MULITDIMENSIONAL HARDY TYPE INEQUALITIES FOR $p < 0$ AND $0 < p < 1$

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Abstract. In this paper we establish some new multidimensional Hardy type inequalities for the cases $p < 0$ and $0 < p < 1$. These inequalities complement, generalize and unify most of the existing results of this type in the literature e.g. those in [4] and [9]. Some of the results are new also for the one dimensional case.

1. Introduction

In [5] Hardy announced and proved in [6] the following integral inequality (see also [8, Chapter 9, Theorem 328]): If $p > 1$, $f(x) \geq 0$, and $F(x) = \int_0^x f(t)dt$, then

$$\int_0^\infty \left( \frac{F(x)}{x} \right)^p dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty f^p dx.$$ (1.1)

The constant $\left( \frac{p}{p-1} \right)^p$ is the best possible.

Moreover, in 1928 Hardy [7] (see also [8, Chapter 9, Theorem 330, p. 245]) proved a generalized form of (1.1), namely that if $p > 1$, $m \neq 1$, and $F(x)$ is defined by

$$F(x) = \begin{cases} \int_0^x f(t)dt, & m > 1, \\ \int_x^\infty f(t)dt, & m < 1, \end{cases}$$ (1.2)

then

$$\int_0^\infty x^{-m}F^p dx \leq \left( \frac{p}{|m-1|} \right)^p \int_0^\infty x^{-m}(xf)^p dx.$$ (1.3)

The constant is the best possible.


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Furthermore, Hardy [7] (see also [8, Chapter 9, Theorem 347, p. 256]) pointed out that if \( m \) and \( F \) satisfy the conditions of the above result, but \( 0 < p < 1 \), then

\[
\int_0^\infty x^{-m} F^p dx \geq \left( \frac{P}{m-1} \right)^p \int_0^\infty x^{-m} (xf)^p dx.
\]  \( \text{(1.4)} \)

For further remarks concerning the history, development, generalizations and applications of inequalities (1.1) and (1.3) see for instance, [1], [2], [8], [11], [12], [15] and the references cited therein.

The first result for the unweighted one dimensional Hardy’s inequality in the discrete case for \( p < 0 \) was obtained in 1928 by Knopp [10] (see also [12] and the references cited therein). The weighted Hardy inequalities for negative powers appeared in the papers of Beesack and Heinig [1] and Heinig [2] where the cases \( p, q < 0 \) and \( 0 < p, q < 1 \) are considered. They studied the reverse Hardy inequality

\[
\left( \int_0^\infty [f(x)v(x)]^p dx \right)^{\frac{1}{p}} \leq C \left( \int_0^\infty \left[ u(x) \left( \int_0^x f(t) dt \right) \right]^q dx \right)^{\frac{1}{q}}
\]  \( \text{(1.5)} \)

and its dual version by deriving some necessary as well as some sufficient conditions for their validity.

The unweighted multidimensional Hardy-type inequalities for the cases \( p < 0 \) and \( 0 < p < 1 \) were studied in [9] by using a convexity argument.

In this paper we prove and discuss some weighted multidimensional Hardy type inequalities for the cases \( p < 0 \) and \( 0 < p < 1 \) of the type (1.3) for different values of \( m \). Some results are new also for the one dimensional case. The techniques that will be used in the proofs are mainly a convexity argument, which is very different from the classical methods used e.g. by Beesack and Heinig [1], Heinig [2] and Hardy [8].

The paper is organized as follows: In order not to disturb our discussions later on we use Section 2 to present some preliminaries, including some convexity results from the paper [14]. The main results are given in Section 3, while our concluding examples and remarks are presented in Section 4.

