

CERTAIN GEOMETRIC PROPERTIES OF GENERALIZED k -BESSEL FUNCTIONS

ERHAN DENİZ*, TAYFUN ÇOBAN AND ADEM KIZILTEPE

Abstract. In this paper our aim is to establish some geometric properties (like starlikeness, convexity and close-to-convexity) for the generalized k -Bessel functions. In order to prove our main results, we use some inequalities for ratio of these functions in normalized form and classical result of Fejer [15].

1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m,$$

which are analytic in the open unit disk

$$U = \{z : |z| < 1, z \in \mathbb{C}\}.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are univalent in U . A function $f \in \mathcal{S}$ is said to be starlike of order β ($0 \leq \beta < 1$) if and only if

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \beta.$$

We denote by $\mathcal{S}^*(\beta)$ the class of all such functions. Also, we note that $\mathcal{S}^*(0) = \mathcal{S}^*$ is the usual class of starlike functions in U . A function $f \in \mathcal{S}$ is said to be convex of order β ($0 \leq \beta < 1$) if and only if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta.$$

We denote by $\mathcal{C}(\beta)$ the class of all such functions. Also, we note that $\mathcal{C}(0) = \mathcal{C}$ is the usual class of convex functions in U . A function $f \in \mathcal{S}$, $g \in \mathcal{S}^*$ is said to be close-to-convex of order β ($0 \leq \beta < 1$) if and only if

$$\operatorname{Re} \left(\frac{zf'(z)}{g(z)} \right) > \beta.$$

Mathematics subject classification (2020): Primary 33C10; Secondary 30C45.

Keywords and phrases: Generalized k -Bessel functions, convex functions, starlike functions, close-to-convex functions.

* Corresponding author.

We denote by $K(\beta)$ the class of all such functions. Also, we note that $K(0) = K$ is the usual class of close-to-convex functions in U .

In applied mathematics and engineering disciplines, Bessel functions of the first kind are frequently utilised. Many academic publications and scientific investigations have examined the properties of Bessel functions. It is a particular solution of the differential equation

$$z^2 \omega''(z) + z\omega'(z) + [z^2 - \nu^2] \omega(z) = 0,$$

and given by the following series representation

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(\nu + m + 1)} \left(\frac{z}{2}\right)^{2m+\nu} \quad (z \in \mathbb{C})$$

where Γ denotes the Euler gamma function. The function J_ν is called the Bessel function of the first kind of order ν . The modified Bessel function of the first kind of order ν is a solution of the differential equation

$$z^2 \omega''(z) + z\omega'(z) - [z^2 + \nu^2] \omega(z) = 0$$

and it is represented by the series form as

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(\nu + m + 1)} \left(\frac{z}{2}\right)^{2m+\nu} \quad (z \in \mathbb{C}).$$

Another function j_ν is the spherical Bessel function of the first kind of order ν defined by

$$j_\nu(z) = \sqrt{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(\nu + m + 3/2)} \left(\frac{z}{2}\right)^{2m+\nu} \quad (z \in \mathbb{C}).$$

The aforementioned functions' inherent generalizations are another important feature. There are some natural generalizations of the Bessel functions in the literature.

Let us consider the following second-order linear homogeneous differential equation [3, p. 159]:

$$z^2 \omega''(z) + az\omega'(z) + [dz^2 - \nu^2 + (1-a)\nu] \omega(z) = 0 \quad (a, d, \nu \in \mathbb{C}). \quad (1.1)$$

The function $\omega_{\nu,a,d}(z)$, which is called the generalized Bessel function of the first kind of order ν , is defined as a particular solution of (1.1). We have the following familiar series representation for the function $\omega_{\nu,a,d}(z)$

$$\omega_{\nu,a,d}(z) = \sum_{m=0}^{\infty} \frac{(-d)^m}{m! \Gamma(\nu + m + (a+1)/2)} \left(\frac{z}{2}\right)^{2m+\nu} \quad (z \in \mathbb{C}) \quad (1.2)$$

where Γ stands for the Euler gamma function. The series (1.2) permits the study of Bessel, modified Bessel and spherical Bessel functions in a unified manner. Each of these particular cases of the function $\omega_{\nu,a,d}(z)$ is worthy of mention here. Another

generalization of Bessel functions was obtained by Mondal and Akel [18, p. 217] given by the following infinite-series representation

$$W_{v,d}^k(z) = \sum_{m=0}^{\infty} \frac{(-d)^m}{\Gamma_k(mk + v + k)\Gamma(m+1)} \left(\frac{z}{2}\right)^{2m + \frac{v}{k}} \quad (z \in \mathbb{C}), \quad (1.3)$$

where $k > 0$, $v > -1$, $d \in \mathbb{R}$ and Γ_k stands for the k -gamma functions studied in [13] defined by

