NORM INEQUALITIES FOR THE CHAOTICALLY GEOMETRIC MEAN AND ITS REVERSE

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Abstract. Let A and B be strictly positive operators on a Hilbert space H such that 0 < m ≤ B ≤ M for some scalars 0 < m < M and \( h_B = \frac{M}{m} \). We prove a norm inequality for the chaotically geometric mean \( A \diamond_\alpha B = e^{(1-\alpha) \log A + \alpha \log B} \) and its reverse: For each real number \( \alpha \in \mathbb{R} \)
\[
S(h_B) - 1 \left\| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{-\frac{1-\alpha}{2}} \right\| \leq \left\| A \diamond_\alpha B \right\| \leq \left\| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{-\frac{1-\alpha}{2}} \right\|
\]
where the constant \( S(h_B) \) is the Specht ratio and \( \left\| \cdot \right\| \) is the operator norm.

1. Introduction

A (bounded linear) operator A on a Hilbert space H is said to be positive (in symbol: \( A \geq 0 \)) if \( (Ax, x) \geq 0 \) for all \( x \in H \) and strictly positive (in symbol: \( A > 0 \)) if A is positive and invertible. Let A and B be two strictly positive operators on a Hilbert space H. In [6], the chaotically geometric mean \( A \diamond_\alpha B \) is defined by
\[
A \diamond_\alpha B = \exp((1-\alpha) \log A + \alpha \log B) \quad \text{for all } \alpha \in \mathbb{R}.
\]
If A and B commute, then \( A \diamond_\alpha B = A^{1-\alpha} B^\alpha \) for all \( \alpha \in \mathbb{R} \).

In the preceding paper [10, 11], Nakamoto and the author obtained the following norm inequality and its reverse for the chaotically geometric mean: Let A and B be strictly positive operators such that \( 0 < m \leq A, B \leq M \) for some scalars \( 0 < m < M \) and \( h = \frac{M}{m} \). Then
\[
K(h^2, \alpha) \| A^{1-\alpha} B^\alpha \| \leq \| A \diamond_\alpha B \| \leq \| A^{1-\alpha} B^\alpha \| \quad \text{for all } \alpha \in [0, 1],
\]
where \( K(h, \alpha) \) is a generalized Kantorovich constant.

In this paper, we show a slight improvement of the norm inequality (1) for the chaotically geometric mean and its reverse: Let A and B be strictly positive operators such that \( 0 < m \leq B \leq M \) for some scalars \( 0 < m < M \) and \( h_B = \frac{M}{m} \). Then
\[
S(h_B) - 1 \left\| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{-\frac{1-\alpha}{2}} \right\| \leq \left\| A \diamond_\alpha B \right\| \leq \left\| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{-\frac{1-\alpha}{2}} \right\| \quad \text{for all } \alpha \in \mathbb{R},
\]
where the constant \( S(h_B) \) is the Specht ratio. Our main tools are Araki’s inequality and its reverse.

Key words and phrases: Positive operator, Araki’s inequality, chaotically geometric mean, Specht ratio, a generalized Kantorovich constant, reverse inequality.
2. Results

First of all, we cite Araki’s inequality [1] and its reverse [7]:

**Theorem A.** If A and B are positive operators such that $0 < m \leq A \leq M$ for some scalars $0 < m < M$, then

$$K(h, p) \| BAB \|^p \leq \| B^p A^p B^p \| \leq \| BAB \|^p$$

for all $p \in [0, 1]$, where $h = \frac{M}{m}$ is a generalized condition number of A in the sense of Turing [13] and a generalized Kantorovich constant $K(h, p)$ [9, Definition 2.2] is defined by

$$K(h, p) = \frac{h^p - h}{(p - 1)(h - 1)} \left( \frac{p - 1}{h^p - h} \right)^p$$

for any real number $p \in \mathbb{R}$.

We state some properties of $K(h, p)$ [9, Theorem 2.54]:

**Lemma 1.** Let $h > 0$ be given. Then a generalized Kantorovich constant $K(h, p)$ has the following properties.

(i) $K(h, p) = K(h^{-1}, p)$ for all $p \in \mathbb{R}$.

(ii) $K(h, p) = K(h, 1 - p)$ for all $p \in \mathbb{R}$.

(iii) $K(h, 0) = K(h, 1) = 1$ and $K(1, p) = 1$ for all $p \in \mathbb{R}$.

(iv) $K(h', \frac{p}{r})^r = K(h^p, \frac{r}{p})^{-\frac{1}{r}}$ for $pr \neq 0$.