### 2. Preliminaries

Throughout the paper all functions are assumed to be measurable. Here and in the sequel the notations \( b, x, (0, b), (b, \infty), [b, \infty) \) as usual means \( b = (b_1, ..., b_n) \), \( x = (x_1, ..., x_n) \), \( (0, b) = \{ x \in \mathbb{R}^n : 0 < x_j < b_j, j = 1, 2, ..., n \} \), \( (b, \infty) = \{ x \in \mathbb{R}^n : b_j < x_j \leq \infty, j = 1, 2, ..., n \} \), \( [b, \infty) = \{ x \in \mathbb{R}^n : b_j \leq x_j < \infty, j = 1, 2, ..., n \} \) and \( b < x \) means that \( b_j < x_j, j = 1, 2, ..., n \). \( n \in \mathbb{Z}_+ \).

We now present some results in the recent paper [14], which are crucial to the proofs of our main results.
LEMMA 2.1. Let $b \in (0, \infty)$, $-\infty \leq a < c \leq \infty$ and let $\Phi$ be a positive function on $[a, c]$. Suppose that the weight function $u$ defined on $(0, b)$ is nonnegative such that $\frac{u(x_1, \ldots, x_n)}{x_1 \cdots x_n}$ is locally integrable on $(0, b)$ and the weight function $v$ is defined by

$$v(t_1, \ldots, t_n) = \frac{1}{t_1 \cdots t_n} \int_{t_1}^{b_1} \cdots \int_{t_n}^{b_n} u(x_1, \ldots, x_n) dx_1 \cdots dx_n, \quad t \in (0, b).$$

(i) If $\Phi$ is convex, then

$$\int_{0}^{b_1} \cdots \int_{0}^{b_n} u(x_1, \ldots, x_n) \Phi \left( \frac{1}{x_1 \cdots x_n} \int_{0}^{x_1} \cdots \int_{0}^{x_n} f(t_1, \ldots, t_n) dt_1 \cdots dt_n \right) dx_1 \cdots dx_n \leq \int_{0}^{b_1} \cdots \int_{0}^{b_n} v(x_1, \ldots, x_n) \Phi \left( f(x_1, \ldots, x_n) \right) dx_1 \cdots dx_n \tag{2.1}$$

holds for every function $f$ on $(0, b)$ such that $a < f(x_1, \ldots, x_n) < c$.

(ii) If $\Phi$ is concave, then

$$\int_{0}^{b_1} \cdots \int_{0}^{b_n} u(x_1, \ldots, x_n) \Phi \left( \frac{1}{x_1 \cdots x_n} \int_{0}^{x_1} \cdots \int_{0}^{x_n} f(t_1, \ldots, t_n) dt_1 \cdots dt_n \right) dx_1 \cdots dx_n \geq \int_{0}^{b_1} \cdots \int_{0}^{b_n} v(x_1, \ldots, x_n) \Phi \left( f(x_1, \ldots, x_n) \right) dx_1 \cdots dx_n \tag{2.2}$$

holds for every function $f$ on $(0, b)$ such that $a < f(x_1, \ldots, x_n) < c$.

**Proof.** The proof is easy and just a consequence of Jensen’s inequality and Fubini’s theorem (for details see [14]). □

LEMMA 2.2. Let $b \in [0, \infty)$, $-\infty \leq a < c \leq \infty$ and $\Phi$ be a positive function on $[a, c]$. Assume that the weight function $u$ defined on $[b, \infty)$ is nonnegative such that $\frac{u(x_1, \ldots, x_n)}{x_1 \cdots x_n}$ is locally integrable on $[b, \infty)$ and the weight function $v$ is defined by

$$v(t_1, \ldots, t_n) = \frac{1}{t_1 \cdots t_n} \int_{t_1}^{b_1} \cdots \int_{t_n}^{b_n} u(x_1, \ldots, x_n) dx_1 \cdots dx_n < \infty, \quad t \in (b, \infty).$$
(i) If $\Phi$ is convex, then
\[
\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} u(x_1, \ldots, x_n) \Phi \left( \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(t_1, \ldots, t_n) \frac{dt_1 \cdots dt_n}{t_1^2 \cdots t_n^2} \right) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}
\]
\leq \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} v(x_1, \ldots, x_n) \Phi (f(x_1, \ldots, x_n)) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}
\] (2.3)
holds for every function $f$ on $[b, \infty)$ such that $a < f(x_1, \ldots, x_n) < c$.