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-\frac{t^k}{k}} dt,$$

where $\Re(z) > 0$ and $k > 0$. For more information about the k -gamma function, the interested reader is referred to [12–14] and references therein. The function $W_{v,d}^k(z)$ is called as k -Bessel function. Moreover for $z \in \mathbb{C}$ and $k \in \mathbb{R}^+$ the k -gamma function and the classical Gamma function have the following relation:

$$\Gamma_k(z) = k^{\frac{z}{k}-1} \Gamma\left(\frac{z}{k}\right). \quad (1.4)$$

We also note that for $x \in \mathbb{C}$, $n \in \mathbb{N}^+$, $k \in \mathbb{R}^+$, the function $\Gamma_k(z)$ satisfies the following relations:

$$\begin{aligned} \Gamma_k(z+k) &= z\Gamma_k(z), \quad \Gamma_k(k) = 1, \\ \frac{1}{\Gamma_k(z)} &= zk^{-\frac{z}{k}} e^{\frac{z}{k}\gamma} \prod_{n=1}^{\infty} \left(1 + \frac{z}{nk}\right) e^{-\frac{z}{nk}}, \\ (x)_{n,k} &= \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}, \end{aligned}$$

with Euler's constant γ and $(x)_{n,k}$ stands for the Pochhammer k -symbol defined by

$$(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k).$$

Note that $(x)_{n,1} = (x)_n$ is the classical Pochhammer symbol. In special case of $d = k = 1$ in (1.3) we obtain the classical Bessel function defined with (1.1) (see, for details, [22, Ch. III]).

Very recently, Zayed [23] has defined generalized k -Bessel functions as follows

$$W_{v,a,d}^k(z) = \sum_{m=0}^{\infty} \frac{(-d)^m}{m!\Gamma_k(mk + v + \frac{a+1}{2}k)} \left(\frac{z}{2}\right)^{2m + \frac{v}{k}} \quad (z \in \mathbb{C}) \quad (1.5)$$

for $k > 0$ and $v, a, d \in \mathbb{C}$. The function $W_{v,a,d}^k$ is the generalization of the Bessel-type functions given in above:

$$\begin{aligned} W_{v,1,1}^1(z) &= J_v(z) && \text{(classical Bessel function)} \\ W_{v,1,-1}^1(z) &= I_v(z) && \text{(modified Bessel function)} \\ \sqrt{\frac{\pi}{2}} W_{v,2,1}^1(z) &= j_v(z) && \text{(spherical Bessel function)} \\ W_{v,a,d}^1(z) &= \omega_{v,a,d}(z) && \text{(generalized Bessel function)} \\ W_{v,1,d}^k(z) &= W_{v,d}^k(z) && \text{(\textit{k}-Bessel function).} \end{aligned}$$

Note that $W_{v,a,d}^k(z)$ does not belongs to class $\mathcal{A}$. In this paper, we define the normalized forms of the generalized k -Bessel functions defined by

$$h_{k,v,a,d}(z) = 2^{\frac{v}{k}} \Gamma_k \left(v + \frac{a+1}{2} k \right) z^{1-\frac{v}{2k}} W_{v,a,d}^k(\sqrt{z}).$$

From the definition of $h_{k,v,a,d}(z)$ we obtain series representation as

$$h_{k,v,a,d}(z) = z + \sum_{m=1}^{\infty} \frac{(-d)^m}{4^m m! (x)_{m,k}} z^{m+1},$$

where $x := v + \frac{a+1}{2} \notin \mathbb{Z}_0^-$.

Many authors have studied the some geometric properties such as univalence, starlikeness, convexity and close-to-convexity of the hypergeometric functions, Bessel functions and its generalizations (see, [1–11, 16, 19–21]).

To prove our main results we need the following Lemmas.

LEMMA 1.1. [15] *If the function $f(z) = \sum_{m=1}^{\infty} A_m z^m$, where $A_1 = 1$ and $A_m \geq 0$ for all $m \geq 2$ is analytic in U , and the sequences $\{mA_m - (m+1)A_{m+1}\}_{m \geq 1}$ and $\{mA_m\}_{m \geq 1}$ both are decreasing, then f is starlike in U .*

LEMMA 1.2. [17] *If $f \in \mathcal{A}$ satisfy $\left| \frac{f(z)}{z} - 1 \right| < 1$ for each $z \in U$, then f is starlike in $U_{1/2} = \{z : |z| < \frac{1}{2}\}$.*

LEMMA 1.3. [17] *If $f \in \mathcal{A}$ satisfy $|f'(z) - 1| < 1$ for each $z \in U$, then f is convex in $U_{1/2} = \{z : |z| < \frac{1}{2}\}$.*

LEMMA 1.4. [17] *If $f \in \mathcal{A}$ satisfy $|f'(z) - 1| < \frac{2}{\sqrt{5}}$ for each $z \in U$, then f is starlike in $U = \{z : |z| < 1\}$.*

LEMMA 1.5. [20] *If $0 \leq \beta < 1$, $f \in \mathcal{A}$ satisfy $|zf''(z)| < \frac{1-\beta}{4}$ for each $z \in U$, then $\operatorname{Re} f'(z) > \frac{1+\beta}{4}$ in $U = \{z : |z| < 1\}$.*

Recently Zayed et al [23] studied the functional inequalities and properties of zeros for the generalized k -Bessel functions.

