\theta(x)) dx, \quad (2)$$

where  $\Theta = (p, q, \nu, \lambda)$  stands for the four dimensional parameter vector with the parameter space  $\lambda > 0$ ,  $\min\{\Re(p), \Re(q)\} > 0$ ;  $2\min\{s, t\} > \lambda > 0$ , whilst  $\nu \in \mathbb{R}$ . The modified argument

$$h_\theta(x) = \frac{p}{x^\lambda} + \frac{q}{(1-x)^\lambda}, \quad \theta = (p, q, \lambda), \quad (3)$$

possesses singularities and the endpoints of the unit integration interval. The modified Bessel  $I_\nu(x)$  is real for all  $\nu \in \mathbb{R}$  and  $\arg(z) = 0$  and because

$$K_{\nu+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} (1 + \mathcal{O}(z^{-1})), \quad z \rightarrow \infty, \quad (4)$$

is the reason for including the multiplication factor for  $K_{\nu+\frac{1}{2}}$  in the definition (2). In the same paper, Parmar and Pogány used generalized Beta function (2) to extend the Gaussian and the Kummer's confluent hypergeometric functions in the form [19, p. 3, Definition 2, Eq. (4)]

$$F_{p,q,\nu}^\lambda(a, b; c; z) := {}_2F_1(a, b; c; z, \Theta) = \sum_{k \geq 0} (a)_k \frac{B_{p,q,\nu}^\lambda(b+k, c-b)}{B(b, c-b)} \frac{z^k}{k!}, \quad (5)$$

in which the parameters  $\lambda > 0$ ,  $\min\{p, q\} \geq 0$ ;  $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$ ,  $\Re(c) > \Re(b) > 0$  and  $|z| < 1$ ; also the extension of the Kummer function reads [19, p. 3, Definition 2, Eq. (5)]

$$\Phi_{p,q,v}^\lambda(b; c; z) := {}_1F_1(b; c; z, \Theta) = \sum_{k \geq 0} \frac{B_{p,q,v}^\lambda(b+k, c-b)}{B(b, c-b)} \frac{z^k}{k!}, \quad (6)$$

where  $\lambda > 0$ ,  $\min\{p, q\} \geq 0$ ;  $\Re(c) > \Re(b) > 0$ . They establish their functional bounding inequalities, Turán inequalities and studied their raw moments and moment inequalities by defining a novel probability Beta distribution. Specifying the values of parameters  $p, q, \lambda$  and  $v$ , the generalized Beta function  $B_{p,q,v}^\lambda(s, t)$  (2) covers a whole spectrum of various well-known form of extended Beta functions.

Setting the values of the parameters  $p, q, v, \lambda$  in (2) we get various known and frequently studied members of the Beta functions' family. So, when  $\lambda = 1$  and  $q = p$  we arrive at the so-called  $(p, v)$ -extended Beta function introduced by Parmar *et al.* [17, p. 93, Eq. (13)]:

$$B_{p,v}(s, t) = \sqrt{\frac{2p}{\pi}} \int_0^1 x^{s-\frac{3}{2}} (1-x)^{t-\frac{3}{2}} K_{v+\frac{1}{2}} \left( \frac{p}{x(1-x)} \right) dx,$$

where  $\Re(p) \geq 0$ ;  $2 \min\{\Re(s), \Re(t)\} > 1$  and  $\sqrt{p}$  takes its principal value. This type of Beta function is recently considered by Milovanović *et al.* in [13] for establishing Gautschi–Pinelis type upper bounds for the Macdonald function and the  $(p, v)$ -extended Beta function. For complete details of numerous other special cases, we refer to the recent articles [20, 21]. Finally, if use the fact (4) and set  $v = 0$  and  $p, q \searrow 0$ , then  $B_{p,q,v}^\lambda(s, t)$  approaches  $B(s, t)$  giving Euler's integral (1).

The research conduction in this paper will be carried out through the following plans:

1. introducing the four parametric  $\Theta = (p, q, v, \lambda)$ -extension of Appell's double  $F_1[x, y, \Theta]$  and Srivastava's triple hypergeometric  $H_A[x, y, z, \Theta]$  functions by considering the definition of extended Beta function  $B_{p,q,v}^\lambda(s, t) := B(s, t, \Theta)$  in (2) involving the Macdonald kernel  $K_{n+\frac{1}{2}}$ ;

2. the related integral representations, such as Euler's and Laplace-Mellin type, as well as some integral representations involving Bessel  $J_\nu(z)$  and Modified Bessel functions  $I_\nu(z)$  are established;

3. a number of functional upper bounds are derived either by series representations or integral representations for extended Appell's double  $F_{1,p,q,v}^\lambda$  and Srivastava's triple hypergeometric  $H_{A,p,q,v}^\lambda$  functions by using Luke's, von Lommel's, Minakshisundaram and Szász and Olenko bounds given in the literature;

4. in the final section some conclusions and observations are mentioned with applications to certain contiguous relations and recursive formulae.

## 2. Extended Appell's $F_{1,p,q,v}^\lambda[x,y]$ and Srivastava's $H_{A,p,q,v}^\lambda[x,y,z]$ functions

In this section, we first introduce generalized extended first Appell's hypergeometric function of two variables  $F_{1,p,q,v}^\lambda[x,y]$  and extended Srivastava's triple hypergeometric function  $H_{A,p,q,v}^\lambda[x,y,z]$ . Next, we derive their various integral representations of Euler's and Laplace–Mellin type, as well as integral representations involving Bessel  $J_\nu(z)$  and modified Bessel functions  $I_\nu(z)$  of the first kind.

In terms of the extended beta function  $B_{p,q,v}^\lambda(s,t)$  defined by (2) we introduce the generalized extended first Appell function for  $\alpha, \beta, \beta' \in \mathbb{C}$  and  $\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$  by the following double power series

$$F_{1,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma; x, y] = \sum_{m,n \geq 0} (\beta)_m (\beta')_n \frac{B_{p,q,v}^\lambda(\alpha + m + n, \gamma - \alpha)}{B(\alpha, \gamma - \alpha)} \frac{x^m}{m!} \frac{y^n}{n!}, \quad (7)$$

provided  $\lambda > 0$ ,  $\min\{\Re(p), \Re(q)\} > 0$ ;  $\Re(\gamma) > \Re(\alpha) > 0$ ,  $\max\{|x|, |y|\} < 1$  and  $v \in \mathbb{R}$ .

The special case  $F_{1,0,0,0}^1[x,y] = F_1[x,y]$  reduces to first Appell's hypergeometric function of two variables  $F_1[x,y]$  [5, 26]

$$\begin{aligned} F_1[\alpha, \beta, \beta'; \gamma; x, y] &= \sum_{m,n \geq 0} (\beta)_m (\beta')_n \frac{B(\alpha + m + n, \gamma - \alpha)}{B(\alpha, \gamma - \alpha)} \frac{x^m}{m!} \frac{y^n}{n!} \\ &= \sum_{m,n \geq 0} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad \max\{|x|, |y|\} < 1. \end{aligned}$$