Also, Specht [12] estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1, \ldots, x_n \in [m, M]$ with $0 < m \leq M$

$$\frac{x_1 + \cdots + x_n}{n} \leq S(h)^{\frac{1}{q}} x_1 \cdots x_n,$$

where $h = \frac{M}{m} (\geq 1)$ and the Specht ratio $S(h)$ [9, page 71] is defined by

$$S(h) = \left( \frac{(h - 1)h^{\frac{1}{h-1}}}{e \log h} \right)^{\frac{1}{h-1}} (h \neq 1) \quad \text{and} \quad S(1) = 1. \quad (2)$$

Yamazaki and Yanagida [14] showed the following close relation between the Specht ratio and a generalized Kantorovich constant, also see [4, 8].

**Lemma 2.** Let $h > 0$ be given. Then

$$K\left( h^p, \frac{r + p}{r} \right) \geq S(h^p) \quad \text{for all } p > 0 \text{ and } r > 0 \quad (3)$$

and

$$K\left( h^p, \frac{1}{p} \right) \to S(h) \quad \text{as } p \to 0. \quad (4)$$
The following lemma shows the Golden-Thompson type inequality for the operator norm and its reverse, also see [3, 2].

**Lemma 3.** Let $A$ and $B$ be selfadjoint operators such that $m \leq B \leq M$ for some scalars $m < M$. Then

$$S(e^{M-m})^{-1} \left\| e^{\frac{A}{2}} e^{B} e^{\frac{A}{2}} \right\| \leq \left\| e^{A+B} \right\| \leq \left\| e^{\frac{A}{2}} e^{B} e^{\frac{A}{2}} \right\|,$$

where $S(e^{M-m})$ is the Specht ratio defined by (2).

**Proof.** Since $0 < e^m \leq e^B \leq e^M$ and a generalized condition number of $e^B$ is $e^{M-m}$, it follows from Theorem A that

$$K(e^{M-m}, p) \left\| e^{\frac{A}{2}} e^{pB} e^{\frac{A}{2}} \right\|^{p} \leq \left\| e^{\frac{pA}{2}} e^{pB} e^{\frac{pA}{2}} \right\| \leq \left\| e^{\frac{A}{2}} e^{B} e^{\frac{A}{2}} \right\|$$

for all $p \in [0, 1]$.

Taking $\frac{1}{p}$-th power of both sides, we have

$$K(e^{M-m}, p) \frac{1}{p} \left\| e^{\frac{A}{2}} e^{pB} e^{\frac{A}{2}} \right\| \leq \left\| e^{\frac{pA}{2}} e^{pB} e^{\frac{pA}{2}} \right\| \frac{1}{p} \leq \left\| e^{\frac{A}{2}} e^{B} e^{\frac{A}{2}} \right\| .$$

(5)

It follows from (iv) of Lemma 1 and (4) of Lemma 2 that

$$K(e^{M-m}, p) \frac{1}{p} = K \left( e^{pM-pm}, \frac{1}{p} \right) \rightarrow S(e^{M-m})^{-1} \text{ as } p \rightarrow 0 .$$

By the Lie-Trotter formula, we have $\left\| e^{\frac{pA}{2}} e^{pB} e^{\frac{pA}{2}} \right\|^{\frac{1}{p}} \rightarrow \left\| e^{A+B} \right\|$ as $p \rightarrow 0$ and hence by (5) it follows that

$$S(e^{M-m})^{-1} \left\| e^{\frac{A}{2}} e^{B} e^{\frac{A}{2}} \right\| \leq \left\| e^{A+B} \right\| \leq \left\| e^{\frac{A}{2}} e^{B} e^{\frac{A}{2}} \right\|,$$

as desired. □

By Lemma 3, we have the following theorem which is a slight improvement of (1):

**Theorem 4.** Let $A$ and $B$ be strictly positive operators such that $0 < m \leq B \leq M$ for some scalars $0 < m < M$, $h_B = \frac{M}{m}$. Then for each real number $\alpha \in \mathbb{R}$

$$S(h_B^{-\alpha})^{-1} \left\| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \right\| \leq \left\| A \hat{\alpha} B \right\| \leq \left\| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \right\| ,$$

where $S(h)$ is the Specht ratio defined by (2).