In this paper, we study certain geometric properties of the function $h_{k,v,a,d}(z)$.

2. Main results

Throughout our present investigation, we tacitly assume that $x := v + \frac{a+1}{2}$.

THEOREM 2.1. *Let $k \in \mathbb{R}^+$, $a \in \mathbb{R}$, $d \in \mathbb{C}$, $x \in \mathbb{R} - \{\mathbb{Z}^- \cup \{0\}\}$, $\beta \in [0, 1)$ and $z \in U$. If the inequality*

$$\beta < \frac{8x(x+k) - |d|(x+6(x+k))}{8x(x+k) - |d|(x+2(x+k))}$$

holds, then $h_{k,v,a,d}(z) \in \mathcal{S}^(\beta)$.*

Proof. We use the inequality $\left| \frac{zh'_{k,v,a,d}(z)}{h_{k,v,a,d}(z)} - 1 \right| < 1 - \beta$ to prove the starlikeness of order β for the function $h_{k,v,a,d}(z)$. So by using the well-known triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

and following inequalities

$$(m-1)! \geq 2^{m-2}, \quad (x)_{m,k} \geq x(x+k)^{m-1}, \quad m \in \mathbb{N} \quad (2.1)$$

we obtain

$$\begin{aligned} \left| h'_{k,v,a,d}(z) - \frac{h_{k,v,a,d}(z)}{z} \right| &= \left| \sum_{m=1}^{\infty} \frac{(-d)^m}{4^m (m-1)! (x)_{m,k}} z^m \right| \\ &\leq \frac{|d|}{4x} \sum_{m=1}^{\infty} \frac{1}{4^{m-1} 2^{m-2} (x+k)^{m-1}} |d|^{m-1} \\ &\leq \frac{|d|}{2x} \sum_{m=1}^{\infty} \left[\frac{|d|}{8(x+k)} \right]^{m-1} \\ &= \frac{4|d|(x+k)}{x(8(x+k) - |d|)}. \end{aligned}$$

Furthermore, if we use reverse triangle inequality

$$|z_1 + z_2| \geq ||z_1| - |z_2||$$

and the inequalities (2.1) then we get

$$\begin{aligned} \left| \frac{h_{k,v,a,d}(z)}{z} \right| &\geq 1 - \left| \sum_{m=1}^{\infty} \frac{(-d)^m}{4^m m! (x)_{m,k}} z^m \right| \\ &\geq 1 - \frac{|d|}{4x} \sum_{m=1}^{\infty} \left[\frac{|d|}{8(x+k)} \right]^{m-1} \\ &= \frac{x(8(x+k) - |d|) - 2|d|(x+k)}{x(8(x+k) - |d|)}. \end{aligned}$$

By combining the above inequalities, we obtain

$$\left| \frac{zh'_{k,v,a,d}(z)}{h_{k,v,a,d}(z)} - 1 \right| \leq \frac{4|d|(x+k)}{8x(x+k) - |d|(x+2(x+k))}.$$

Consequently, by the hypothesis of theorem we have

$$\frac{4|d|(x+k)}{8x(x+k) - |d|(x+2(x+k))} < 1 - \beta$$

that is,

$$\left| \frac{zh'_{k,v,a,d}(z)}{h_{k,v,a,d}(z)} - 1 \right| < 1 - \beta.$$

So, $h_{k,v,a,d}(z)$ is starlike function of order β . $\square$

THEOREM 2.2. *If the following inequality*

$$|d| \leq \frac{8x(x+k)}{3\sqrt{5}(x+k)+x}$$

holds, then $h_{k,v,a,d}(z) \in \mathcal{S}^*$.