Next, we introduce in terms of the extended beta function  $B_{p,q,v}^\lambda(s,t)$  defined by (2) the extended Srivastava's triple hypergeometric function for all  $\alpha, \beta, \beta' \in \mathbb{C}$  and  $\gamma, \gamma' \in \mathbb{C} \setminus \mathbb{Z}_0^-$  as

$$\begin{aligned} H_{A,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] \\ = \sum_{k,m,n \geq 0} \frac{(\alpha)_{k+n} (\beta)_{k+m}}{(\gamma)_k} \frac{B_{p,q,v}^\lambda(\beta' + m + n, \gamma' - \beta')}{B(\beta', \gamma' - \beta')} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!}, \end{aligned} \quad (8)$$

where  $\lambda > 0$ ,  $\min\{\Re(p), \Re(q)\} > 0$ ;  $\Re(\gamma') > \Re(\beta') > 0$ ,  $|x| < r$ ,  $|y| < s$ ,  $|z| < t$ ;  $r = (1-s)(1-t)$  and  $v \in \mathbb{R}$ , which special case  $H_{A,0,0,0}^1 = H_A$  reduces to Srivastava's triple hypergeometric function  $H_A$ , viz. [23, 24, 25]

$$\begin{aligned} H_A[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] &= \sum_{k,m,n \geq 0} \frac{(\alpha)_{k+n} (\beta)_{k+m}}{(\gamma)_k} \frac{B(\beta' + m + n, \gamma' - \beta')}{B(\beta', \gamma' - \beta')} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!} \\ &= \sum_{k,m,n \geq 0} \frac{(\alpha)_{k+n} (\beta)_{k+m} (\beta')_{m+n}}{(\gamma)_k (\gamma')_{m+n}} \frac{x^k}{k!} \frac{y^m}{m!} \frac{z^n}{n!}, \end{aligned} \quad (9)$$

where  $|x| < r$ ,  $|y| < s$ ,  $|z| < t$ ;  $r = (1-s)(1-t)$ , compare also [26, p. 43, Eq. (11)].

Now, we establish various categories of results upon the newly defined special functions  $F_{1,p,q,v}^\lambda$  and  $H_{A,p,q,v}^\lambda$  which are integral representations, a set of bounding inequalities via series and integral representations and we close the presentation with the contiguous relations, recursive formulae.

## 2.1. Integral representations

We begin the main results exposition with presenting a set of integral representations of Euler and Laplace type for  $F_{1,p,q,v}^\lambda$  and  $H_{A,p,q,v}^\lambda$ . We also obtain certain integral representations involving Bessel and modified Bessel functions. Firstly, we establish an integral representation for Appell's function.

**THEOREM 1.** *For all  $\lambda > 0$ ,  $\min\{\Re(p), \Re(q)\} > 0$ ,  $v \in \mathbb{R}$  we have*

$$F_{1,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma; x, y] = \sqrt{\frac{2}{\pi}} \int_0^1 \frac{t^{\alpha-1} (1-t)^{\gamma-\alpha-1} \sqrt{h_\theta(t)}}{B(\alpha, \gamma-\alpha) (1-xt)^\beta (1-yt)^{\beta'}} K_{v+\frac{1}{2}}(h_\theta(t)) dt, \quad (10)$$

provided  $\max\{|\arg(1-x)|, |\arg(1-y)|\} < \pi$ ;  $\Re(\gamma) > \Re(\alpha) > 0$ .

*Proof.* Consider right-hand-side integral (10) and assume that  $\max\{|x|, |y|\} < 1$ . Then, upon expanding both binomials  $(1-xt)^{-\beta}$ ,  $(1-yt)^{-\beta'}$  in the integrand into Maclaurin series, inverting the order of summation and integration (which can be justified by absolute and uniform convergence) we find that for all  $\Re(\gamma) > \Re(\alpha) > 0$ :

$$\Lambda(x, y, \Theta) = \frac{\sqrt{\frac{2}{\pi}}}{B(\alpha, \gamma-\alpha)} \sum_{m,n \geq 0} (\beta)_m (\beta')_n \frac{x^m}{m!} \frac{y^n}{n!} \cdot \int_0^1 t^{\alpha+m+n-1} (1-t)^{\gamma-\alpha-1} \sqrt{h_\theta(t)} K_{v+\frac{1}{2}}(h_\theta(t)) dt,$$

which, in view of the definitions (2) and (7) yields (10).  $\square$

**THEOREM 2.** *For all  $\lambda > 0$ ,  $\min\{\Re(p), \Re(q)\} > 0$ ,  $v \in \mathbb{R}$  and for all  $\Re(\gamma') > \Re(\beta') > 0$ , we have*

$$H_{A,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] = \frac{\sqrt{\frac{2}{\pi}}}{B(\beta', \gamma' - \beta')} \int_0^1 \frac{v^{\beta'-1} (1-v)^{\gamma'-\beta'-1} \sqrt{h_\theta(v)}}{(1-vy)^\beta (1-vz)^\alpha} \cdot {}_2F_1\left(\alpha, \beta; \gamma; \frac{x}{(1-vy)(1-vz)}\right) K_{v+\frac{1}{2}}(h_\theta(v)) dv. \quad (11)$$

provided  $|y| < 1$ ,  $|z| < 1$ ,  $|x| < |(1-y)(1-z)| < 1$ .

*Proof.* The definition of extended Srivastava's triple hypergeometric functions (8) can be expressed as a single series of the extended Appell function (7) as:

$$H_{A,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] = \sum_{k \geq 0} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} F_{1,p,q,v}^\lambda[\beta', \beta+k, \alpha+k; \gamma'; y, z] \frac{x^k}{k!}. \quad (12)$$

Application of the integral representation (10) in (12) yields

$$H_{A,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] = \frac{\sqrt{2/\pi}}{B(\beta', \gamma' - \beta')} \sum_{k \geq 0} \int_0^1 v^{\beta'-1} (1-v)^{\gamma'-\beta'-1} (1-vy)^{-\beta} \cdot (1-vz)^{-\alpha} \sqrt{h_{\theta}(v)} K_{v+\frac{1}{2}}(h_{\theta}(v)) \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} \left\{ \frac{x}{(1-vy)(1-vz)} \right\}^k dv.$$

Changing the order of summation and integration we obtain the formula (11).  $\square$

Now, we expose two equivalent double integral expressions related to  $H_{A,p,q,v}^{\lambda}$ .

**THEOREM 3.** *For all  $\lambda > 0$ ,  $\min\{\Re(p), \Re(q)\} > 0$ ,  $v \in \mathbb{R}$  and for all  $\Re(\gamma) > \Re(\beta) > 0$ ,  $\Re(\gamma') > \Re(\beta') > 0$ , we have*

$$H_{A,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] = \frac{\sqrt{2/\pi}}{B(\beta, \gamma - \beta) B(\beta', \gamma' - \beta')} \int_0^1 \int_0^1 u^{\beta-1} v^{\beta'-1} \cdot \frac{(1-u)^{\gamma-\beta-1} (1-v)^{\gamma'-\beta'-1} (1-vy)^{\alpha-\beta}}{(1-ux-vy-vz+v^2yz)^{\alpha}} \sqrt{h_{\theta}(v)} K_{v+\frac{1}{2}}(h_{\theta}(v)) du dv. \quad (13)$$

Moreover, for the same parameter space there holds

$$H_{A,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] = \frac{\sqrt{2/\pi}}{B(\beta, \gamma - \beta) B(\beta', \gamma' - \beta')} \int_0^1 \int_0^1 u^{\beta-1} v^{\beta'-1} \cdot \frac{(1-u)^{\gamma-\beta-1} (1-v)^{\gamma'-\beta'-1} (1-vy)^{-\beta}}{(1-ux-vz)^{\alpha}} \left( 1 - \frac{uvxy}{(1-vy)(1-ux-vz)} \right)^{-\alpha} \cdot \sqrt{h_{\theta}(v)} K_{v+\frac{1}{2}}(h_{\theta}(v)) du dv. \quad (14)$$

*Proof.* Using the integral representation [16, p. 388, Eq. 15.6.1]

$${}_2F_1(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt,$$

when there holds  $\Re(c) > \Re(b) > 0$ ;  $|\arg(1-z)| \leq \pi - \varepsilon$ ,  $0 < \varepsilon < \pi$  in (11), we are led to the asserted integral (13). Finally, after some algebraic manipulations in (13), we arrive at the last statement (14).  $\square$

**REMARK 1.** If  $p = q = 0$ , then (14) follows under the identical conditions upon the other involved parameters, that is, when  $\Re(\gamma) > \Re(\beta) > 0$  and  $\Re(\gamma') > \Re(\beta') > 0$ . Indeed, the case  $v = 0$ , and  $p = q = 0$  holds asymptotically because then the kernel (3)  $h_{\theta}(z) \equiv 0$  controls the asymptotic behaviour of  $K_{v+\frac{1}{2}}(z)$  when  $z \rightarrow 0+$ .