**Proof.** For each $\alpha > 0$, replacing $A$ and $B$ by $(1 - \alpha) \log A$ and $\alpha \log B$ in Lemma 3 respectively, we have the desired inequality since $\alpha \log m \leq \alpha \log B \leq \alpha \log M$ and $e^{\alpha \log M - \alpha \log m} = h_B^{\alpha \log M \alpha}$. In the case of $\alpha < 0$, we have $\alpha \log M \leq \alpha \log B \leq \alpha \log m$ and $e^{\alpha \log M - \alpha \log m} = h_B^{-\alpha \log M}$. By the property of the Specht ratio [9, Lemma 2.47], it follows that $S(h_B^{-\alpha}) = S(h_B^{\alpha})$ and hence we have this theorem. □
The following corollary is a complementary result for Theorem 4:

**COROLLARY 5.** Let $A$ and $B$ be strictly positive operators such that $0 < m \leq A \leq M$ for some scalars $0 < m < M$, $h_A = \frac{M}{m}$. Then for each real number $\alpha \in \mathbb{R}$

$$S(h_A^{1-\alpha})^{-1} \left\| A^{\frac{1-\alpha}{2}} B^\alpha A \frac{1-\alpha}{2} \right\| \leq \|A \diamond_\alpha B\| \leq \left\| A^{\frac{1-\alpha}{2}} B^\alpha A \frac{1-\alpha}{2} \right\|.$$

**Proof.** If we apply $B \diamond_{1-\alpha} A$ to Theorem 4, then it follows that

$$S(h_A^{1-\alpha})^{-1} \left\| B^\alpha A^{1-\alpha} B^\frac{2}{1-\alpha} \right\| \leq \|B \diamond_{1-\alpha} A\|$$

and hence we have this corollary. $\square$

**REMARK.** Let $A$ and $B$ be strictly positive operators such that $0 < m \leq A, B \leq M$ for some scalars $0 < m < M$, $h = \frac{M}{m}$. Since $\left\| A^{\frac{1-\alpha}{2}} B^\alpha A \frac{1-\alpha}{2} \right\| \leq \|A^{1-\alpha} B^\alpha\|$, the expression (1) in §1 implies

$$K(h^2, \alpha) \left\| A^{\frac{1-\alpha}{2}} B^\alpha A \frac{1-\alpha}{2} \right\| \leq \|A \diamond_\alpha B\| \quad \text{for all } \alpha \in [0, 1].$$

(6)

By combining Theorem 4 and Corollary 5, we have

$$\max\{S(h^\alpha)^{-1}, S(h^{1-\alpha})^{-1}\} \left\| A^{\frac{1-\alpha}{2}} B^\alpha A \frac{1-\alpha}{2} \right\| \leq \|A \diamond_\alpha B\| \quad \text{for all } \alpha \in \mathbb{R}.$$  

(7)

Then (7) is an improvement of (6). As a matter of fact, we have

$$K(h^2, \alpha) \leq S(h^\alpha)^{-1} \quad \text{for all } 0 \leq \alpha \leq \frac{1}{2}.$$  

(i)

$$K(h^2, \alpha) \leq S(h^{1-\alpha})^{-1} \quad \text{for all } \frac{1}{2} \leq \alpha \leq 1.$$  

(ii)

To prove (i), it is sufficient to show $K(h, \alpha)^{-1} \geq S(h^\frac{2}{\alpha})$ for all $0 \leq \alpha \leq \frac{1}{2}$. By Lemma 1 and (3) of Lemma 2, we have

$$K(h, \alpha)^{-1} = K(h, 1-\alpha)^{-1} = K\left(h^{1-\alpha}, \frac{1}{1-\alpha}\right)^{1-\alpha}$$

$$= K\left(h^{1-\alpha}, \frac{\alpha + 1 - \alpha}{1-\alpha}\right)^{1-\alpha} \geq S(h^\alpha)^{1-\alpha}.$$  

Since $S(h^s)^{\frac{1}{s}}$ is increasing for $0 \leq s \leq 1$ by [5, Lemma 9], it follows that $S(h^\alpha) \geq S(h^\frac{2}{\alpha})^2$ and hence we have

$$S(h^\alpha)^{1-\alpha} \geq S(h^\frac{2}{\alpha})^2(1-\alpha) \geq S(h^\frac{2}{\alpha})$$

since $0 \leq \alpha \leq \frac{1}{2}$. Therefore, it follows that $K(h^2, \alpha) \leq S(h^\alpha)^{-1}$ for all $0 \leq \alpha \leq \frac{1}{2}$. Similarly, we have (ii). Therefore we have

$$K(h^2, \alpha) \leq \max\{S(h^\alpha)^{-1}, S(h^{1-\alpha})^{-1}\} \quad \text{for all } \alpha \in [0, 1].$$


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