A VARIANT OF JENSEN’S INEQUALITY FOR
CONVEX FUNCTIONS OF SEVERAL VARIABLES

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Abstract. Two Jensen’s type inequalities for convex functions defined on \( \mathbb{R}^k \) which involve elements of convex hulls are given. As their consequences the comparison theorem for weighted \( L^p \)-conjugate means and a Mercer’s result are obtained.

1. Introduction

Let \( U \) be a convex subset of \( \mathbb{R}^k \) and \( f : U \to \mathbb{R} \) a function. We define \( f \) to be convex on \( U \) if

\[
f (\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)
\]

for all \( x, y \in U \) and \( \lambda \in (0, 1) \). \( f \) is concave if the reversed inequality of \( (1) \) holds.

For a convex function \( f : U \to \mathbb{R} \), \( n \) \( k \)-tuples \( x_i \) in \( U \) and \( n \) positive real numbers \( \lambda_i \) with \( \Lambda_n = \sum_{i=1}^{n} \lambda_i \), Jensen’s inequality

\[
f \left( \frac{1}{\Lambda_n} \sum_{i=1}^{n} \lambda_i x_i \right) \leq \frac{1}{\Lambda_n} \sum_{i=1}^{n} \lambda_i f(x_i)
\]

holds.

If we set the following conditions:

\[
\lambda_1 > 0, \quad \lambda_i \leq 0 \quad (i = 2, \ldots, n), \quad \Lambda_n > 0
\]

and

\[
\frac{1}{\Lambda_n} \sum_{i=1}^{n} \lambda_i x_i \in U,
\]

then

\[
f \left( \frac{1}{\Lambda_n} \sum_{i=1}^{n} \lambda_i x_i \right) \geq \frac{1}{\Lambda_n} \sum_{i=1}^{n} \lambda_i f(x_i)
\]

holds. This inequality is known as the reversed Jensen’s inequality and it is a simple consequence of the inequality \( (2) \) (see [5]).

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We denote by $H \left( \{x_1, \ldots, x_n\} \right)$ the convex hull of the set $\{x_1, \ldots, x_n\} \subset U$. If $y \in H \left( \{x_1, \ldots, x_n\} \right)$ then it can be written in a unique way as a convex combination of $x_1, \ldots, x_n$ i.e., $y = \sum_{i=1}^{n} \lambda_i x_i$ where $\lambda_i \geq 0$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \lambda_i = 1$ (see [6]).

The aim of this paper is to establish two new Jensen’s type inequalities for convex functions defined on convex subsets of the space $\mathbb{R}^k$ which involve elements of convex hulls. As their direct consequences the comparison theorem for weighted $L$-conjugate means and a Mercer’s result are obtained.

2. Main results

The next theorem gives a Jensen’s type inequality which involves elements of a convex hull.

**Theorem 1.** Let $U$ be a convex subset of $\mathbb{R}^k$, $x_1, \ldots, x_n \in U$ and $y_1, \ldots, y_m \in H \left( \{x_1, \ldots, x_n\} \right)$. If $f : U \rightarrow \mathbb{R}$ is convex on $U$ then the inequality

$$f \left( \frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j y_j}{P_n - W_m} \right) \leq \frac{\sum_{i=1}^{n} p_i f(x_i) - \sum_{j=1}^{m} w_j f(y_j)}{P_n - W_m}$$

(4)

holds for all positive real numbers $p_1, \ldots, p_n$ and $w_1, \ldots, w_m$ satisfying the condition

$$p_i \geq W_m \text{ for all } i = 1, \ldots, n,$$

(5)

where $P_n = \sum_{i=1}^{n} p_i$ and $W_m = \sum_{j=1}^{m} w_j$. If $f$ is concave on $U$ the inequality (4) is reversed.

