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NORM INEQUALITIES FOR THE CHAOTICALLY GEOMETRIC MEAN AND ITS REVERSE

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Abstract. Let A and B be strictly positive operators on a Hilbert space H such that $0 < m \leqslant B \leqslant M$ for some scalars $0 < m < M$ and $h_B = \frac{M}{m}$. We prove a norm inequality for the chaotically geometric mean $A \diamondsuit_\alpha B = e^{(1-\alpha)\log A + \alpha \log B}$ and its reverse: For each real number $\alpha \in \mathbb{R}$

$$S(h_B^\alpha)^{-1} \left\| A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}} \right\| \leqslant \|A \diamondsuit_\alpha B\| \leqslant \left\| A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}} \right\|$$

where the constant $S(h)$ is the Specht ratio and $\|\cdot\|$ is the operator norm.

1. Introduction

A (bounded linear) operator A on a Hilbert space H is said to be positive (in symbol: $A \geqslant 0$) if $(Ax, x) \geqslant 0$ for all $x \in H$ and strictly positive (in symbol: $A > 0$) if A is positive and invertible. Let A and B be two strictly positive operators on a Hilbert space H. In [6], the chaotically geometric mean $A \diamondsuit_\alpha B$ is defined by

$$A \diamondsuit_\alpha B = \exp((1-\alpha)\log A + \alpha \log B) \qquad \text{for all } \alpha \in \mathbb{R}.$$

If A and B commute, then $A \diamondsuit_\alpha B = A^{1-\alpha}B^\alpha$ for all $\alpha \in \mathbb{R}$.

In the preceding paper [10, 11], Nakamoto and the author obtained the following norm inequality and its reverse for the chaotically geometric mean: Let A and B be strictly positive operators such that $0 < m \leqslant A, B \leqslant M$ for some scalars $0 < m < M$ and $h = \frac{M}{m}$. Then

$$K(h^2, \alpha)\|A^{1-\alpha}B^\alpha\| \leqslant \|A \diamondsuit_\alpha B\| \leqslant \|A^{1-\alpha}B^\alpha\| \qquad \text{for all } \alpha \in [0,1], \tag{1}$$

where $K(h, \alpha)$ is a generalized Kantorovich constant.

In this paper, we show a slight improvement of the norm inequality (1) for the chaotically geometric mean and its reverse: Let A and B be strictly positive operators such that $0 < m \leqslant B \leqslant M$ for some scalars $0 < m < M$ and $h_B = \frac{M}{m}$. Then

$$S(h^\alpha)^{-1} \left\| A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}} \right\| \leqslant \|A \diamondsuit_\alpha B\| \leqslant \left\| A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}} \right\| \qquad \text{for all } \alpha \in \mathbb{R},$$

where the constant $S(h)$ is the Specht ratio. Our main tools are Araki's inequality and its reverse.

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2. Results

First of all, we cite Araki's inequality [1] and its reverse [7]:

THEOREM A. *If A and B are positive operators such that $0 < m \leqslant A \leqslant M$ for some scalars $0 < m < M$, then*

$$K(h_A, p)\,\|BAB\|^p \leqslant \|B^pA^pB^p\| \leqslant \|BAB\|^p \qquad \text{for all } p \in [0,1],$$

where $h_A = \frac{M}{m}$ is a generalized condition number of A in the sense of Turing [13] and a generalized Kantorovich constant $K(h,p)$ [9, Definition 2.2] is defined by

$$K(h,p) = \frac{h^p - h}{(p-1)(h-1)}\left(\frac{p-1}{p}\cdot\frac{h^p-1}{h^p-h}\right)^p$$

for any real number $p \in \mathbb{R}$.