(ii) If $\Phi$ is concave, then
\[
\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} u(x_1, \ldots, x_n) \Phi \left( \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(t_1, \ldots, t_n) \frac{dt_1 \cdots dt_n}{t_1^2 \cdots t_n^2} \right) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}
\]
\geq \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} v(x_1, \ldots, x_n) \Phi (f(x_1, \ldots, x_n)) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}
\] (2.4)
holds for every function $f$ on $[b, \infty)$ such that $a < f(x_1, \ldots, x_n) < c$.

Proof. The proof follows by applying Jensen’s inequality and Fubini’s theorem (for details see [14]). □

3. Main results

Our first result reads:

THEOREM 3.1. Let $p < 0$, $b \in (0, \infty]$, and let $f$ be a nontrival and nonnegative function on $(0, b)$ and assume that

\[
0 < \int_{0}^{b_1} \cdots \int_{0}^{b_n} \int_{0}^{p-m} \cdots \int_{0}^{p-m} f^p(x_1, \ldots, x_n) dx_1 \cdots dx_n < \infty.
\]

If $m < 1$, then
\[
\int_{0}^{b_1} \cdots \int_{0}^{b_n} \int_{0}^{1-m} \cdots \int_{0}^{1-m} f(t_1, \ldots, t_n) dt_1 \cdots dt_n \]
\[
\leq \left( \frac{p}{m-1} \right)^{p n} \int_{0}^{b_1} \cdots \int_{0}^{b_n} \left[ 1 - \left( \frac{x_1}{b_1} \right)^{m-1} \right] \cdots \left[ 1 - \left( \frac{x_n}{b_n} \right)^{m-1} \right] \times
\]
\[
x_1^{p-m} \cdots x_n^{p-m} f^p(x_1, \ldots, x_n) dx_1 \cdots dx_n.
\] (3.1)
Proof. We use Lemma 2.1 (i) with the convex function $\Phi(x) = x^p$ and the weight function $u(x_1, ..., x_n) \equiv 1$ (so that $v(x_1, ..., x_n) = \left(1 - \frac{x_1}{b_1}\right) ... \left(1 - \frac{x_n}{b_n}\right)$). Then inequality (2.1) yields

$$\int_0^{b_1} \cdots \int_0^{b_n} \left( \frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, ..., t_n) \, dt_1 \cdots dt_n \right) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} \leq \int_0^{b_1} \cdots \int_0^{b_n} \left(1 - \frac{x_1}{b_1}\right) ... \left(1 - \frac{x_n}{b_n}\right) f^p(x_1, ..., x_n) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}. \quad (3.2)$$

Now, replace $b_j$ by $a_j = \frac{m-1}{b_j} p$ and choose for $f$ the function

$$x \mapsto f \left(\frac{x_1^p}{a_1^{m-1}}, ..., \frac{x_n^p}{a_n^{m-1}}\right) x_1^{m-1} \cdots x_n^{m-1} \frac{x_1^{m-1} \cdots x_n^{m-1}}{a_1^{m-1} \cdots a_n^{m-1}}. \quad \text{Thereafter, by using the substitutions}$$

$s_j = t_j^{\frac{p}{m-1}}$ and $y_j = x_j^{\frac{p}{m-1}}$, respectively, the left hand side of (3.2) becomes

$$\int_0^{a_1} \cdots \int_0^{a_n} \left( \frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f \left(\frac{x_1^p}{a_1^{m-1}}, ..., \frac{x_n^p}{a_n^{m-1}}\right) \frac{x_1^{m-1} \cdots x_n^{m-1}}{a_1^{m-1} \cdots a_n^{m-1}} \, dt_1 \cdots dt_n \right) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} \quad (3.3)$$

