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# ON WEIGHTED SIMPSON TYPE INEQUALITIES AND APPLICATIONS

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*Abstract.* In this paper we establish some weighted Simpson type inequalities and give several applications for the $r-moments$ and the expectation of a continuous random variable. An approximation for Euler's Beta mapping is given as well.

## 1. Introduction

The *Simpson's inequality*, states that if $f^{(4)}$ exists and is bounded on $(a,b)$, then

$$\left|\int_a^b f(t)dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leqslant \frac{(b-a)^5}{2880}\left\|f^{(4)}\right\|_\infty, \tag{1.1}$$

where

$$\left\|f^{(4)}\right\|_\infty := \sup_{t\in(a,b)}\left|f^{(4)}(t)\right| < \infty.$$

Now if we assume that $I_n : a = x_0 < x_1 < \cdots < x_n = b$ is a partition of the interval $[a,b]$ and $f$ is as above, then we can approximate the integral $\int_a^b f(t)\,dt$ by the *Simpson's quadrature formula* $A_S(f,I_n)$, having an error given by $R_S(f,I_n)$, where

$$A_S(f,I_n) := \sum_{i=0}^{n-1}\frac{l_i}{3}\left[\frac{f(x_i)+f(x_{i+1})}{2} + 2f\left(\frac{x_i+x_{i+1}}{2}\right)\right], \tag{1.2}$$

and the remainder $R_S(f,I_n) = \int_a^b f(t)\,dt - A_S(f,I_n)$ satisfies the estimation

$$|R_S(f,I_n)| \leqslant \frac{1}{2880}\left\|f^{(4)}\right\|_\infty \sum_{i=0}^{n-1} l_i^5, \tag{1.3}$$

with $l_i := x_{i+1} - x_i$ for $i = 0,1,\ldots,n-1$.

For some recent results which generalize, improve and extend this classic inequality (1.1), see the papers [2] – [7] and [9] – [12].

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Recently, Dragomir [6], (see also the survey paper authored by Dragomir, Agarwal and Cerone [7]) has proved the following two Simpson type inequalities for functions of bounded variation:

THEOREM 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation. Then*

$$\left|\int_a^b f(t)dt-\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\leqslant\frac{1}{3}(b-a)\bigvee_a^b(f), \tag{1.4}$$

*where $\bigvee_a^b(f)$ denotes the total variation of $f$ on the interval $[a,b]$. The constant $\frac{1}{3}$ is the best possible.*

Let $I_n$, $l_i$ $(i=0,1,\ldots,n-1)$, $A_S(f,I_n)$ and $R_S(f,I_n)$ be as above. We have the following result concerning the approximation of the integral $\int_a^b f(t)dt$ in terms of $A_S(f,I_n)$.

THEOREM 2. *Let $f$ be defined as in Theorem 1. Then the remainder*

$$R_S(f,I_n)=\int_a^b f(x)dx-A_S(f,I_n) \tag{1.5}$$

*satisfies the estimate*

$$|R_S(f,I_n)|\leqslant\frac{1}{3}\nu(l)\bigvee_a^b(f), \tag{1.6}$$

*where $\nu(l):=\max\{l_i\,|i=0,1,\ldots,n-1\}$. The constant $\frac{1}{3}$ is best posible in (1.6).*

In this paper, we establish some generalizations of Theorems 1–2, and give several applications for the $r-moments$ and expectation of a continuous random variable. Approximations for Euler's Beta mapping are also provided.