**THEOREM 4.** *Let  $\lambda > 0$ ,  $v \in \mathbb{R}$ ,  $\min\{\Re(p), \Re(q)\} > 0$  and  $x > 0$ ,  $\max\{\Re(y), \Re(z)\} < 1$  and  $\min\{\Re(\alpha), \Re(\beta)\} > 0$ , we have*

$$H_{A,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\infty} \int_0^{\infty} e^{-s-t} t^{\alpha-1} s^{\beta-1} {}_0F_1(-; \gamma; xst) \cdot \Phi_{p,q,v}^{\lambda}(\beta'; \gamma'; ys+zt) ds dt. \quad (15)$$

*Proof.* Using the integral form of the Pochhammer symbols  $(\alpha)_{k+n}$  and  $(\beta)_{k+m}$  and the elementary series identity [26, p. 52, Eq. 1.6(2)]:

$$\sum_{m_1, m_2 \geq 0} \Omega(m_1 + m_2) \frac{x_1^{m_1} x_2^{m_2}}{m_1! m_2!} = \sum_{m \geq 0} \Omega(m) \frac{(x_1 + x_2)^m}{m!},$$

in (8) and afterward applying definition of the extended confluent hypergeometric function (6), we complete the proof.  $\square$

The Bessel function  $J_\nu(z)$  and the modified Bessel function  $I_\nu(z)$  of the first kind of the order  $\nu$  are expressible in terms of hypergeometric functions  ${}_0F_1$  as follows [16, p. 228, Entries 10.16.9, 10.39.9] (see e.g. [29])

$${}_0F_1\left(-; \nu + 1; -\frac{z^2}{4}\right) = \Gamma(\nu + 1) \left(\frac{2}{z}\right)^\nu J_\nu(z), \quad (16)$$

$${}_0F_1\left(-; \nu + 1; \frac{z^2}{4}\right) = \Gamma(\nu + 1) \left(\frac{2}{z}\right)^\nu I_\nu(z), \quad (17)$$

where  $-\nu \notin \mathbb{N}$  in both cases. Now, using (16) and (17) in (15), we yield integral expressions for extended Srivastava's triple hypergeometric functions  $H_{A,p,q,\nu}^\lambda$ .

**COROLLARY 1.** *Let  $\lambda > 0$ ,  $\nu \in \mathbb{R}$ ,  $\min\{\Re(p), \Re(q)\} > 0$  and  $x > 0$ ,  $\max\{\Re(y), \Re(z)\} < 1$ ,  $\Re(\gamma) > 1$  and  $\min\{\Re(\alpha), \Re(\beta)\} > 0$ , we have*

$$\begin{aligned} H_{A,p,q,\nu}^\lambda[\alpha, \beta, \beta'; \gamma, \gamma'; \mp x, y, z] &= \frac{\Gamma(\gamma) x^{\frac{1-\gamma}{2}}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty e^{-s-t} t^{\alpha-\frac{\gamma-1}{2}-1} s^{\beta-\frac{\gamma-1}{2}-1} \\ &\quad \times \left\{ \frac{J_{\gamma-1}(2\sqrt{xst})}{I_{\gamma-1}(2\sqrt{xst})} \right\} \Phi_{p,q,\nu}^\lambda(\beta'; \gamma'; ys + zt) ds dt. \quad (18) \end{aligned}$$

### 3. Functional bounds for $F_{1,p,q,\nu}^\lambda[x, y]$ and $H_{A,p,q,\nu}^\lambda[x, y, z]$ functions

This section explores bounding inequalities for the extended Appell's double hypergeometric function  $F_{1,p,q,\nu}^\lambda[x, y]$  and Srivastava's triple hypergeometric  $H_{A,p,q,\nu}^\lambda[x, y, z]$ . Our main aim in this section is to find bounding inequalities for the above considered cases of these function.

#### 3.1. Functional bounds established via series representations

In this introductory part of this subsection we present sharp estimates for the extended Beta, hypergeometric and confluent hypergeometric functions  $B_{p,q,\nu}^\lambda$ ,  $F_{p,q,\nu}^\lambda$  and  $\Phi_{p,q,\nu}^\lambda$ .

The first auxiliary lemma describes an estimate for  $B_{p,q,\nu}^\lambda$  defined in (2).

LEMMA 1. [19, p. 5, Theorem 2, Eq. (8)] Let  $p, q > 0$ ,  $\lambda \in (0, 1) \cup (1, \infty)$ ,  $v \in \mathbb{R}$ . Then for all  $2 \min\{s, t\} > \lambda > 0$  we have

$$\begin{aligned} |B_{p,q,v}^\lambda(s, t)| &\leq \frac{\sqrt{2pq} K_{v+\frac{1}{2}} \left( (p^{\frac{1}{\lambda+1}} + q^{\frac{1}{\lambda+1}})^{\lambda+1} \right)}{\sqrt{\pi} (p^{\frac{1}{\lambda-1}} + q^{\frac{1}{\lambda-1}})^{\frac{\lambda-1}{2}}} B(s - \frac{\lambda}{2}, t - \frac{\lambda}{2}) \\ &=: \Omega_v^\lambda(p, q) B(s - \frac{\lambda}{2}, t - \frac{\lambda}{2}), \end{aligned} \quad (19)$$

where

$$\Omega_v^\lambda(p, q) = \frac{\sqrt{2pq} K_{v+\frac{1}{2}} \left( (p^{\frac{1}{\lambda+1}} + q^{\frac{1}{\lambda+1}})^{\lambda+1} \right)}{\sqrt{\pi} (p^{\frac{1}{\lambda-1}} + q^{\frac{1}{\lambda-1}})^{\frac{\lambda-1}{2}}}.$$

The second auxiliary lemma gives an estimate for  $F_{p,q,v}^\lambda$  and  $\Phi_{p,q,v}^\lambda$  presented in (5) and (6)

LEMMA 2. [21, p. 7, Theorem 3.2] For all  $p, q > 0$  and  $\lambda \in (0, 1) \cup (1, \infty)$ ,  $v \in \mathbb{R}$  and  $\Re(t) > \Re(s) > 0$ , we have

$$\begin{aligned} |F_{p,q,v}^\lambda(a, s; t; z)| &\leq \Omega_v^\lambda(p, q) \frac{B(s - \frac{\lambda}{2}, t - s - \frac{\lambda}{2})}{B(s, t - s)} {}_2F_1(a, s - \frac{\lambda}{2}; t - \lambda; |z|), \\ |\Phi_{p,q,v}^\lambda(s; t; z)| &\leq \Omega_v^\lambda(p, q) \frac{B(s - \frac{\lambda}{2}, t - s - \frac{\lambda}{2})}{B(s, t - s)} \Phi(s - \frac{\lambda}{2}; t - \lambda; |z|), \end{aligned} \quad (20)$$

provided  $2 \min\{s, t - s\} > \lambda$  and  ${}_2F_1$  and  $\Phi$  denote the Gauss hypergeometric function and the Kummer confluent hypergeometric function, respectively.

The following theorem provides bounding inequalities for extended Appell's  $F_{1,p,q,v}^\lambda$  and extended Srivastava's triple hypergeometric  $H_{A,p,q,v}^\lambda$ . We present now the inferred results concerning these functional bounds.