Similarly, the right hand side of (3.2) yields

$$\int_0^{\frac{b_1}{a_1}} \cdots \int_0^{\frac{b_n}{a_n}} \left(1 - \frac{x_1}{a_1}\right) ... \left(1 - \frac{x_n}{a_n}\right) f^p \left(\frac{x_1^p}{a_1^{m-1}}, ..., \frac{x_n^p}{a_n^{m-1}}\right) x_1^{m-1} \cdots x_n^{m-1} \frac{x_1^{m-1} \cdots x_n^{m-1}}{a_1^{m-1} \cdots a_n^{m-1}} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} \quad (3.4)$$

$$= \left(\frac{m-1}{p}\right)^n \int_0^{\frac{b_1}{a_1}} \cdots \int_0^{\frac{b_n}{a_n}} \left[1 - \left(\frac{y_1}{b_1}\right)^\frac{m-1}{p}\right] ... \left[1 - \left(\frac{y_n}{b_n}\right)^\frac{m-1}{p}\right] \times$$

$$f^p(y_1, ..., y_n) y_1^{p-m+1} \cdots y_n^{p-m+1} \frac{dy_1 \cdots dy_n}{y_1 \cdots y_n}$$
\[
\begin{align*}
&= \left( \frac{m-1}{p} \right)^n \int_{b_1}^{b_1} \ldots \int_{b_n}^{b_n} \left[ 1 - \left( \frac{y_1}{b_1} \right)^{m-1} \right] \ldots \left[ 1 - \left( \frac{y_n}{b_n} \right)^{m-1} \right] \times \\
&\quad y_1^{p-m} \ldots y_n^{p-m} f^p(y_1, \ldots, y_n) dy_1 \ldots dy_n.
\end{align*}
\] (3.4)

(3.1) follows by just combining (3.3) and (3.4). The proof is complete. \(\square\)

In the next theorem we state the dual of Theorem 3.1.

**Theorem 3.2.** Let \( p < 0, b \in [0, \infty) \), let \( f \) be a nontrivial and nonnegative function on \( (b, \infty) \). If \( m > 1 \) and

\[
0 < \int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} x_1^{p-m} \ldots x_n^{p-m} f^p(x_1, \ldots, x_n) dx_1 \ldots dx_n < \infty,
\]

then

\[
\int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} x_1^{p-m} \ldots x_n^{p-m} \left( \int_{x_1}^{\infty} \ldots \int_{x_n}^{\infty} f(t_1, \ldots, t_n) dt_1 \ldots dt_n \right)^p dx_1 \ldots dx_n
\]

\[
\leq \left( \frac{p}{1-m} \right)^{pn} \int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} \left[ 1 - \left( \frac{b_1}{x_1} \right)^{1-m} \right] \ldots \left[ 1 - \left( \frac{b_n}{x_n} \right)^{1-m} \right] \times \\
&\quad x_1^{p-m} \ldots x_n^{p-m} f^p(x_1, \ldots, x_n) dx_1 \ldots dx_n.
\]

**Proof.** We use Lemma 2.2 (i) with the convex function \( \Phi(x) = x^p \) and the weight function \( u(x_1, \ldots, x_n) \equiv 1 \) (so that \( v(x_1, \ldots, x_n) = \left( 1 - \frac{b_1}{x_1} \right) \ldots \left( 1 - \frac{b_n}{x_n} \right) \)). Then inequality (2.3) becomes

\[
\int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} \left( x_1 \ldots x_n \int_{x_1}^{\infty} \ldots \int_{x_n}^{\infty} f(t_1, \ldots, t_n) \frac{dt_1 \ldots dt_n}{t_1^{1-m} t_n^{1-m}} \right)^p x_1^{1-m} \ldots x_n^{1-m} dx_1 \ldots dx_n
\]

\[
\leq \int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} \left( 1 - \frac{b_1}{x_1} \right) \ldots \left( 1 - \frac{b_n}{x_n} \right) f^p(x_1, \ldots, x_n) x_1^{1-m} \ldots x_n^{1-m} dx_1 \ldots dx_n.
\]