We state some properties of $K(h,p)$ [9, Theorem 2.54]:

LEMMA 1. *Let $h > 0$ be given. Then a generalized Kantorovich constant $K(h,p)$ has the following properties.*
(i) $K(h,p) = K(h^{-1},p)$ *for all* $p \in \mathbb{R}$.
(ii) $K(h,p) = K(h,1-p)$ *for all* $p \in \mathbb{R}$.
(iii) $K(h,0) = K(h,1) = 1$ *and* $K(1,p) = 1$ *for all* $p \in \mathbb{R}$.
(iv) $K(h^r, \frac{p}{r})^{\frac{1}{p}} = K(h^p, \frac{r}{p})^{-\frac{1}{r}}$ *for* $pr \neq 0$.

Also, Specht [12] estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1, \cdots, x_n \in [m,M]$ with $0 < m \leqslant M$

$$\frac{x_1 + \cdots + x_n}{n} \leqslant S(h)\,\sqrt[n]{x_1 \cdots x_n},$$

where $h = \frac{M}{m} (\geqslant 1)$ and the Specht ratio $S(h)$ [9, page 71] is defined by

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \quad (h \neq 1) \qquad \text{and} \quad S(1) = 1. \tag{2}$$

Yamazaki and Yanagida [14] showed the following close relation between the Specht ratio and a generalized Kantorovich constant, also see [4, 8].

LEMMA 2. *Let $h > 0$ be given. Then*

$$K\left(h^r, \frac{r+p}{r}\right) \geqslant S(h^p) \qquad \text{for all } p > 0 \text{ and } r > 0 \tag{3}$$

and

$$K\left(h^p, \frac{1}{p}\right) \to S(h) \qquad \text{as } p \to 0. \tag{4}$$

The following lemma shows the Golden-Thompson type inequality for the operator norm and its reverse, also see [3, 2].

LEMMA 3. *Let A and B be selfadjoint operators such that $m \leqslant B \leqslant M$ for some scalars $m < M$. Then*

$$S(e^{M-m})^{-1}\left\|e^{\frac{A}{2}}e^{B}e^{\frac{A}{2}}\right\| \leqslant \|e^{A+B}\| \leqslant \left\|e^{\frac{A}{2}}e^{B}e^{\frac{A}{2}}\right\|,$$

where $S(e^{M-m})$ is the Specht ratio defined by (2).

Proof. Since $0 < e^m \leqslant e^B \leqslant e^M$ and a generalized condition number of e^B is e^{M-m}, it follows from Theorem A that

$$K(e^{M-m},p)\left\|e^{\frac{A}{2}}e^{B}e^{\frac{A}{2}}\right\|^{p} \leqslant \left\|e^{\frac{pA}{2}}e^{pB}e^{\frac{pA}{2}}\right\| \leqslant \left\|e^{\frac{A}{2}}e^{B}e^{\frac{A}{2}}\right\|^{p} \qquad \text{for all } p \in [0,1].$$

Taking $\frac{1}{p}$-th power of both sides, we have

$$K(e^{M-m},p)^{\frac{1}{p}}\left\|e^{\frac{A}{2}}e^{B}e^{\frac{A}{2}}\right\| \leqslant \left\|e^{\frac{pA}{2}}e^{pB}e^{\frac{pA}{2}}\right\|^{\frac{1}{p}} \leqslant \left\|e^{\frac{A}{2}}e^{B}e^{\frac{A}{2}}\right\|. \tag{5}$$

It follows from (iv) of Lemma 1 and (4) of Lemma 2 that

$$K(e^{M-m},p)^{\frac{1}{p}} = K\left(e^{pM-pm},\frac{1}{p}\right)^{-1} \to S(e^{M-m})^{-1} \qquad \text{as } p \to 0.$$

By the Lie-Trotter formula, we have $\left\|e^{\frac{pA}{2}}e^{pB}e^{\frac{pA}{2}}\right\|^{\frac{1}{p}} \to \|e^{A+B}\|$ as $p \to 0$ and hence by (5) it follows that

$$S(e^{M-m})^{-1}\left\|e^{\frac{A}{2}}e^{B}e^{\frac{A}{2}}\right\| \leqslant \|e^{A+B}\| \leqslant \left\|e^{\frac{A}{2}}e^{B}e^{\frac{A}{2}}\right\|,$$

as desired. □

By Lemma 3, we have the following theorem which is a slight improvement of (1):