THEOREM 5. Assume that  $p, q > 0$  and  $\lambda \in (0, 1) \cup (1, \infty)$ ,  $v \in \mathbb{R}$  and the constraints

$$\min\{\alpha, \gamma - \alpha\} > \frac{\lambda}{2}; \min\{\beta', \gamma' - \beta'\} > \frac{\lambda}{2}.$$

Then we have the following functional bounds for  $\max\{|\arg(1-x)|, |\arg(1-y)|\} < \pi$ ;  $\Re(\gamma) > \Re(\alpha) > 0$ ,  $\Re(\beta) > 0$ , it follows

$$\begin{aligned} |F_{1,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma; x, y]| &\leq \Omega_v^\lambda(p, q) \frac{B(\alpha - \frac{\lambda}{2}, \gamma - \alpha - \frac{\lambda}{2})}{B(\alpha, \gamma - \alpha)} \\ &\quad F_1(\alpha - \frac{\lambda}{2}, \beta, \beta'; \gamma - \lambda; |x|, |y|); \end{aligned} \quad (21)$$

while for  $|x| < r$ ,  $|y| < s$ ,  $|z| < t$  and  $r = (1-s)(1-t)$ , there holds

$$\begin{aligned} |H_{A,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z]| &\leq \Omega_v^\lambda(p, q) \frac{B(\beta' - \frac{\lambda}{2}, \gamma' - \beta' - \frac{\lambda}{2})}{B(\beta', \gamma' - \beta')} \\ &\quad \times H_A[\alpha, \beta, \beta' - \frac{\lambda}{2}; \gamma, \gamma' - \lambda; |x|, |y|, |z|]. \end{aligned} \quad (22)$$

*Proof.* Applying the bound of  $B_{p,q,v}^\lambda(s,t)$  in Lemma 1 to the extended Appell's function (7) and simplification, we get

$$\begin{aligned} \left| F_{1,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma; x, y] \right| &= \left| \sum_{m,n \geq 0} (\beta)_m (\beta')_n \frac{B_{p,q,v}^\lambda(\alpha + m + n, \gamma - \alpha)}{B(\alpha, \gamma - \alpha)} \frac{x^m}{m!} \frac{y^n}{n!} \right| \\ &\leq \Omega_v^\lambda(p, q) \sum_{m,n \geq 0} (\beta)_m (\beta')_n \frac{B(\alpha + m + n - \frac{\lambda}{2}, \gamma - \alpha - \frac{\lambda}{2})}{B(\alpha, \gamma - \alpha)} \frac{|x|^m}{m!} \frac{|y|^n}{n!} \\ &= \Omega_v^\lambda(p, q) \frac{B(\alpha - \frac{\lambda}{2}, \gamma - \alpha - \frac{\lambda}{2})}{B(\alpha, \gamma - \alpha)} F_1[\alpha - \frac{\lambda}{2}, \beta, \beta'; \gamma - \lambda; |x|, |y|]. \end{aligned}$$

This proves the inequality (21). Moreover, by similar arguments as in this proof verify (22). The details are omitted here.  $\square$

### 3.2. Bounds obtained via integral representations

In the previous section simple integral representation (Theorem 1 and Theorem 2 in terms of  ${}_2F_1$ ), two double-integral representations (Theorem 3), while two double improper integral representations (Theorem 4 and Corollary 1 in terms of Bessel function of the first kind  $J_v$  and with the modified Bessel function of the first kind  $I_v$  respectively) have been obtained. Since the integrands of the integral expressions consist either  $\varphi(x) = \sqrt{h_\theta(x)} K_{v+\frac{1}{2}}(h_\theta(x))$  extension or extended Kummer's  $\Phi_{p,q,v}^\lambda$  together with the modified Bessel functions (Corollary 1), we need certain auxiliary tools in bounding these special functions including Lemma 1 and Lemma 2.

- For  $\beta_j \geq \alpha_j > 0$ ,  $j = \overline{1, r}$  and  $t \geq 0$ , there exist Luke's bilateral functional inequalities for the generalized hypergeometric function  ${}_rF_r$  [12, Theorem 16, Eq. (5.6)]

$$e^{\omega t} \leq {}_rF_r(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r; t) \leq 1 - \omega(1 - e^t),$$

where

$$\omega = \frac{\max_{1 \leq j \leq r} \alpha_j}{\min_{1 \leq j \leq r} \beta_j},$$

and the equality holds for  $t = 0$ . In the case when  $r = 1$  we have the Kummer confluent hypergeometric function  $\Phi(\alpha; \beta; t) = {}_1F_1(\alpha; \beta; t)$  in that case when  $\beta \geq \alpha > 0$  we have

$$e^{\frac{\alpha}{\beta}t} \leq \Phi(\alpha; \beta; t) \leq 1 - \frac{\alpha}{\beta}(1 - e^t), \quad t \geq 0. \quad (23)$$

- The set of bounding inequalities for  $J_v$  and  $I_v$  read as

(i) von Lommel's bounds [29, pp. 31 and 406], [10], [11, pp. 548–549]

$$|J_v(t)| \leq 1, \quad |J_{v+1}(t)| \leq \frac{1}{\sqrt{2}}, \quad v \in \mathbb{R}_+, \quad t \in \mathbb{R}; \quad (24)$$

- (ii) Minakshisundaram and Szász bound [6, Eq. (1.8)], [14, pp. 36–37]; cf. [29, p. 16]

$$|J_\nu(t)| \leq \frac{1}{\Gamma(\nu+1)} \left( \frac{|t|}{2} \right)^\nu, \quad \nu \geq 0, \quad t \in \mathbb{R}; \quad (25)$$

- (iii) For  $\nu \geq 0$  and  $t \in \mathbb{R}$  there are the bounds by Landau [8]

$$|J_\nu(t)| \leq b_L \nu^{-1/3}, \quad b_L = \sqrt[3]{2} \sup_{t \geq 0} \text{Ai}(t), \quad (26)$$

$$|J_\nu(t)| \leq c_L |t|^{-1/3}, \quad c_L = \sup_{t \geq 0} t^{1/3} J_0(t), \quad (27)$$

where  $\text{Ai}(\cdot)$  stands for the Airy function

$$\text{Ai}(-t) = \frac{\pi}{2} \sqrt{\frac{t}{3}} \left[ J_{1/3} \left\{ 2 \left( \frac{t}{3} \right)^{3/2} \right\} + J_{-1/3} \left\{ 2 \left( \frac{t}{3} \right)^{3/2} \right\} \right]. \quad (28)$$

- (iv) Olenko's bound [15, Theorem 2.1]

$$\sup_{t \geq 0} \sqrt{t} |J_\nu(t)| \leq b_L \sqrt{\nu^{1/3} + \frac{\tau_1}{\nu^{1/3}} + \frac{3\tau_1^2}{10\nu}} =: d_O, \quad \nu > 0, \quad (29)$$

where  $\tau_1$  is the smallest positive zero of the Airy-function  $\text{Ai}$  in (28) and  $b_L$  is the Landau's constant in (26). This bound is asymptotically precise and the constant  $b_L$  is the best possible.

- (v) Luke [12, p. 55, Eq. (6.25)] obtained the following result

$$I_\nu(t) < \frac{\left(\frac{t}{2}\right)^\nu}{\Gamma(\nu+1)} \cosh t, \quad t > 0, \quad \nu > -\frac{1}{2}. \quad (30)$$

Now we are ready to state and prove our second set of bounding inequality results.