Proof. To prove this theorem, we use Lemma 1.4. From the well-known triangle inequality and the inequalities (2.1) we have

$$\begin{aligned} |h'_{k,v,a,d}(z) - 1| &\leq \sum_{m=1}^{\infty} \frac{(m+1)}{4^m m! (x)_{m,k}} |d|^m \\ &\leq \left[\sum_{m=1}^{\infty} \frac{1}{4^m (m-1)! (x)_{m,k}} |d|^m + \sum_{m=1}^{\infty} \frac{1}{4^m m! (x)_{m,k}} |d|^m \right] \\ &\leq \frac{|d|}{2x} \sum_{m=1}^{\infty} \left[\frac{|d|}{8(x+k)} \right]^{m-1} + \frac{|d|}{4x} \sum_{m=1}^{\infty} \left[\frac{|d|}{8(x+k)} \right]^{m-1} \\ &= \frac{6|d|(x+k)}{x(8(x+k)-|d|)}. \end{aligned}$$

Therefore from hypothesis of theorem we obtain

$$|h'_{k,v,a,d}(z) - 1| \leq \frac{6|d|(x+k)}{x(8(x+k)-|d|)} < \frac{2}{\sqrt{5}}.$$

So, from Lemma 1.4 $h_{k,v,a,d}(z)$ is starlike function. $\square$

THEOREM 2.3. *Let* $k \in \mathbb{R}^+$, $a \in \mathbb{R}$, $d \in \mathbb{C}$, $x \in \mathbb{R} - \{\mathbb{Z}^- \cup \{0\}\}$, $\beta \in [0, 1)$ and $z \in U$. *If the inequality*

$$\beta < \frac{10|d|^2(x+k) - 64|d|(x+k)^2 + (8-|d|)(x(8(x+k)-|d|))}{6|d|^2(x+k) + (8-|d|)(x(8(x+k)-|d|))}$$

holds, then $h_{k,v,a,b,c}(z) \in \mathcal{C}(\beta)$.

Proof. To prove the convexity of order β of function $h_{k,v,a,d}(z)$, we have to show that $\left| \frac{zh''_{k,v,a,d}(z)}{h'_{k,v,a,d}(z)} \right| < 1 - \beta$. By using the well-known triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

and the inequalities (2.1), we have

$$\begin{aligned}
 |z^2 h''_{k,v,a,d}(z)| &= \left| \sum_{m=1}^{\infty} \frac{m(m+1)(-d)^m}{4^m m! (x)_{m,k}} z^{m+1} \right| \\
 &\leq \frac{|d|}{4x} \sum_{m=1}^{\infty} \frac{m}{4^{m-1} (m-1)! (x+k)^{m-1}} |d|^{m-1} + \frac{|d|}{4x} \sum_{m=1}^{\infty} \frac{1}{4^{m-1} (m-1)! (x+k)^{m-1}} |d|^{m-1} \\
 &\leq \frac{|d|}{2x} \sum_{m=1}^{\infty} m \left[\frac{|d|}{8(x+k)} \right]^{m-1} + \frac{|d|}{2x} \sum_{m=1}^{\infty} \left[\frac{|d|}{8(x+k)} \right]^{m-1} \\
 &= \frac{-4|d|^2(x+k) + |d|(8(x+k))^2}{x(8(x+k) - |d|)^2}.
 \end{aligned} \tag{2.2}$$

Moreover, if we use reverse triangle inequality

$$|z_1 + z_2| \geq ||z_1| - |z_2||$$

and the inequalities (2.1), we have

$$\begin{aligned}
 |zh'_{k,v,a,d}(z)| &= \left| z + \sum_{m=1}^{\infty} \frac{(m+1)(-d)^m}{4^m m! (x)_{m,k}} z^m \right| \\
 &\geq 1 - \sum_{m=1}^{\infty} \frac{(m+1)}{4^m m! (x)_{m,k}} |d|^m \\
 &\geq 1 - \left[\sum_{m=1}^{\infty} \frac{1}{4^m (m-1)! (x)_{m,k}} |d|^m + \sum_{m=1}^{\infty} \frac{1}{4^m m! (x)_{m,k}} |d|^m \right] \\
 &\geq 1 - \left[\frac{|d|}{2x} \sum_{m=1}^{\infty} \left[\frac{|d|}{8(x+k)} \right]^{m-1} + \frac{|d|}{4x} \sum_{m=1}^{\infty} \left[\frac{|d|}{8(x+k)} \right]^{m-1} \right] \\
 &= \frac{-6|d|(x+k) + x(8(x+k) - |d|)}{x(8(x+k) - |d|)}.
 \end{aligned} \tag{2.3}$$

Consequently, from (2.2), (2.3) and by the hypothesis of theorem we get

$$\left| \frac{zh''_{k,v,a,d}(z)}{h'_{k,v,a,d}(z)} \right| \leq \frac{-4|d|^2(x+k) + 64|d|(x+k)^2}{6|d|^2(x+k) + (8-|d|)(x(8(x+k) - |d|))} < 1 - \beta$$

and so, $h_{k,v,a,d}(z)$ is convex function of order β . $\square$

THEOREM 2.4. Let $k \in \mathbb{R}^+$, $a \in \mathbb{R}$, $d \in \mathbb{C}$, $x \in \mathbb{R} - \{\mathbb{Z}^- \cup \{0\}\}$, $\beta \in [0, 1)$ and $z \in U$. If the inequality

$$|d| < \frac{x(8(x+k))}{2(x+k) + x}$$

holds, then $h_{k,v,a,d}(z)$ is starlike in $U_{1/2}$.