## 2. Some Integral Inequalities

We may state and prove the following main result:

THEOREM 3. *Let $g:[a,b]\to\mathbb{R}$ be positive and continuous and let $h(x)=\int_a^x g(t)dt, x\in[a,b]$. Let $f$ be as in Theorem 3. Then*

$$\left|\int_a^b f(t)g(t)\,dt-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(h^{-1}(x)\right)\right]\int_a^b g(t)\,dt\right|$$
$$\leqslant\left[\frac{1}{3}h(b)+\left|x-\frac{h(b)}{2}\right|\right]\cdot\bigvee_a^b(f), \tag{2.1}$$

*for all $x\in\left[\frac{h(b)}{6},\frac{5h(b)}{6}\right]$, where $\bigvee_a^b(f)$ denotes the total variation of $f$ on the interval $[a,b]$. The constant $\frac{1}{3}$ is the best possible.*

*Proof.* Fix $x \in \left[\frac{h(b)}{6}, \frac{5h(b)}{6}\right]$. Define

$$s(t) := \begin{cases} h(t) - \frac{h(b)}{6}, & t \in \left[a, h^{-1}(x)\right) \\ h(t) - \frac{5h(b)}{6}, & t \in \left[h^{-1}(x), b\right] \end{cases}.$$

By integration by parts, we have the following identity

$$\begin{aligned} \int_a^b s(t)\,df(t) &= \left[\left(h(t) - \frac{h(b)}{6}\right) f(t) \,|_a^{h^{-1}(x)} - \int_a^{h^{-1}(x)} f(t) g(t)\,dt\right] \\ &\quad + \left[\left(h(t) - \frac{5h(b)}{6}\right) f(t) \,|_{h^{-1}(x)}^{b} - \int_{h^{-1}(x)}^{b} f(t) g(t)\,dt\right] \\ &= \frac{1}{3} h(b) \left[\frac{f(a)+f(b)}{2} + 2f\left(h^{-1}(x)\right)\right] - \int_a^b f(t) g(t)\,dt \\ &= \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(h^{-1}(x)\right)\right] \int_a^b g(t)\,dt - \int_a^b f(t) g(t)\,dt. \end{aligned} \tag{2.2}$$

It is well known (see for instance [1, p. 159]) that, if $\mu, \nu : [a,b] \to \mathbb{R}$ are such that $\mu$ is continuous on $[a,b]$ and $\nu$ is of bounded variation on $[a,b]$, then $\int_a^b \mu(t)\,d\nu(t)$ exists and [1, p. 177]

$$\left|\int_a^b \mu(t)\,d\nu(t)\right| \leqslant \sup_{t\in[a,b]} |\mu(t)| \bigvee_a^b (\nu). \tag{2.3}$$

Now, using (2.2) and (2.3), we have

$$\begin{aligned} &\left|\int_a^b f(t) g(t)\,dt - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(h^{-1}(x)\right)\right] \int_a^b g(t)\,dt\right| \\ &\qquad \leqslant \sup_{t\in[a,b]} |s(t)| \bigvee_a^b (f). \end{aligned} \tag{2.4}$$

Since $h(t) - \frac{h(b)}{6}$ is increasing on $\left[a, h^{-1}(x)\right)$, $h(t) - \frac{5h(b)}{6}$ is increasing on $\left[h^{-1}(x), b\right]$ and the fact that $\max\{c,d\} = \frac{c+d}{2} + \frac{1}{2}|c-d|$ for any real $c$ and $d$, hence we have

$$\sup_{t\in[a,b]} |s(t)| = \max\left\{\frac{h(b)}{6}, x - \frac{h(b)}{6}, \frac{5h(b)}{6} - x\right\}$$

and

$$\begin{aligned}\sup_{t\in[a,b]}|s(t)| &= \max\left\{\frac{h(b)}{6}, x-\frac{h(b)}{6}, \frac{5h(b)}{6}-x\right\}\\ &= \max\left\{x-\frac{h(b)}{6}, \frac{5h(b)}{6}-x\right\}\\ &= \frac{1}{2}\left[\left(x-\frac{h(b)}{6}\right)+\left(\frac{5h(b)}{6}-x\right)\right]+\frac{1}{2}\left|\left(x-\frac{h(b)}{6}\right)-\left(\frac{5h(b)}{6}-x\right)\right|\\ &= \frac{h(b)}{3}+\left|x-\frac{h(b)}{2}\right|\\ &= \frac{1}{3}\int_a^b g(t)\,dt+\left|x-\frac{1}{2}\int_a^b g(t)dt\right|. \qquad (2.5)\end{aligned}$$

Thus, by (2.4) and (2.5), we obtain the desired inequality (2.1).