**THEOREM 6.** Assume  $\min\{\Re(p), \Re(q)\} > 0$  and in the case  $p = q = 0$  there holds  $\Re(y) > 0$ ,  $\Re(z) > 0$ ,  $\min\{\Re(\alpha), \Re(\beta)\} > 0$ . Then under  $2\min\{\alpha, \beta\} + 1 > 0$ ,  $\gamma \geq 2$  and  $\min\{\beta', \gamma' - \beta'\} > \frac{\lambda}{2}$ , we have

$$\begin{aligned} & \left| H_{A,p,q,\nu}^\lambda[\alpha, \beta, \beta'; \gamma, \gamma'; -x, y, z] \right| \\ & \leq \Omega_\nu^\lambda(p, q) \frac{\text{B}(\beta' - \frac{\lambda}{2}, \gamma' - \beta' - \frac{\lambda}{2})}{\sqrt{2} \text{B}(\beta', \gamma' - \beta')} \frac{\Gamma(\gamma) \Gamma(\alpha - \frac{\gamma-1}{2}) \Gamma(\beta - \frac{\gamma-1}{2})}{\Gamma(\alpha) \Gamma(\beta) |x|^{\frac{\gamma-1}{2}}} \\ & \quad \times \left\{ 1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} \left( 1 - (1-y)^{-\beta + \frac{\gamma-1}{2}} (1-z)^{-\alpha + \frac{\gamma-1}{2}} \right) \right\}. \quad (31) \end{aligned}$$

In the same parameter range for all  $x > 0$  there holds true

$$\begin{aligned} & \left| H_{A,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma, \gamma'; -x, y, z] \right| \\ & \leq \Omega_v^{\lambda}(p, q) \frac{B(\beta' - \frac{\lambda}{2}, \gamma' - \beta' - \frac{\lambda}{2})}{B(\beta', \gamma' - \beta')} \frac{\Gamma(\gamma) \Gamma(\alpha - \frac{\gamma-1}{2}) \Gamma(\beta - \frac{\gamma-1}{2}) b_L}{\sqrt[3]{\gamma-1} \Gamma(\alpha) \Gamma(\beta) |x|^{\frac{\gamma-1}{2}}} \\ & \quad \times \left\{ 1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} \left( 1 - (1-y)^{-\beta + \frac{\gamma-1}{2}} (1-z)^{-\alpha + \frac{\gamma-1}{2}} \right) \right\}. \quad (32) \end{aligned}$$

Next, for all  $\min\{\alpha, \beta, \gamma\} > 0$  and for  $x > 0$ ;  $y, z < 1$  we have

$$\begin{aligned} & \left| H_{A,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma, \gamma'; -x, y, z] \right| \leq \Omega_v^{\lambda}(p, q) \frac{B(\beta' - \frac{\lambda}{2}, \gamma' - \beta' - \frac{\lambda}{2})}{B(\beta', \gamma' - \beta')} \\ & \quad \times \left\{ 1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} \left( 1 - (1-y)^{-\beta} (1-z)^{-\alpha} \right) \right\}. \quad (33) \end{aligned}$$

When  $\min\{\alpha, \beta\} > 0$ ;  $\gamma \geq 2$  and for  $x > 0$ ;  $y, z < 1$  it follows

$$\begin{aligned} & \left| H_{A,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma, \gamma'; -x, y, z] \right| \leq \Omega_v^{\lambda}(p, q) \frac{B(\beta' - \frac{\lambda}{2}, \gamma' - \beta' - \frac{\lambda}{2})}{B(\beta', \gamma' - \beta')} \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} \\ & \quad \times \left\{ \begin{aligned} & \frac{c_L x^{-\frac{\gamma}{2} + \frac{1}{3}}}{\sqrt[3]{2}} \Gamma\left(\alpha - \frac{\gamma}{2} + \frac{1}{3}\right) \Gamma\left(\beta - \frac{\gamma}{2} + \frac{1}{3}\right) \left( 1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} + \frac{\frac{2\beta' - \lambda}{2(\gamma' - \lambda)}}{(1-y)^{\beta - \frac{\gamma}{2} + \frac{1}{3}} (1-z)^{\alpha - \frac{\gamma}{2} + \frac{1}{3}}} \right) \\ & \frac{d_O x^{-\frac{\gamma}{2} + \frac{1}{4}}}{\sqrt{2}} \Gamma\left(\alpha - \frac{\gamma}{2} + \frac{1}{4}\right) \Gamma\left(\beta - \frac{\gamma}{2} + \frac{1}{4}\right) \left( 1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} + \frac{\frac{2\beta' - \lambda}{2(\gamma' - \lambda)}}{(1-y)^{\beta - \frac{\gamma}{2} + \frac{1}{4}} (1-z)^{\alpha - \frac{\gamma}{2} + \frac{1}{4}}} \right) \end{aligned} \right\}; \end{aligned}$$

here the bound above holds if  $6\alpha - 3\gamma + 2 > 0$ , while the expression below appears for  $4\alpha - 2\gamma + 1 > 0$ .

Moreover, for all  $\min\{\alpha, \beta, \gamma\} > 0$  and for all  $y, z \in (0, 1 - \sqrt{x})$ ,  $0 < x < 1$  there holds

$$\begin{aligned} & \left| H_{A,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] \right| \leq \Omega_v^{\lambda}(p, q) \frac{B(\beta' - \frac{\lambda}{2}, \gamma' - \beta' - \frac{\lambda}{2})}{B(\beta', \gamma' - \beta')} \\ & \quad \times \left\{ \frac{1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)}}{(1 - \sqrt{x})^{\alpha + \beta}} + \frac{\frac{2\beta' - \lambda}{2(\gamma' - \lambda)}}{(1 - \sqrt{x} - y)^{\beta} (1 - \sqrt{x} - z)^{\alpha}} \right\}. \quad (34) \end{aligned}$$

*Proof.* Considering the double integral representation (18), we conclude by the second estimate in Lemma 2, we obtain

$$\begin{aligned}
 |H_{\lambda,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma, \gamma'; -x, y, z]| &\leq \Omega_v^{\lambda}(p, q) \frac{\Gamma(\gamma) \mathbf{B}(\beta' - \frac{\lambda}{2}, \gamma' - \beta' - \frac{\lambda}{2})}{\Gamma(\alpha) \Gamma(\beta) \mathbf{B}(\beta', \gamma' - \beta')} |x|^{\frac{1-\gamma}{2}} \\
 &\times \int_0^{\infty} \int_0^{\infty} e^{-s-t} t^{\alpha - \frac{\gamma-1}{2} - 1} s^{\beta - \frac{\gamma-1}{2} - 1} \left\{ \frac{|J_{\gamma-1}(2\sqrt{xst})|}{I_{\gamma-1}(2\sqrt{xst})} \right\} \\
 &\times \Phi(\beta' - \frac{\lambda}{2}; \gamma' - \lambda; ys + zt) ds dt \\
 &= \Omega_v^{\lambda}(p, q) \frac{\Gamma(\gamma) \mathbf{B}(\beta' - \frac{\lambda}{2}, \gamma' - \beta' - \frac{\lambda}{2})}{\Gamma(\alpha) \Gamma(\beta) \mathbf{B}(\beta', \gamma' - \beta')} |x|^{\frac{1-\gamma}{2}} =: \mathfrak{J}_1. \quad (35)
 \end{aligned}$$

Next, Luke's upper bound (23) transforms again the integral, therefore we arrive at the following important estimate

$$\begin{aligned}
 \mathfrak{J}_1 &\leq \int_0^{\infty} \int_0^{\infty} e^{-s-t} t^{\alpha - \frac{\gamma-1}{2} - 1} s^{\beta - \frac{\gamma-1}{2} - 1} \left\{ \frac{|J_{\gamma-1}(2\sqrt{xst})|}{I_{\gamma-1}(2\sqrt{xst})} \right\} \\
 &\times \left[ 1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} (1 - e^{ys+zt}) \right] ds dt =: \mathfrak{J}_2. \quad (36)
 \end{aligned}$$

Now, we can apply in getting final results the bounding inequalities for the modulus of the Bessel functions in the integrand of  $\mathfrak{J}_2$ . So, by the von Lommel's uniform bound (24), valid for all  $\gamma > 0$ , from (35) and (36), we arrive at (31). In similar manner, (32) is established by virtue of the first Landau's result (26), (35) and (36).