Proof. By using the well-known triangle inequality and the second inequality in (2.1) we get

$$\begin{aligned} \left| \frac{h_{k,v,a,d}(z)}{z} - 1 \right| &\leq \left| \sum_{m=1}^{\infty} \frac{(-d)^m}{4^m m! (x)_{m,k}} z^m \right| \\ &\leq \frac{|d|}{4x} \sum_{m=1}^{\infty} \frac{|d|^{m-1}}{4^{m-1} m! (x+k)^{m-1}} \\ &\leq \frac{|d|}{4x} \sum_{m=1}^{\infty} \left[\frac{|d|}{8(x+k)} \right]^{m-1} \\ &= \frac{2|d|(x+k)}{x(8(x+k) - |d|)}. \end{aligned}$$

Since the hypothesis of theorem we obtain

$$\frac{2|d|(x+k)}{x(8(x+k) - |d|)} < 1.$$

Consequently, from Lemma 1.2 $h_{k,v,a,d}(z)$ is starlike in $U_{1/2}$. $\square$

THEOREM 2.5. Let $k \in \mathbb{R}^+$, $a \in \mathbb{R}$, $d \in \mathbb{C}$, $x \in \mathbb{R} - \{\mathbb{Z}^- \cup \{0\}\}$ with $\beta \in [0, 1)$ and $z \in U$. If

$$|d| < \frac{x(8(x+k))}{3\sqrt{5}(x+k) + x}$$

then $h_{k,v,a,d}(z)$ is convex in $U_{1/2}$.

Proof. By using the well-known triangle inequality and the inequalities (2.1) for $n \in \mathbb{N}$, we have

$$\begin{aligned} |h'_{k,v,a,d}(z) - 1| &\leq \left| 1 + \sum_{m=1}^{\infty} \frac{(m+1)(-d)^m}{4^m m! (x)_{m,k}} z^m - 1 \right| \\ &\leq \sum_{m=1}^{\infty} \frac{1}{4^m (m-1)! (x)_{m,k}} |d|^m + \sum_{m=1}^{\infty} \frac{1}{4^m m! (x)_{m,k}} |d|^m \\ &\leq \frac{|d|}{2x} \sum_{m=1}^{\infty} \left[\frac{|d|}{8(x+k)} \right]^{m-1} + \frac{|d|}{4x} \sum_{m=1}^{\infty} \left[\frac{|d|}{8(x+k)} \right]^{m-1} \\ &= \frac{6|d|(x+k)}{x(8(x+k) - |d|)}. \end{aligned}$$

Thus, from the hypothesis of theorem we obtain

$$\frac{6|d|(x+k)}{x(8(x+k) - |d|)} < 1$$

and so we have required result by using Lemma 1.3. $\square$

THEOREM 2.6. Let $k \in \mathbb{R}^+$, $a \in \mathbb{R}$, $d \in \mathbb{C}$, $x \in \mathbb{R} - \{\mathbb{Z}^- \cup \{0\}\}$, $\beta \in [0, 1)$ and $z \in U$. If the inequality

$$\frac{|d|}{8(x+k)} < 1$$

holds, then $\operatorname{Re} h'_{k,v,a,d}(z) > \frac{1+\beta}{4}$ or $h_{k,v,a,d}(z)$ is close-to-convexity of order $\frac{1+\beta}{4}$ where,

$$\beta < \frac{-|d|(80(x+k)+x)+x(8(x+k))}{x(8(x+k)-|d|)}$$

Proof. By using the well-known triangle inequality and the inequalities (2.1) and $8(m-1)!(x+k)^{m-1} \geq m[2(x+k)]^{m-1}$, we have

$$\begin{aligned} |zh''_{k,v,a,d}(z)| &\leq \left| \sum_{m=1}^{\infty} \frac{m(m+1)(-d)^m}{4^m m! (x)_{m,k}} z^m \right| \\ &\leq \frac{|d|}{4x} \sum_{m=1}^{\infty} \frac{m}{4^{m-1} (m-1)! (x+k)^{m-1}} |d|^{m-1} \\ &\quad + \frac{|d|}{4x} \sum_{n=1}^{\infty} \frac{1}{4^{m-1} (m-1)! (x+k)^{m-1}} |d|^{m-1} \\ &\leq \frac{2|d|}{x} \sum_{m=1}^{\infty} \left[\frac{|d|}{8(x+k)} \right]^{m-1} + \frac{|d|}{2x} \sum_{m=1}^{\infty} \left[\frac{|d|}{8(x+k)} \right]^{m-1} \\ &= \frac{20|d|8(x+k)}{x(8(x+k)-|d|)}. \end{aligned}$$

Thus, from the hypothesis of theorem we obtain

$$\frac{20|d|8(x+k)}{x(8(x+k)-|d|)} < \frac{1-\beta}{4}.$$

Consequently by using Lemma 1.5, we have required result. $\square$

THEOREM 2.7. Let $k \in \mathbb{R}^+$, $a \in \mathbb{R}$, $x > 0$ and $d < 0$. If $x+d > 0$, then $h_{k,v,a,d}(z) \in \mathcal{S}^*$.