(3.6)

Now, replace \( b_j \) by \( a_j = \frac{b_j^p}{p} \) and choose for \( f \) the function

\[ x \mapsto f \left( \frac{p}{x_1^{1-m}}, \ldots, \frac{p}{x_n^{1-m}} \right) x_1^{\frac{p}{1-m}+1} \ldots x_n^{\frac{p}{1-m}+1}. \] Thereafter, use the substitutions \( s_j = \frac{p}{t_j^{1-m}} \). \]
and \( y_j = x_j^{\frac{p}{m}} \), respectively. Then the left hand side of (3.6) yields

\[
\int_{a_1}^{\infty} \ldots \int_{a_n}^{\infty} \left( \int_{x_1}^{\infty} \ldots \int_{x_n}^{\infty} f \left( t_1^{\frac{p}{m}} \ldots t_n^{\frac{p}{m}} \right) \left( \frac{t_1^{\frac{p}{m}} + 1}{t_1^{\frac{p}{m}} + 1} \ldots \frac{t_n^{\frac{p}{m}} + 1}{t_n^{\frac{p}{m}} + 1} \right) dt_1 \ldots dt_n \right) \frac{dx_1 \ldots dx_n}{x_1 \ldots x_n} \]

\[
= \left( \frac{1 - m}{p} \right)^{pn} \int_{a_1}^{\infty} \ldots \int_{a_n}^{\infty} \left( \int_{y_1}^{\infty} \ldots \int_{y_n}^{\infty} f \left( s_1, \ldots, s_n \right) ds_1 \ldots ds_n \right) x_1^{p-1} \ldots x_n^{p-1} dy_1 \ldots dy_n
\]

\[
= \left( \frac{1 - m}{p} \right)^{pn + n} \int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} \left( \int_{y_1}^{\infty} \ldots \int_{y_n}^{\infty} f \left( s_1, \ldots, s_n \right) ds_1 \ldots ds_n \right) x_1^{p-1} \ldots x_n^{p-1} dy_1 \ldots dy_n.
\]

Similarly, the right hand side of (3.6) reads

\[
\int_{a_1}^{\infty} \ldots \int_{a_n}^{\infty} \left( 1 - \frac{a_1}{x_1} \right) \ldots \left( 1 - \frac{a_n}{x_n} \right) f^{p} \left( x_1^{\frac{p}{m}} \ldots x_n^{\frac{p}{m}} \right) \left( 1 - \frac{b_1}{y_1} \right)^{\frac{1 - m}{p}} \ldots \left( 1 - \frac{b_n}{y_n} \right)^{\frac{1 - m}{p}} \frac{dx_1 \ldots dx_n}{x_1 \ldots x_n}
\]

\[
= \left( \frac{1 - m}{p} \right)^{n} \int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} \left[ 1 - \left( \frac{b_1}{y_1} \right)^{\frac{1 - m}{p}} \right] \ldots \left[ 1 - \left( \frac{b_n}{y_n} \right)^{\frac{1 - m}{p}} \right] y_1^{p-m} \ldots y_n^{p-m} f^{p} \left( y_1, \ldots, y_n \right) dy_1 \ldots dy_n.
\]

Inequality (3.5) follows by combining (3.7) and (3.8). The proof is complete. \( \square \)

Our next result, which deals with the case \( 0 < p < 1 \), is the following:

THEOREM 3.3. \( \text{Let } 0 < p < 1, \ b \in (0, \infty], \) and let \( f \) be a nontrival and nonnegative function on \((0, b)\) and assume that

\[
0 < \int_{0}^{b_1} \ldots \int_{0}^{b_n} x_1^{p-m} \ldots x_n^{p-m} f^{p} \left( x_1, \ldots, x_n \right) dx_1 \ldots dx_n < \infty.
\]
If $m > 1$, then
\[
\int_{\mathbf{b}} \cdots \int_{\mathbf{b}} \left( \int_{\mathbf{b}} \cdots \int_{\mathbf{b}} f(t_1, \ldots, t_n) \, dt_1 \cdots dt_n \right) \, dx_1 \cdots dx_n
\]
\[
\geq \left( \frac{p}{m - 1} \right)^{p^n} \int_{\mathbf{b}} \cdots \int_{\mathbf{b}} \left[ 1 - \left( \frac{x_1}{b_1} \right)^{\frac{m-1}{p}} \right] \cdots \left[ 1 - \left( \frac{x_n}{b_n} \right)^{\frac{m-1}{p}} \right] \times
\]
\[
x_1^{p-m} \cdots x_n^{p-m} f^p(x_1, \ldots, x_n) \, dx_1 \cdots dx_n.
\]

**Proof.** The proof is completely similar to that of Theorem 3.1 and, hence, the details are omitted. In this case we note that the function $\Phi(x) = x^p$, for $0 < p < 1$ is concave and thus only the inequality signs are reversed. □

Our final result in this section is the following:

**Theorem 3.4.** Let $0 < p < 1$, $\mathbf{b} \in [0, \infty)$, let $f$ be a nontrivial and nonnegative function on $(\mathbf{b}, \infty)$, and let $m < 1$, and
\[
0 < \int_{\mathbf{b}_1}^{\infty} \cdots \int_{\mathbf{b}_n}^{\infty} x_1^{p-m} \cdots x_n^{p-m} f^p(x_1, \ldots, x_n) \, dx_1 \cdots dx_n < \infty.
\]

Then
\[
\int_{\mathbf{b}_1}^{\infty} \cdots \int_{\mathbf{b}_n}^{\infty} x_1^{p-m} \cdots x_n^{p-m} \left( \int_{\mathbf{x}_1}^{\infty} \cdots \int_{\mathbf{x}_n}^{\infty} f(t_1, \ldots, t_n) \, dt_1 \cdots dt_n \right) \, dx_1 \cdots dx_n
\]
\[
\geq \left( \frac{p}{1 - m} \right)^{p^n} \int_{\mathbf{b}_1}^{\infty} \cdots \int_{\mathbf{b}_n}^{\infty} \left[ 1 - \left( \frac{b_1}{x_1} \right)^{\frac{1-m}{p}} \right] \cdots \left[ 1 - \left( \frac{b_n}{x_n} \right)^{\frac{1-m}{p}} \right] \times
\]
\[
x_1^{p-m} \cdots x_n^{p-m} f^p(x_1, \ldots, x_n) \, dx_1 \cdots dx_n.
\]

**Proof.** The proof is completely similar to the proof of Theorem 3.2. In this case we note that the function $\Phi(x) = x^p$, for $0 < p < 1$ is concave and, hence, only the inequality signs are reversed. □

4. Concluding examples and remarks

By using our Theorems 3.1 and Corollary 2.1 (i) in [14] for the cases $b_j = \infty$, $j = 1, 2, \ldots, n$, and $m = p$, respectively, we obtain the following multidimensional Hardy type inequalities:
EXAMPLE 4.1. If $n \in \mathbb{Z}_+$ and $p < 0$, $m < 1$ or $p > 1$, $m > 1$, then
\[
\int_0^\infty \cdots \int_0^\infty x_1^{-m} \cdots x_n^{-m} \left( \int_0^\infty \cdots \int_0^\infty f(t_1, \ldots, t_n) dt_1 \cdots dt_n \right)^p dx_1 \cdots dx_n \\
\leq \left( \frac{p}{m-1} \right)^{pn} \int_0^\infty \cdots \int_0^\infty x_1^{-p} \cdots x_n^{-p} f^p(x_1, \ldots, x_n) dx_1 \cdots dx_n. \quad (4.1)
\]