The bound (25) by Minakshisundaram and Szász is of the magnitude  $|J_{\gamma-1}(t)| \leq \mathfrak{E} t^{\kappa}$  where  $\mathfrak{E} \in \{[2^{\gamma-1}\Gamma(\gamma)]^{-1}, c_L, d_O\}$  and  $\kappa \in \{\gamma - 1, -\frac{1}{3}, -\frac{1}{2}\}$ , respectively and so do the second Landau's (27) and Olenko's (29) inequalities, where  $\mathfrak{E} > 0$ ,  $\kappa \in \{\gamma - 1, -\frac{1}{3}, -\frac{1}{2}\}$  respectively. Thus, by these three inequalities we get

$$\begin{aligned}
 \mathfrak{J}_2 &\leq \mathfrak{E} |x|^{\frac{\kappa}{2}} \int_0^{\infty} \int_0^{\infty} e^{-s-t} t^{\alpha + \frac{\kappa-\gamma+1}{2} - 1} s^{\beta + \frac{\kappa-\gamma+1}{2} - 1} \left\{ 1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} (1 - e^{ys+zt}) \right\} ds dt \\
 &= \mathfrak{E} |x|^{\frac{\kappa}{2}} \left\{ \left( 1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} \right) \int_0^{\infty} \int_0^{\infty} e^{-s-t} t^{\alpha + \frac{\kappa-\gamma+1}{2} - 1} s^{\beta + \frac{\kappa-\gamma+1}{2} - 1} ds dt \right. \\
 &\quad \left. + \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} \int_0^{\infty} \int_0^{\infty} e^{-(1-y)s - (1-z)t} t^{\alpha + \frac{\kappa-\gamma+1}{2} - 1} s^{\beta + \frac{\kappa-\gamma+1}{2} - 1} ds dt \right\} \\
 &= \mathfrak{E} |x|^{\frac{\kappa}{2}} \Gamma\left(\alpha + \frac{\kappa-\gamma+1}{2}\right) \Gamma\left(\beta + \frac{\kappa-\gamma+1}{2}\right) \\
 &\quad \times \left\{ 1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} + \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} \frac{1}{(1-y)^{\beta + \frac{\kappa-\gamma+1}{2}} (1-z)^{\alpha + \frac{\kappa-\gamma+1}{2}}} \right\}.
 \end{aligned}$$

Now, choosing  $(\mathfrak{E}, \kappa) \in \{(1/[2^{\gamma-1}\Gamma(\gamma)], \gamma - 1), (c_L, -\frac{1}{3}), (d_O, -\frac{1}{2})\}$  we realize the bounds affiliated to the Minakshisundaram and Szász, the second Landau's, whilst Olenko's estimates, respectively.

As to the use of the bound (30) to  $\mathfrak{J}_2$ , we remark that  $\cosh u \leq e^u$ ,  $u > 0$ , therefore by the arithmetic mean–geometric mean inequality we deduce

$$\begin{aligned} \mathfrak{J}_2 &\leq \frac{|x|^{\frac{\gamma-1}{2}}}{\Gamma(\gamma)} \int_0^\infty \int_0^\infty e^{-s-t+2\sqrt{xs}t} t^{\alpha-1} s^{\beta-1} \left\{ 1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} (1 - e^{ys+zt}) \right\} ds dt \\ &\leq \frac{|x|^{\frac{\gamma-1}{2}}}{\Gamma(\gamma)} \int_0^\infty \int_0^\infty e^{-(1-\sqrt{x})s - (1-\sqrt{x})t} t^{\alpha-1} s^{\beta-1} \left\{ 1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)} (1 - e^{ys+zt}) \right\} ds dt \\ &=: \mathfrak{J}_3. \end{aligned}$$

Finally, we have

$$\mathfrak{J}_3 = \frac{\Gamma(\alpha)\Gamma(\beta)|x|^{\frac{\gamma-1}{2}}}{\Gamma(\gamma)} \left\{ \frac{1 - \frac{2\beta' - \lambda}{2(\gamma' - \lambda)}}{(1 - \sqrt{x})^{\alpha+\beta}} + \frac{\frac{2\beta' - \lambda}{2(\gamma' - \lambda)}}{(1 - \sqrt{x} - y)^\beta (1 - \sqrt{x} - z)^\alpha} \right\},$$

which proves the upper bound (34).  $\square$

#### 4. Concluding remarks and further observations

We introduced the four parametric extension of Appell's first  $F_1$  and Srivastava's triple hypergeometric function  $H_A$  by making use of the definition of extended Beta function  $B_{p,q,v}^\lambda(s,t)$  in (2) which involves the Macdonald function  $K_{n+\frac{1}{2}}$  in the kernel of integral. We systematically studied the Euler's and Laplace type associated integral representations with other integral representations involving Bessel  $J_\nu(z)$  and Modified Bessel functions  $I_\nu(z)$ . Further applying several functional upper bounds such as Luke's, von Lommel's, Minakshisundaram and Szász and Olenko bounds, we derived several bounds for defined extended hypergeometric functions. Moreover certain contiguous and recursion formulas are also presented of defined extended hypergeometric functions. Particular choices of parameters of obtained findings below reduces to the results studied in [28].

#### A. Contiguous relations and recursion formulae

We first present the contiguous relations recursion formulae for extended Appell's function  $F_{1,p,q,v}^\lambda[x,y]$ . Next, contiguous recursion formulae for the extended Srivastava's triple hypergeometric function  $H_{A,p,q,v}^\lambda[x,y,z]$  are exposed with respect to both numerator and denominator parameters. By the definition of  $F_{1,p,q,v}^\lambda[x,y]$  taking into account the Pochhammer symbol's and the Beta function's transformations:

$$(\beta + 1)_m = (\beta)_m \left( 1 + \frac{m}{\beta} \right); \quad B(b, c - b) = \frac{c}{b} B(b + 1, c - b),$$

we infer the following contiguous relations, where for the second relation we employ  $\beta'$ , viz.

$$\begin{aligned} F_{1,p,q,v}^{\lambda}[\alpha, \beta + 1, \beta'; \gamma; x, y] &= F_{1,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma; x, y] \\ &\quad + \frac{\alpha x}{\gamma} F_{1,p,q,v}^{\lambda}[\alpha + 1, \beta + 1, \beta'; \gamma + 1; x, y], \\ F_{1,p,q,v}^{\lambda}[\alpha, \beta, \beta' + 1; \gamma; x, y] &= F_{1,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma; x, y] \\ &\quad + \frac{\alpha y}{\gamma} F_{1,p,q,v}^{\lambda}[\alpha + 1, \beta, \beta' + 1; \gamma + 1; x, y]. \end{aligned} \quad (37)$$

The Beta function transformation results in the formulae

$$\begin{aligned} F_{1,p,q,v}^{\lambda}[\alpha, \beta + r, \beta'; \gamma; x, y] &= F_{1,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma; x, y] \\ &\quad + \frac{\alpha x}{\gamma} \sum_{\ell=1}^r F_{1,p,q,v}^{\lambda}[\alpha + 1, \beta + \ell, \beta'; \gamma + 1; x, y], \\ F_{1,p,q,v}^{\lambda}[\alpha, \beta, \beta' + r; \gamma; x, y] &= F_{1,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma; x, y] \\ &\quad + \frac{\alpha y}{\gamma} \sum_{\ell=1}^r F_{1,p,q,v}^{\lambda}[\alpha + 1, \beta, \beta' + \ell; \gamma + 1; x, y], \\ F_{1,p,q,v}^{\lambda}[\alpha, \beta - r, \beta'; \gamma; x, y] &= F_{1,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma; x, y] \\ &\quad - \frac{\alpha x}{\gamma} \sum_{\ell=0}^{r-1} F_{1,p,q,v}^{\lambda}[\alpha + 1, \beta - \ell, \beta'; \gamma + 1; x, y], \\ F_{1,p,q,v}^{\lambda}[\alpha, \beta, \beta' - r; \gamma; x, y] &= F_{1,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma; x, y] \\ &\quad - \frac{\alpha y}{\gamma} \sum_{\ell=0}^{r-1} F_{1,p,q,v}^{\lambda}[\alpha + 1, \beta, \beta' - \ell; \gamma + 1; x, y]. \end{aligned} \quad (38)$$