THEOREM 4. *Let A and B be strictly positive operators such that $0 < m \leqslant B \leqslant M$ for some scalars $0 < m < M$, $h_B = \frac{M}{m}$. Then for each real number $\alpha \in \mathbb{R}$*

$$S(h_B^{\alpha})^{-1}\left\|A^{\frac{1-\alpha}{2}}B^{\alpha}A^{\frac{1-\alpha}{2}}\right\| \leqslant \|A \diamondsuit_{\alpha} B\| \leqslant \left\|A^{\frac{1-\alpha}{2}}B^{\alpha}A^{\frac{1-\alpha}{2}}\right\|,$$

where $S(h)$ is the Specht ratio defined by (2).

Proof. For each $\alpha > 0$, replacing A and B by $(1-\alpha)\log A$ and $\alpha \log B$ in Lemma 3 respectively, we have the desired inequality since $\alpha \log m \leqslant \alpha \log B \leqslant \alpha \log M$ and $e^{\alpha \log M - \alpha \log m} = h_B^{\alpha}$. In the case of $\alpha < 0$, we have $\alpha \log M \leqslant \alpha \log B \leqslant \alpha \log m$ and $e^{\alpha \log m - \alpha \log M} = h_B^{-\alpha}$. By the property of the Specht ratio [9, Lemma 2.47], it follows that $S(h_B^{-\alpha}) = S(h_B^{\alpha})$ and hence we have this theorem. □

The following corollary is a complementary result for Theorem 4:

COROLLARY 5. *Let A and B be strictly positive operators such that $0 < m \leqslant A \leqslant M$ for some scalars $0 < m < M$, $h_A = \frac{M}{m}$. Then for each real number $\alpha \in \mathbb{R}$*

$$S(h_A^{1-\alpha})^{-1} \left\| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \right\| \leqslant \| A \diamondsuit_{\alpha} B \| \leqslant \left\| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \right\|.$$

Proof. If we apply $B \diamondsuit_{1-\alpha} A$ to Theorem 4, then it follows that

$$S(h_A^{1-\alpha})^{-1} \left\| B^{\frac{\alpha}{2}} A^{1-\alpha} B^{\frac{\alpha}{2}} \right\| \leqslant \| B \diamondsuit_{1-\alpha} A \|$$

and hence we have this corollary. □

REMARK. Let A and B be strictly positive operators such that $0 < m \leqslant A, B \leqslant M$ for some scalars $0 < m < M$, $h = \frac{M}{m}$. Since $\left\| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \right\| \leqslant \| A^{1-\alpha} B^{\alpha} \|$, the expression (1) in §1 implies

$$K(h^2, \alpha) \left\| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \right\| \leqslant \| A \diamondsuit_{\alpha} B \| \qquad \text{for all } \alpha \in [0,1]. \tag{6}$$

By combining Theorem 4 and Corollary 5, we have

$$\max\{S(h^{\alpha})^{-1}, S(h^{1-\alpha})^{-1}\} \left\| A^{\frac{1-\alpha}{2}} B^{\alpha} A^{\frac{1-\alpha}{2}} \right\| \leqslant \| A \diamondsuit_{\alpha} B \| \qquad \text{for all } \alpha \in \mathbb{R}. \tag{7}$$

Then (7) is an improvement of (6). As a matter of fact, we have

$$K(h^2, \alpha) \leqslant S(h^{\alpha})^{-1} \qquad \text{for all } 0 \leqslant \alpha \leqslant \tfrac{1}{2}. \tag{i}$$

$$K(h^2, \alpha) \leqslant S(h^{1-\alpha})^{-1} \qquad \text{for all } \tfrac{1}{2} \leqslant \alpha \leqslant 1. \tag{ii}$$