Proof. From the definition of the function $h_{k,v,a,d}$ we write

$$h_{k,v,a,d}(z) = \sum_{m=1}^{\infty} A_m z^m,$$

where

$$A_m = \frac{(-d)^{m-1}}{4^{m-1} (m-1)! (x)_{m-1,k}}, \quad A_1 = 1.$$

Since $d < 0$ we have $A_m > 0$ for all $m \geq 1$.

To prove the theorem we use Lemma 1.1. Namely we must prove that the sequences $\{mA_m\}_{m \geq 1}$ and $\{mA_m - (m+1)A_{m+1}\}_{m \geq 1}$ are decreasing.

First we prove that the sequence $\{mA_m\}_{m \geq 1}$ is decreasing. We have

$$\begin{aligned} mA_m - (m+1)A_{m+1} &= \frac{m(-d)^{m-1}}{4^{m-1}(m-1)!(x)_{m-1,k}} - \frac{(m+1)(-d)^m}{4^m m!(x)_{m,k}} \\ &= \frac{(-d)^{m-1}}{4^{m-1}(m-1)!(x)_{m-1,k}} \left[m + \frac{(m+1)d}{4m(x+(m-1)k)} \right] \\ &= A_m \left[\frac{4m^3k + 4m^2(x-k) + md + d}{4m(x+(m-1)k)} \right]. \end{aligned}$$

Let $T_1(m) = 4m^3k + 4m^2(x-k) + md + d$. Since $d < 0$, $x+d > 0$, $k > 0$ and $m^3 \geq 3m^2 - 3m + 1$, $m^2 \geq 2m - 1$ for $m \geq 1$, we obtain

$$\begin{aligned} T_1(m) &= 4m^3k + 4m^2(x-k) + md + d \\ &\geq 4(3m^2 - 3m + 1)k + 4m^2(x-k) + md + d \\ &= m^2(8k + 4x) + m(d - 12k) + 4k + d \\ &\geq m(8x + 4k + d) - 4x - 4k + d \\ &\geq 2(2x + d) > 0. \end{aligned}$$

Consequently we have shown that $T_1(m) > 0$. Therefore the sequence $\{mA_m\}_{m \geq 1}$ is strictly decreasing.

Second, we prove that the sequence $\{mA_m - (m+1)A_{m+1}\}_{m \geq 1}$ is decreasing. For convenience we denote $S_m = mA_m - (m+1)A_{m+1}$ for each $m \geq 1$. Thus we get

$$\begin{aligned} S_m - S_{m+1} &= mA_m - 2(m+1)A_{m+1} + (m+2)A_{m+2} \\ &= \frac{m(-d)^{m-1}}{4^{m-1}(m-1)!(x)_{m-1,k}} - \frac{2(m+1)(-d)^m}{4^m m!(x)_{m,k}} + \frac{(m+2)(-d)^{m+1}}{4^{m+1}(m+1)!(x)_{m+1,k}} \\ &= \frac{(-d)^{m-1}}{4^{m-1}(m-1)!(x)_{m-1,k}} \left[m + \frac{2d(m+1)}{4m(x+(m-1)k)} \right. \\ &\quad \left. + \frac{(m+2)d^2}{16m(m+1)(x+mk)(x+(m-1)k)} \right] \\ &= A_m \left[\frac{\left[\begin{aligned} &16m^5k^2 + 32m^4kx + m^3(16x^2 + 16kx - 16k^2 + 8dk) \\ &+ m^2(16x^2 - 16kx + 8dx + 16dk) \\ &+ m(16dx + 8dk + d^2) + 8dx + 2d^2 \end{aligned} \right]}{16m(m+1)(x+mk)(x+(m-1)k)} \right]. \end{aligned}$$

Let

$$\begin{aligned} T_2(m) &= 16m^5k^2 + 32m^4kx + 8m^3(2x^2 + 2kx - 2k^2 + dk) \\ &\quad + 8m^2(2x^2 - 2kx + dx + 2dk) + m(16dx + 8dk + d^2) + 8dx + 2d^2. \end{aligned}$$