EXAMPLE 4.2. If $n \in \mathbb{Z}_+$ and $p < 0$ or $p > 1$, then
\[
\int_0^\infty \cdots \int_0^\infty \left( \int_0^{b_1} \cdots \int_0^{b_n} \left( \int_0^\infty \cdots \int_0^\infty f(t_1, \ldots, t_n) dt_1 \cdots dt_n \right) dx_1 \cdots dx_n \right)^p dx_1 \cdots dx_n \\
\leq \left( \frac{p}{p-1} \right)^{pn} \int_0^\infty \cdots \int_0^\infty \left[ 1 - \left( \frac{x_1}{b_1} \right)^{p-1} \right] \cdots \left[ 1 - \left( \frac{x_n}{b_n} \right)^{p-1} \right] f^p(x_1, \ldots, x_n) dx_1 \cdots dx_n. \quad (4.2)
\]

REMARK 4.1. By using Theorem 3.3 we see that if $0 < p < 1$, $m > 1$, then (4.1) holds in the reversed direction and if $0 < p < 1$, then (4.2) holds in the reversed direction.

By using our Theorem 3.2 and Corollary 2.1 (ii) in [14] for the cases $b_j = \infty$, $j = 1, 2, \ldots, n$, we obtain the following dual version of Example 4.1:

EXAMPLE 4.3. If $n \in \mathbb{Z}_+$ and $p < 0$, $m > 1$ or $p > 1$, $m < 1$, then
\[
\int_0^\infty \cdots \int_0^\infty x_1^{-m} \cdots x_n^{-m} \left( \int_0^\infty \cdots \int_0^\infty f(t_1, \ldots, t_n) dt_1 \cdots dt_n \right)^p dx_1 \cdots dx_n \\
\leq \left( \frac{p}{1-m} \right)^{pn} \int_0^\infty \cdots \int_0^\infty x_1^{-p} \cdots x_n^{-p} f^p(x_1, \ldots, x_n) dx_1 \cdots dx_n. \quad (4.3)
\]

REMARK 4.2. By using Theorem 3.4 we find that (4.3) holds in the reversed direction if $0 < p < 1, m < 1$.

REMARK 4.3. We remark that for the case $n = 1, p > 1$ (4.1)-(4.2) coincides with the weighted Hardy inequality (1.3). Moreover, according to Remarks 4.1 and 4.2, we obtain the reversed inequality (1.4) for the case $n = 1, 0 < p < 1$.

REMARK 4.4. For the special case $m = p$, $0 < p < 1$ in Theorem 3.4, the inequality (3.10) coincides with inequality (2.4) in [9, Corollary 2.2 (b)]. In particular, for $n = 1, b_1 = 0$ (3.10) reduces to [8, Theorem 337, p. 251].
We believe that the result in Theorem 3.4 is new also in the case \( n = 1 \):

**EXAMPLE 4.4.** If \( 0 \leq b < \infty, \ 0 < p < 1 \) and \( m < 1 \), then

\[
\int_{b}^{\infty} x^{-m} \left( \int_{x}^{\infty} f(t) \, dt \right)^{p} \, dx \geq \left( \frac{p}{1 - m} \right)^{p} \int_{b}^{\infty} \left[ 1 - \left( \frac{b}{x} \right)^{\frac{1-m}{p}} \right] x^{-m} f^{p}(x) \, dx. \tag{4.4}
\]

In particular, if \( m = 0 \) (4.4) reads:

\[
\int_{b}^{\infty} \left( \int_{x}^{\infty} f(t) \, dt \right)^{p} \, dx \geq p^{p} \int_{b}^{\infty} \left[ 1 - \left( \frac{b}{x} \right)^{\frac{1}{p}} \right] x^{p} f^{p}(x) \, dx.
\]

**REMARK 4.5.** Some complementary results to those obtained in this paper for the case \( p > 1 \) are recently proved and discussed in [14].

**REMARK 4.6.** For the case \( p > 1 \) some multidimensional Hardy type inequalities were also proved in [3]. They used another mixed mean inequality technique and it may very well be possible to prove some of the results in this paper by using this technique.

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