Indeed, by replacing  $\beta \rightarrow \beta + 1$  in the contiguous relation (37) we conclude

$$\begin{aligned} F_{1,p,q,v}^{\lambda}[\alpha, \beta + 2, \beta'; \gamma; x, y] &= F_{1,p,q,v}^{\lambda}[\alpha, \beta + 1, \beta'; \gamma; x, y] + \frac{\alpha x}{\gamma} F_{1,p,q,v}^{\lambda}[\alpha + 1, \beta + 2, \beta'; \gamma + 1; x, y], \\ &= F_{1,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma; x, y] + \frac{\alpha x}{\gamma} \{ F_{1,p,q,v}^{\lambda}[\alpha + 1, \beta + 1, \beta'; \gamma + 1; x, y] \\ &\quad + F_{1,p,q,v}^{\lambda}[\alpha + 1, \beta + 2, \beta'; \gamma + 1; x, y] \}, \end{aligned} \quad (39)$$

where (37) was used in the second equality. Iterating this method  $r$  times for the parameter  $\beta + r$ , we get the desired result (38). A similar argument gives other results.

Further, by the definition of  $H_{A,p,q,v}^{\lambda}[x, y, z]$  and using the Pochhammer symbol's identities:

$$(\alpha + 1)_{k+n} = (\alpha)_{k+n} \left( 1 + \frac{k}{\alpha} + \frac{n}{\alpha} \right), \quad (\beta + 1)_{k+m} = (\beta)_{k+m} \left( 1 + \frac{k}{\beta} + \frac{m}{\beta} \right),$$

$$\frac{1}{(\gamma-1)_k} = \frac{1}{(\gamma)_k} \left(1 + \frac{k}{\gamma-1}\right),$$

and using the similar arguments as above, we arrive at the following below given contiguous relations for  $H_{A,p,q,v}^\lambda[x, y, z]$ .

$$\begin{aligned} H_{A,p,q,v}^\lambda[\alpha+1, \beta, \beta'; \gamma, \gamma'; x, y, z] &= H_{A,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] \\ &\quad + \frac{\beta x}{\gamma} H_{A,p,q,v}^\lambda[\alpha+1, \beta+1, \beta'; \gamma+1, \gamma'; x, y, z] \\ &\quad + \frac{\beta' z}{\gamma'} H_{A,p,q,v}^\lambda[\alpha+1, \beta, \beta'+1; \gamma, \gamma'+1; x, y, z], \\ H_{A,p,q,v}^\lambda[\alpha, \beta+1, \beta'; \gamma, \gamma'; x, y, z] &= H_{A,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] \\ &\quad + \frac{\alpha x}{\gamma} H_{A,p,q,v}^\lambda[\alpha+1, \beta+1, \beta'; \gamma+1, \gamma'; x, y, z] \\ &\quad + \frac{\beta' y}{\gamma'} H_{A,p,q,v}^\lambda[\alpha, \beta+1, \beta'+1; \gamma, \gamma'+1; x, y, z], \\ H_{A,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma-1, \gamma'; x, y, z] &= H_{A,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] \\ &\quad + \frac{\alpha \beta x}{\gamma(\gamma-1)} H_{A,p,q,v}^\lambda[\alpha+1, \beta+1, \beta'; \gamma+1, \gamma'; x, y, z]. \end{aligned}$$

Iterating this method  $r$  times for the parameter  $\alpha+r$ ,  $\beta+r$  and  $\gamma-r$ , we get below given the desired set of recursion formulae for  $H_{A,p,q,v}^\lambda[x, y, z]$  which can be useful in further research and applications.

$$\begin{aligned} H_{A,p,q,v}^\lambda[\alpha+r, \beta, \beta'; \gamma, \gamma'; x, y, z] &= H_{A,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] \\ &\quad + \frac{\beta x}{\gamma} \sum_{\ell=1}^r H_{A,p,q,v}^\lambda[\alpha+\ell, \beta+1, \beta'; \gamma+1, \gamma'; x, y, z] \\ &\quad + \frac{\beta' z}{\gamma'} \sum_{\ell=1}^r H_{A,p,q,v}^\lambda[\alpha+\ell, \beta, \beta'+1; \gamma, \gamma'+1; x, y, z], \\ H_{A,p,q,v}^\lambda[\alpha-r, \beta, \beta'; \gamma, \gamma'; x, y, z] &= H_{A,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] \\ &\quad - \frac{\beta x}{\gamma} \sum_{\ell=0}^{r-1} H_{A,p,q,v}^\lambda[\alpha-\ell, \beta+1, \beta'; \gamma+1, \gamma'; x, y, z] \\ &\quad - \frac{\beta' z}{\gamma'} \sum_{\ell=0}^{r-1} H_{A,p,q,v}^\lambda[\alpha-\ell, \beta, \beta'+1; \gamma, \gamma'+1; x, y, z], \\ H_{A,p,q,v}^\lambda[\alpha, \beta+r, \beta'; \gamma, \gamma'; x, y, z] &= H_{A,p,q,v}^\lambda[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] \\ &\quad + \frac{\alpha x}{\gamma} \sum_{\ell=1}^r H_{A,p,q,v}^\lambda[\alpha+1, \beta+\ell, \beta'; \gamma+1, \gamma'; x, y, z] \\ &\quad + \frac{\beta' y}{\gamma'} \sum_{\ell=1}^r H_{A,p,q,v}^\lambda[\alpha, \beta+\ell, \beta'+1; \gamma, \gamma'+1; x, y, z], \end{aligned}$$

$$\begin{aligned}
H_{A,p,q,v}^{\lambda}[\alpha, \beta - r, \beta'; \gamma, \gamma'; x, y, z] &= H_{A,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] \\
&\quad - \frac{\alpha x}{\gamma} \sum_{\ell=0}^{r-1} H_{A,p,q,v}^{\lambda}[\alpha + 1, \beta - \ell, \beta'; \gamma + 1, \gamma'; x, y, z] \\
&\quad - \frac{\beta' y}{\gamma'} \sum_{\ell=0}^{r-1} H_{A,p,q,v}^{\lambda}[\alpha, \beta - \ell, \beta' + 1; \gamma, \gamma' + 1; x, y, z], \\
H_{A,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma - r, \gamma'; x, y, z] &= H_{A,p,q,v}^{\lambda}[\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z] \\
&\quad + \alpha \beta x \sum_{\ell=1}^r \frac{H_{A,p,q,v}^{\lambda}[\alpha + 1, \beta + 1, \beta'; \gamma - \ell + 2, \gamma'; x, y, z]}{(\gamma - \ell)(\gamma - \ell + 1)}.
\end{aligned}$$

## B. Refined bilateral functional bounds

In our present investigation, with the help of the extended beta function  $B_{p,q,v}^{\lambda}(x, y)$ , we have introduced the extended Appell's double hypergeometric function  $F_{1,p,q,v}^{\lambda}$  and Srivastava triple hypergeometric function  $H_{A,p,q,v}^{\lambda}$  and investigated their diverse properties such mainly as various integral representations involving integral representations involving Bessel  $J_{\nu}(z)$  and modified Bessel functions  $I_{\nu}(z)$  of the first kind along with contiguous relations and recursive formulae. The special cases of given in section 2 and section 3 presented here when using the fact  $K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$  in (4) and set  $v = 0$  and  $p, q \searrow 0$  would reduce to the corresponding well-known results for the Appell's double hypergeometric function  $F_1[x, y]$  and Srivastava's triple hypergeometric function (see, for details, [18, 23, 24, 25, 26, 27]).