Let us consider the particular functions:

$$g(t)\equiv 1, \quad t\in[a,b],$$
$$h(t)=t-a, \quad t\in[a,b],$$
$$f(t)=\begin{cases}1 & \text{as } t\in\left[a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right]\\ -1 & \text{as } t=\frac{a+b}{2}\end{cases}$$

and $x=\frac{b-a}{2}$. Since for these choices we get equality in (2.1), it is easy to see that the constant $\frac{1}{3}$ is the best possible constant in (2.1). This completes the proof. □

REMARK 1.

(1) If we choose $g(t)\equiv 1$, $h(t)=t-a$ on $[a,b]$ and $x=\frac{b-a}{2}$, then the inequality (2.1) reduces to (1.4).

(2) If we choose $x=\frac{h(b)}{2}$, then we get

$$\left|\int_a^b f(t)g(t)\,dt-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(h^{-1}\left(\frac{h(b)}{2}\right)\right)\right]\int_a^b g(t)\,dt\right| \leqslant \frac{1}{3}\int_a^b g(t)\,dt\cdot\bigvee_a^b(f). \qquad (2.6)$$

Under the conditions of Theorem 3, we have the following corollaries.

COROLLARY 1. *Let $f\in C^{(1)}[a,b]$. Then we have the inequality*

$$\left|\int_a^b f(t)g(t)\,dt-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(h^{-1}(x)\right)\right]\int_a^b g(t)\,dt\right| \leqslant \left[\frac{1}{3}\int_a^b g(t)\,dt+\left|x-\frac{h(b)}{2}\right|\right]\|f'\|_1, \qquad (2.7)$$

*for all* $x \in \left[\frac{h(b)}{6}, \frac{5h(b)}{6}\right]$, *where* $\|\cdot\|_1$ *is the* $L_1-$*norm, namely*

$$\|f'\|_1 := \int_a^b |f'(t)|\, dt.$$

COROLLARY 2. *Let* $f : [a,b] \to \mathbb{R}$ *be a Lipschitzian mapping with the constant* $M > 0$. *Then we have the inequality*

$$\left|\int_a^b f(t)g(t)\,dt - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(h^{-1}(x)\right)\right]\int_a^b g(t)\,dt\right| \leqslant \left[\frac{1}{3}\int_a^b g(t)\,dt + \left|x - \frac{h(b)}{2}\right|\right](b-a)M, \tag{2.8}$$

*for all* $x \in \left[\frac{h(b)}{6}, \frac{5h(b)}{6}\right]$.

COROLLARY 3. *Let* $f : [a,b] \to \mathbb{R}$ *be a monotonic mapping. Then we have the inequality*

$$\left|\int_a^b f(t)g(t)\,dt - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(h^{-1}(x)\right)\right]\int_a^b g(t)\,dt\right| \leqslant \left[\frac{1}{3}\int_a^b g(t)\,dt + \left|x - \frac{h(b)}{2}\right|\right]\cdot |f(b) - f(a)| \tag{2.9}$$

*for all* $x \in \left[\frac{h(b)}{6}, \frac{5h(b)}{6}\right]$.