Our aim is to show that $T_2(m) > 0$. By using the inequality $m^4 \geq 4m^3 - 6m^2 + 4m - 1$, we obtain

$$\begin{aligned}
 T_2(m) &= m \left[\begin{array}{c} 16m^4k^2 + 32m^3kx \\ +8m^2(2x^2 + 2kx - 2k^2 + dk) \\ +8m(2x^2 - 2kx + dx + 2dk) \\ +16dx + 8dk + d^2 \end{array} \right] + 8dx + 2d^2 \\
 &\geq m \left[\begin{array}{c} 16(4m^3 - 6m^2 + 4m - 1)k^2 + 32m^3kx \\ +8m^2(2x^2 + 2kx - 2k^2 + dk) \\ +8m(2x^2 - 2kx + dx + 2dk) \\ +16dx + 8dk + d^2 \end{array} \right] + 8dx + 2d^2 \\
 &= m \left[\begin{array}{c} 32km^3(x + 2k) \\ +8m^2(2x^2 + 2kx + dk - 14k^2) \\ +8m(2x^2 - 2kx + 2dk + dx + 8k^2) \\ -16k^2 + 16dx + 8dk + d^2 \end{array} \right] + 8dx + 2d^2 \\
 &:= X_1(m).
 \end{aligned}$$

Clearly, the coefficient of m^3 in the above expression is nonnegative. Therefore using that $m^3 \geq 3m^2 - 3m + 1$, we obtain

$$\begin{aligned}
 X_1(m) &= m \left[\begin{array}{c} 32km^3(x + 2k) \\ +8m^2(2x^2 + 2kx + dk - 14k^2) \\ +8m(2x^2 - 2kx + 2dk + dx + 8k^2) \\ -16k^2 + 16dx + 8dk + d^2 \end{array} \right] + 8dx + 2d^2 \\
 &\geq m \left[\begin{array}{c} 32k(3m^2 - 3m + 1)(x + 2k) \\ +8m^2(2x^2 + 2kx + dk - 14k^2) \\ +8m(2x^2 - 2kx + 2dk + dx + 8k^2) \\ -16k^2 + 16dx + 8dk + d^2 \end{array} \right] + 8dx + 2d^2 \\
 &= m \left[\begin{array}{c} 8m^2(2x^2 + 14kx + dk + 10k^2) \\ +8m(2x^2 + dx - 16k^2 - 14kx + 2dk) \\ +32kx + 48k^2 + 16dx + 8dk + d^2 \end{array} \right] + 8dx + 2d^2 \\
 &:= X_2(m).
 \end{aligned}$$

Since $x + d > 0$, the coefficient $2x^2 + 14kx + dk + 10k^2 \geq 0$. Therefore using that $m^2 \geq 2m - 1$, we obtain

$$\begin{aligned}
 X_2(m) &= m \left[\begin{array}{c} 8m^2(2x^2 + 14kx + dk + 10k^2) \\ +8m(2x^2 + dx - 16k^2 - 14kx + 2dk) \\ +32kx + 48k^2 + 16dx + 8dk + d^2 \end{array} \right] + 8dx + 2d^2 \\
 &\geq m \left[\begin{array}{c} 8(2m - 1)(2x^2 + 14kx + dk + 10k^2) \\ +8m(2x^2 + dx - 16k^2 - 14kx + 2dk) \\ +32kx + 48k^2 + 16dx + 8dk + d^2 \end{array} \right] + 8dx + 2d^2
 \end{aligned}$$

$$\begin{aligned}
&= m \left[\frac{8m(6x^2 + 4dk + 14kx + 4k^2 + dx)}{+16dx - 32k^2 - 80kx - 16x^2 + d^2} \right] + 8dx + 2d^2 \\
&:= X_3(m)
\end{aligned}$$

Since $6x^2 + 4dk + 14kx + 4k^2 + dx$ is also nonnegative, we have for $m \geq 1$

$$\begin{aligned}
X_3(m) &\geq m[32x^2 + 24dx + 32kx + 32dk + d^2] + 8dx + 2d^2 \\
&:= X_4(m).
\end{aligned}$$

Since $x + d > 0$, we have $32x^2 + 24dx + 32kx + 32dk + d^2 \geq 0$. Thus we get for $m \geq 1$

$$\begin{aligned}
X_4(m) &\geq 32x^2 + 32dx + 32kx + 32dk + 3d^2 \\
&:= X_5.
\end{aligned}$$

Consequently, we have proved a chain of inequalities

$$T_2(m) \geq X_1(m) \geq X_2(m) \geq X_3(m) \geq X_4(m) \geq X_5 > 0.$$

This implies that the sequence $\{S_m\}_{m \geq 1}$, that is, $\{mA_m - (m+1)A_{m+1}\}_{m \geq 1}$ is strictly decreasing.