To prove (i), it is sufficient to show $K(h, \alpha)^{-1} \geqslant S(h^{\frac{\alpha}{2}})$ for all $0 \leqslant \alpha \leqslant \frac{1}{2}$. By Lemma 1 and (3) of Lemma 2, we have

$$\begin{aligned} K(h,\alpha)^{-1} = K(h, 1-\alpha)^{-1} &= K\left(h^{1-\alpha}, \frac{1}{1-\alpha}\right)^{1-\alpha} \\ &= K\left(h^{1-\alpha}, \frac{\alpha + 1 - \alpha}{1-\alpha}\right)^{1-\alpha} \geqslant S(h^{\alpha})^{1-\alpha}. \end{aligned}$$

Since $S(h^s)^{\frac{1}{s}}$ is increasing for $0 \leqslant s \leqslant 1$ by [5, Lemma 9], it follows that $S(h^{\alpha}) \geqslant S(h^{\frac{\alpha}{2}})^2$ and hence we have

$$S(h^{\alpha})^{1-\alpha} \geqslant S(h^{\frac{\alpha}{2}})^{2(1-\alpha)} \geqslant S(h^{\frac{\alpha}{2}})$$

since $0 \leqslant \alpha \leqslant \frac{1}{2}$. Therefore, it follows that $K(h^2, \alpha) \leqslant S(h^{\alpha})^{-1}$ for all $0 \leqslant \alpha \leqslant \frac{1}{2}$. Similarly, we have (ii). Therefore we have

$$K(h^2, \alpha) \leqslant \max\{S(h^{\alpha})^{-1}, S(h^{1-\alpha})^{-1}\} \qquad \text{for all } \alpha \in [0,1].$$

REFERENCES

[1] H. ARAKI, *On an inequality of Lieb and Thirring*, Letters in Math. Phys., **19** (1990), 167–170.
[2] R. BHATIA, *Matrix Analysis*, Springer, Germany, 1996.
[3] J.C. BOURIN AND Y. SEO, *Reverse inequality to Golden-Thompson type inequalities: Comparison of* e^{A+B} *and* $e^{A}e^{B}$, preprint.
[4] J.I. FUJII, M. FUJII, Y. SEO AND M. TOMINAGA, *On generalized Kantorovich inequalities*, Proc. Int. Sym. on Banach and Function Spaces, Kitakyushu, Japan, October 2-4, (2003), 205–213.
[5] J.I. FUJII, Y. SEO AND M. TOMINAGA, *Kantorovich type operator inequalities via the Specht ratio*, Linear Alg. Appl., **377**(2004), 69–81.
[6] M. FUJII AND R. NAKAMOTO, *A geometric mean in the Furuta inequality*, Sci. Math. Japon., **55** (2002), 615–621.
[7] M. FUJII AND Y. SEO, *Reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities*, Nihonkai Math. J., **16** (2005), 145–154.
[8] T. FURUTA, *Specht ratio* $S(1)$ *can be expressed by Kantorovich constant* $K(p)$ *:* $S(1) = \exp K'(1)$ *and its application*, Math. Inequal. Appl., **6** (2003), 521–530.
[9] T. FURUTA, J. MIĆIĆ, J.E. PEČARIĆ AND Y. SEO, *Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 1, Element, Zagreb, 2005.
[10] R. NAKAMOTO AND Y. SEO, *Norm inequalities for the geometric mean and its reverse*, Sci. Math. Japon., **65**, No. 2(2007), 281–286, :e2006, 1209–1214.
[11] R. NAKAMOTO AND Y. SEO, *A complement of the Ando-Hiai inequality and norm inequalities for the geometric mean*, preprint.
[12] W. SPECHT, *Zur Theorie der elementaren Mittel*, Math. Z., **74**(1960), 91–98.
[13] A.M. TURING, *Rounding off-errors in matrix processes*, Quart. J. Mech. Appl. Math., **1**(1948), 287–308.
[14] T. YAMAZAKI AND M. YANAGIDA, *Characterizations of chaotic order associated with Kantorovich inequality*, Sci. Math., **2**(1999), 37–50.

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