## 3. Applications for Quadrature Formulae

Throughout this section, let $g$, $h$ be as in Theorem 3, $f : [a,b] \to \mathbb{R}$, and let $I_n : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$, and $h_i(x) = \int_{x_i}^x g(t)dt$, $x \in [x_i, x_{i+1}]$, $\xi_i \in \left[\frac{h(x_{i+1})}{6}, \frac{5h(x_{i+1})}{6}\right]$ $(i = 0, 1, \ldots, n-1)$ are intermediate points. Put $L_i := h_i(x_{i+1}) = \int_{x_i}^{x_{i+1}} g(t)\,dt$ and define the sum

$$A_S(f,g,I_n,\xi) := \sum_{i=0}^{n-1} \frac{L_i}{3}\left[\frac{f(x_i) + f(x_{i+1})}{2} + 2f\left(h^{-1}(\xi_i)\right)\right]$$

and

$$R_S(f,g,I_n,\xi) = \int_a^b f(t)g(t)dx - A_S(f,g,I_n,\xi).$$

We have the following approximation of the integral $\int_a^b f(t)g(t)\,dt$.

THEOREM 4. *Let* $f$ *be defined as in Theorem 3 and let*

$$\int_a^b f(t)g(t)\,dt = A_S(f,g,I_n,\xi) + R_S(f,g,I_n,\xi). \tag{3.1}$$

*Then, the remainder term* $R_S(f,g,h,I_n,\xi)$ *satisfies the estimate*

$$\begin{aligned}|R_S(f,g,h,I_n,\xi)| &\leqslant \left[\frac{1}{3}\nu(L)+\max_{i=0,1,\dots,n-1}\left|\xi_i-\frac{h_i(x_{i+1})}{2}\right|\right]\bigvee_a^b(f)\\ &\leqslant \frac{2}{3}\nu(L)\bigvee_a^b(f),\end{aligned}\tag{3.2}$$

*where* $\nu(L):=\max\{L_i\,|i=0,1,\dots,n-1\}$. *The constant* $\frac{1}{3}$ *in the first inequality of (3.2) is the best possible.*

*Proof.* Apply Theorem 3 on the intervals $[x_i,x_{i+1}]$ $(i=0,1,\dots,n-1)$ to get

$$\left|\int_{x_i}^{x_{i+1}} f(t)g(t)\,dt-\frac{l_i}{3}\left[\frac{f(x_i)+f(x_{i+1})}{2}+2f\left(h_i^{-1}(\xi_i)\right)\right]\right| \leqslant \left[\frac{1}{3}L_i+\left|\xi_i-\frac{h_i(x_{i+1})}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}(f),$$

for all $i=0,1,\dots,n-1$. Using this and the generalized triangle inequality, we have

$$\begin{aligned}|R_S(f,g,I_n,\xi)| &\leqslant \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(t)g(t)\,dt-\frac{L_i}{3}\left[\frac{f(x_i)+f(x_{i+1})}{2}+2f\left(h_i^{-1}(\xi_i)\right)\right]\right|\\ &\leqslant \sum_{i=0}^{n-1}\left[\frac{1}{3}L_i+\left|\xi_i-\frac{h_i(x_{i+1})}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}(f)\\ &\leqslant \max_{i=0,1,\dots,n-1}\left[\frac{1}{3}L_i+\left|\xi_i-\frac{h_i(x_{i+1})}{2}\right|\right]\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)\\ &\leqslant \left[\frac{1}{3}\nu(L)+\max_{i=0,1,\dots,n-1}\left|\xi_i-\frac{h_i(x_{i+1})}{2}\right|\right]\bigvee_a^b(f)\end{aligned}$$

and the first inequality in (3.2) is proved.

For the second inequality in (3.2), we observe that

$$\left|\xi_i-\frac{h_i(x_{i+1})}{2}\right|\leqslant\frac{1}{3}L_i\ (i=0,1,\dots,n-1);$$

and then

$$\max_{i=0,1,\dots,n-1}\left|\xi_i-\frac{h(x_i)+h(x_{i+1})}{2}\right|\leqslant\frac{1}{3}\nu(L).$$

Thus the theorem is proved. □

REMARK 2. If we choose $g(t)\equiv 1$, then $h(t)=t-a$ on $[a,b]$, $\xi_i=\frac{x_{i+1}-x_i}{2}$ $(i=0,1,\dots,n-1)$, and the first inequality in (3.2) reduces to (1.6).