Further observations for our results derived in section 4 can be refine the bounds presented above, we can apply Luke's companion estimate to (23) [12, p. 57, Theorem 16, Eq. (5.8)]

$$1 + \theta x e^{\frac{\psi}{2}x} < {}_rF_r(a_r; b_r; x) < 1 + \theta x \left(1 - \frac{\psi}{2} + \frac{\psi}{2}e^x\right), \quad (40)$$

where we use the same shorthand notation for  $\theta$  as  $\omega$  in (23) and

$$\psi = \frac{1 + \max_{1 \leq j \leq r} a_j}{1 + \min_{1 \leq j \leq r} b_j},$$

which reduces (40) to

$$1 + \frac{b}{c} x \exp\left\{\frac{1+b}{2(1+c)}x\right\} < \Phi(b; c; x) < 1 + \frac{b}{c} x \left(1 - \frac{1+b}{2(1+c)}(1 - e^x)\right).$$

Now, following the method of derivation in the proof of Theorem 5 now with this upper bound, we can easily establish a novel set of similar inequalities. By obvious reasons their derivation we leave to the interested reader.

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## REFERENCES

- [1] M. A. CHAUDHRY, A. QADIR, M. RAFIQUE AND S. M. ZUBAIR, *Extension of Euler's Beta function*, J. Comput. Appl. Math. **78** (1997), 19–32.
- [2] M. A. CHAUDHRY, A. QADIR, H. M. SRIVASTAVA AND R. B. PARIS, *Extended hypergeometric and confluent hypergeometric functions*, Appl. Math. Comput. **159** (2004), 589–602.
- [3] J. CHOI, A. K. RATHIE AND R. K. PARMAR, *Extension of extended beta, hypergeometric and confluent hypergeometric functions*, Honam Math. J. **36** (2) (2014) 339–367.
- [4] L. EULER, *De serie Lambertiana plurimisque eius insignibus proprietatibus*, Acta Academiae Scientiarum Imperialis Petropolitanae II, (1779–1783), 29–51; reprinted in *Opera Omnia Ser. I*, vol. 6. (Teubner, Leipzig, 1921), 350–369, (in Latin).
- [5] H. EXTON, *Multiple Hypergeometric Functions and Applications*, Halsted Press, Ellis Horwood, Chichester, New York, NY, USA; London, UK; Sydney, Australia; Toronto, QN, Canada, 1976.
- [6] E. K. IFANTIS AND P. D. SIAFARIKAS, *Inequalities involving Bessel and modified Bessel functions*, J. Math. Anal. **147** (1990), 214–227.
- [7] C. KRATTENTHALER AND K. SRINIVASA RAO, *Automatic generation of hypergeometric identities by the beta integral method*, in R. JAGANNATHAN, S. KANEMITSU, G. VAN DEN BERGHE AND V. VAN ASSCHE (Eds.) *Proceedings of the International Conference on Special Functions and their Applications* (Chennai, 2002); J. Comput. Appl. Math. **160** (1–2) (2003), 159–173.
- [8] L. LANDAU, *Monotonicity and bounds on Bessel functions*, in Proceedings of the Symposium on Mathematical Physics and Quantum Field Theory (Berkeley, CA, 1999), 147–154; Electron. J. Differ. Equ. Conf. **4** (Southwest Texas State Univ., San Marcos, TX, 2000).
- [9] D. M. LEE, A. K. RATHIE, R. K. PARMAR AND Y. S. KIM, *Generalization of extended beta function, hypergeometric and confluent hypergeometric functions*, Honam Math. J. **33** (2) (2011), 187–206.
- [10] E. C. J. VON LOMMEL, *Die Beugungserscheinungen einer kreisrunden Öffnung und eines kreisrunden Schirmchens theoretisch und experimentell bearbeitet*, Abh. der math. phys. Classe der k. b. Akad. der Wiss. (München), **15** (1884–1886), 229–328.
- [11] E. C. J. VON LOMMEL, *Die Beugungserscheinungen geradlinig begrenzter Schirme*, Abh. der math. phys. Classe der k. b. Akad. der Wiss. (München), **15** (1884–1886), 529–664.
- [12] Y. L. LUKE, *Inequalities for generalized hypergeometric functions*, J. Approx. Theory **5** (1) (1974), 41–65.
- [13] G. V. MILOVANOVIĆ, R. K. PARMAR AND T. K. POGÁNY, *Bounds for the  $(p, v)$ -extended Beta function and certain consequences*, J. Math. Ineq. **17** (4) (2023), 1433–1441.
- [14] S. MINAKSHISUNDARAM AND O. SZÁSZ, *On absolute convergence of multiple Fourier series*, Trans. Amer. Math. Soc. **61** (1) (1947), 36–53.
- [15] A. YA. OLENKO, *Upper bound on  $\sqrt{x}J_\nu(x)$  and its applications*, Integral Transforms Spec. Funct. **17** (6) (2006), 455–467.
- [16] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT AND C. W. CLARK (Eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
- [17] R. K. PARMAR, R. B. PARIS AND P. CHOPRA, *On an extension of extended beta and hypergeometric functions*, J. Classical Anal. **11** (1) (2017), 91–106.
- [18] R. K. PARMAR AND T. K. POGÁNY, *Extended Srivastava's triple hypergeometric  $H_{4,p,q}$  function and related bounding inequalities*, J. Contemp. Math. Anal. **52** (6) (2017), 261–272.
- [19] R. K. PARMAR AND T. K. POGÁNY, *Bounds for novel extended beta and hypergeometric functions and related results*, J. Inequal. Appl. **2024** (2024), Article No. 77.
- [20] R. K. PARMAR, T. K. POGÁNY AND L. TEOFANOV, *On extended beta function and related inequalities*, Mathematics (Basel), **12** (2024), Article No. 2709.
- [21] R. K. PARMAR, T. K. POGÁNY AND S. PIRIVINA, *Integral forms and functional bounds for certain extended exton's double hypergeometric functions*, J. Math. Ineq. **19** (4) (2025), 1283–1311.
- [22] R. K. PARMAR, T. K. POGÁNY AND S. PIRIVINA, *Generalized Horn function  $H_{4,p,q,\nu}^\lambda$  and related bounding inequalities with applications to statistics*, Rad Hrvat. Akad. Znan. Umjet., Mat. Znan. **570** (2026), 127–140.
- [23] H. M. SRIVASTAVA, *Hypergeometric functions of three variables*, Ganita **15** (1964), 97–108.
- [24] H. M. SRIVASTAVA, *On the reducibility of certain hypergeometric functions*, Rev., Ser. A, Univ. Nac. Tucumán **16** (1966), 7–14.

- [25] H. M. SRIVASTAVA, *Some integrals representing triple hypergeometric functions*, Rend. Circ. Mat. Palermo **16** (2) (1967), 99–115.
- [26] H. M. SRIVASTAVA AND P. W. KARLSSON, *Multiple Gaussian Hypergeometric Series*, Halsted Press – Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.
- [27] J. CHOI, A. HASANOV AND M. TURAEV, *Integral representations for Srivastava's hypergeometric function  $H_A$* , Honam Math. J. **34** (1) (2012), 113–124.
- [28] XIAOXIA WANG, *Recursion formulas for Appell functions*, Integral Transforms Spec. Funct. **23** (6) (2012), 421–433.
- [29] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, second edition, Cambridge University Press, Cambridge, London and New York 1944.

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