Consequently using Lemma 1.1, we obtain that $h_{k,v,a,d}(z) \in \mathcal{S}^*$. $\square$

REFERENCES

- [1] I. AKTAŞ, Á. BARICZ AND S. SINGH, *Geometric and monotonic properties of hyper-Bessel functions*, Ramanujan J. **51** (2020), 275–295.
- [2] I. AKTAS AND L-I. COTIRLA, *Certain geometrical properties and Hardy space of generalized k-Bessel Functions*, Symmetry **16** (12) (2024), 1597.
- [3] Á. BARICZ, *Geometric properties of generalized Bessel functions*, Publ. Math. Debrecen **73** (1–2) (2008), 155–178.
- [4] Á. BARICZ, M. ÇAĞLAR AND E. DENİZ, *Starlikeness of Bessel functions and their derivatives*, Math. Ineq. Appl. **19** (2) (2016), 439–449.
- [5] Á. BARICZ, E. DENİZ AND N. YAĞMUR, *Close-to-convexity of normalized Dini functions*, Math. Narch. **289** (2016), 1721–1726.
- [6] Á. BARICZ, P. KUMAR AND S. SINGH, *On starlikeness of regular Coulomb wave functions*, Proc. Amer. Math. Soc. **151** (6) (2023), 2325–2338.
- [7] Á. BARICZ AND S. PONNUSAMY, *Starlikeness and convexity of generalized Bessel functions*, Integr. Transforms Spec. Funct. **21** (2010), 641–653.
- [8] Á. BARICZ, S. PONNUSAMY AND S. SINGH, *Modified Dini functions: monotonicity patterns and functional inequalities*, Acta Math. Hungrica **149** (2016), 120–142.
- [9] T. BULBOACA AND H. M. ZAYED, *Analytical and geometrical approach to the generalized Bessel function*, J. Inequal. Appl. 2024, **51** (2024).
- [10] E. DENİZ AND Ş. GÖREN, *Geometric properties of generalized Dini functions*, Honam Math. J. **41** (1) (2019), 101–116.
- [11] E. DENİZ, H. ORHAN AND H. M. SRIVASTAVA, *Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions*, Taiwanese J. Math. **15** (2) (2011), 883–917.
- [12] R. DIAZ, C. ORTIZ AND E. PARIGUAN, *On the k-gamma q-distribution*, Centr. Eur. J. Math. **8** (2010), 448–458.

- [13] R. DIAZ AND E. PARIGUAN, *On hypergeometric functions and Poschhammer k -symbol*, Divulg. Math. **15** (2007), 179–192.
- [14] R. DIAZ AND C. TERUEL, *q , k -Generalized gamma and beta functions*, J. Nonlin. Math. Phys. **12** (2005), 118–134.
- [15] L. FEJER, *Untersuchungen über Potenzreihen mit mehrfach monotoner Koeffizientenfolge*, Acta Litterarum ac Scientiarum **8** (1936), 89–115.
- [16] P. HASTO, S. PONNUSAMY AND M. VUORINEN, *Starlikeness of the Gaussian hypergeometric functions*, Complex Var. Elliptic Equ. **55** (1–3) (2010), 173–184.
- [17] T. H. MACGREGOR, *A class of univalent functions*, Proc. Amer. Math. Soc. **15** (1964), 311–317.
- [18] S. R. MONDAL AND M. S. AKEL, *Differential equation and inequalities of the generalized k -Bessel functions*, J. Inequal. Appl. 2018 (14) Article **175** (2018) 1–14.
- [19] S. R. MONDAL AND A. SWAMINATHAN, *Geometric properties of generalized Bessel functions*, Bull. Malays. Math. Sci. Soc. (2) **35** (1) (2012), 179–194.
- [20] S. OWA, M. NUNUKAWA, H. SAITOH AND H. M. SRIVASTAVA, *Close-to-convexity, starlikeness, and convexity of certain analytic functions*, Appl. Math. Lett. **15** (2002), 63–69.
- [21] S. PONNUSAMY AND M. VUORINEN, *Univalence and convexity properties for confluent hypergeometric functions*, Complex Variables Theory Appl. **36** (1) (1998), 73–97.
- [22] G. N. WATSON, *A treatise on the theory of Bessel functions*, Second edition, Cambridge University Press, Cambridge, London and New York, 1944.
- [23] H. M. ZAYED, *Turán-type inequalities for generalized k -Bessel functions*, Ukr. Math. J. **76** (2) (2024) 254–279.

(Received September 2, 2025)

Erhan Deniz

Department of Mathematics

Faculty of Science and Letters, Kafkas University

Kars, Turkey

e-mail: edeniz36@gmail.com

Tayfun Çoban

Department of Mathematics

Faculty of Science and Letters, Kafkas University

Kars, Turkey

e-mail: coban.tayfun36@gmail.com

Adem Kızıltepe

Department of Mathematics

Faculty of Science and Letters, Kafkas University

Kars, Turkey

e-mail: admkz1tp@gmail.com