The following corollaries are useful in practice.

COROLLARY 4. *Let $f : [a,b] \to \mathbb{R}$ be a Lipschitzian mapping with the constant $M > 0$, $I_n$ be defined as above and choose $\xi_i = \frac{h_i(x_{i+1})}{2}$ $(i = 0,1,\ldots,n-1)$. Then we have the formula*

$$\begin{aligned} \int_a^b f(t)g(t)\,dt &= A_S(f,g,I_n,\xi) + R_S(f,g,I_n,\xi) \\ &= \sum_{i=0}^{n-1} \frac{L_i}{3}\left[\frac{f(x_i)+f(x_{i+1})}{2} + 2f\left(h_i^{-1}(\xi_i)\right)\right] + R_S(f,g,I_n,\xi) \end{aligned} \tag{3.3}$$

*and the remainder satisfies the estimate*

$$|R_S(f,g,I_n,\xi)| \leqslant \frac{\nu(L)\cdot M\cdot(b-a)}{3}. \tag{3.4}$$

COROLLARY 5. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic mapping and let $\xi_i$ $(i = 0,1,\ldots,n-1)$ be defined as in Corollary 4. Then we have the formula (3.3) and the remainder satisfies the estimate*

$$|R_S(f,g,I_n,\xi)| \leqslant \frac{\nu(L)}{3}\cdot|f(b)-f(a)|. \tag{3.5}$$

The case of equidistant division is embodied in the following corollary and remark:

COROLLARY 6. *Suppose that $G(x) = \int_a^x g(t)dt, x \in [a,b]$,*

$$x_i = G^{-1}\left(\frac{i}{n}\int_a^b g(t)dt\right)\ (i = 0,1,\ldots,n),$$

$$h_i(x) = \int_{x_i}^x g(t)dt, x \in [x_i,x_{i+1}], (i = 0,1,\ldots,n-1),$$

*and*

$$L_i := h_i(x_{i+1}) = G(x_{i+1}) - G(x_i) = \frac{1}{n}\int_a^b g(t)\,dt \quad (i = 0,1,\ldots,n-1).$$

*Let $f$ be defined as in Theorem 4 and choose $\xi_i = \frac{h_i(x_{i+1})}{2}$ $(i = 0,1,\ldots,n-1)$. Then we have the formula*

$$\begin{aligned} \int_a^b f(t)g(t)\,dt &= A_S(f,g,h,I_n,\xi) + R_S(f,g,h,I_n,\xi) \\ &= \frac{1}{3n}\sum_{i=0}^{n-1}\left[\frac{f(x_i)+f(x_{i+1})}{2} + 2f\left(h_i^{-1}\left(\frac{h_i(x_{i+1})}{2}\right)\right)\right]\int_a^b g(t)\,dt \\ &\quad + R_S(f,g,h,I_n,\xi) \end{aligned} \tag{3.6}$$

*and the remainder satisfies the estimate*

$$|R_S(f,g,h,I_n,\xi)| \leqslant \frac{1}{3n}\bigvee_a^b(f)\int_a^b g(t)\,dt. \tag{3.7}$$

REMARK 3. If we want to approximate the integral $\int_a^b f(t)\,g(t)\,dt$ by $A_S(f,g,h,I_n,\xi)$ with an error less that $\varepsilon > 0$, then we need at least $n_\varepsilon \in N$ points for the partition $I_n$, where

$$n_\varepsilon := \left[ \frac{1}{3\varepsilon} \int_a^b g(t)\,dt \cdot \bigvee_a^b (f) \right] + 1$$

and $[r]$ denotes the Gaussian integer of $r \in \mathbb{R}$.

## 4. Some Inequalities for Random Variables

Throughout this section, let $0 < a < b$, $r \in \mathbb{R}$, and let $X$ be a continuous random variable having the continuous probability density function $g : [a,b] \to [0,\infty)$ and assume the $r$*-moment*, defined by

$$E_r(X) := \int_a^b t^r g(t)\,dt,$$

is finite.

THEOREM 5. *The inequality*

$$\left| E_r(X) - \frac{1}{6}\left[ a^r + 4\left( h^{-1}\left(\frac{1}{2}\right)\right)^r + b^r \right] \right| \leqslant \frac{1}{3}\left| b^r - a^r \right| \tag{4.1}$$

*holds, where* $h(t) = \int_a^t g(x)\,dx \ \ (t \in [a,b])$.

*Proof.* If we put $f(t) = t^r$ and $x = \frac{h(b)}{2} = \frac{1}{2}$ in Corollary 3, then we obtain the inequality

$$\left| \int_a^b f(t) g(t)\,dt - \frac{1}{3}\left[ \frac{f(a)+f(b)}{2} + 2f\left( h^{-1}\left(\frac{1}{2}\right)\right)\right] \int_a^b g(t)\,dt \right| \leqslant \frac{1}{3}\left| f(b) - f(a) \right| \int_a^b g(t)\,dt. \tag{4.2}$$

Since

$$\int_a^b f(t) g(t)\,dt = E_r(X), \qquad \int_a^b g(t)\,dt = 1,$$

$$\frac{f(a)+f(b)}{2} = \frac{a^r+b^r}{2}, \quad \text{and} \quad \left| f(b) - f(a) \right| = \left| b^r - a^r \right|,$$

(4.1) follows from (4.2). □

If we choose $r = 1$ in Theorem 5, then we have the following remark:

REMARK 4. If $E(X)$ is the expectation of random variable $X$, then

$$\left| E(X) - \frac{1}{6}\left[ a + 4h^{-1}\left(\frac{1}{2}\right) + b \right] \right| \leqslant \frac{b-a}{3}. \tag{4.3}$$

## 5. Inequality for the Beta Mapping

The following mapping is well-known in the literature as the *Beta mapping*:

$$\beta(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt,\ \ p>0,\ q>0.$$

The following result may be stated:

THEOREM 6. *Let $p>0,\ q>1$. Then the inequality*

$$\left|\beta(p,q) - \frac{1}{np}\sum_{i=0}^{n-1}\left\{\frac{1}{6}\left(\left[1-\left(\frac{i}{n}\right)^{\frac{1}{p}}\right]^{q-1} + \left[1-\left(\frac{i+1}{n}\right)^{\frac{1}{p}}\right]^{q-1}\right) + \frac{2}{3}\left[1-\left(\frac{2i+1}{2n}\right)^{\frac{1}{p}}\right]^{q-1}\right\}\right| \leqslant \frac{1}{3np} \tag{5.1}$$

*holds for any positive integer $n$.*

*Proof.* If we put $a=0$, $b=1$, $f(t)=(1-t)^{q-1}$, $g(t)=t^{p-1}$ and $G(t)=\frac{t^p}{p}$ $(t\in[0,1])$ in Corollary 6, then,

$$\int_a^b g(t)dt = \frac{1}{p}, x_i = \left(\frac{i}{n}\right)^{\frac{1}{p}} \quad (i=0,1,\ldots,n),$$

$$h_i(x) = \frac{nx^p - i}{np} \quad (x\in[x_i,x_{i+1}],\ i=0,1,\ldots,n-1),$$

$$h_i^{-1}\left(\frac{h_i(x_{i+1})}{2}\right) = \left(\frac{2i+1}{2n}\right)^{\frac{1}{p}} \quad (i=0,1,\ldots,n-1)$$

and $\bigvee_a^b(f)=1$, so that the inequality (5.1) holds. □

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