**Proof.** Let $f : U \rightarrow \mathbb{R}$ be a convex function on $U$ and let $p_1, \ldots, p_n$ and $w_1, \ldots, w_m$ be positive real numbers satisfying the condition (5). Since $y_j \in H \left( \{x_1, \ldots, x_n\} \right)$, there are some $\lambda_{i(j)} \geq 0$ ($i = 1, \ldots, n$) such that $\sum_{i=1}^{n} \lambda_{i(j)} = 1$ and

$$y_j = \sum_{i=1}^{n} \lambda_{i(j)} x_i,$$

for all $j \in \{1, \ldots, m\}$. Since $f$ is convex on $U$,

$$f \left( y_j \right) = f \left( \sum_{i=1}^{n} \lambda_{i(j)} x_i \right) \leq \sum_{i=1}^{n} \lambda_{i(j)} f(x_i)$$

for all $j \in \{1, \ldots, m\}$.

Now, we can write

$$\frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j y_j}{P_n - W_m} = \frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j \sum_{i=1}^{n} \lambda_{i(j)} x_i}{P_n - W_m} = \frac{1}{P_n - W_m} \sum_{i=1}^{n} \left( p_i - \sum_{j=1}^{m} w_{j(i)} \lambda_{i(j)} \right) x_i.$$
We can easily check that
\[
\frac{1}{P_n - W_m} \sum_{i=1}^{n} \left( p_i - \sum_{j=1}^{m} w_j \lambda_i^{(j)} \right) = 1,
\]
and since for all \(i \in \{1, \ldots, n\}\)
\[
p_i \geq W_m \geq \sum_{j=1}^{m} w_j \lambda_i^{(j)},
\]
we also have
\[
\frac{1}{P_n - W_m} \left( p_i - \sum_{j=1}^{m} w_j \lambda_i^{(j)} \right) \geq 0 \quad (i = 1, \ldots, n).
\]
Hence, \(\sum_{i=1}^{n} \frac{p_i \mathbf{x}_i - \sum_{j=1}^{m} w_j \mathbf{y}_j}{P_n - W_m}\) is a convex combination of \(\mathbf{x}_1, \ldots, \mathbf{x}_n \in U\) and it belongs to \(U\) since \(U\) is convex. Since \(f\) is convex on \(U\), we obtain from (2) the following:
\[
f \left( \frac{\sum_{i=1}^{n} p_i \mathbf{x}_i - \sum_{j=1}^{m} w_j \mathbf{y}_j}{P_n - W_m} \right) = f \left( \frac{1}{P_n - W_m} \sum_{i=1}^{n} \left( p_i - \sum_{j=1}^{m} w_j \lambda_i^{(j)} \right) \mathbf{x}_i \right)
\]
\[
\leq \frac{1}{P_n - W_m} \sum_{i=1}^{n} \left( p_i - \sum_{j=1}^{m} w_j \lambda_i^{(j)} \right) f (\mathbf{x}_i)
\]
\[
= \frac{\sum_{i=1}^{n} p_i f (\mathbf{x}_i) - \sum_{i=1}^{n} \sum_{j=1}^{m} w_j \lambda_i^{(j)} f (\mathbf{x}_i)}{P_n - W_m}
\]
\[
= \frac{\sum_{i=1}^{n} p_i f (\mathbf{x}_i) - \sum_{j=1}^{m} \lambda_i^{(j)} f (\mathbf{y}_j)}{P_n - W_m}
\]
\[
\leq \frac{\sum_{i=1}^{n} p_i f (\mathbf{x}_i) - \sum_{j=1}^{m} w_j f (\mathbf{y}_j)}{P_n - W_m}.
\]
It can be easily seen that if \(f\) is concave the inequality (4) is reversed. \(\square\)

The following theorem gives a converse inequality of Jensen’s type which involves elements of a convex hull.

**Theorem 2.** Let \(U\) be a convex subset of \(\mathbb{R}^k\), \(\mathbf{x}_1, \ldots, \mathbf{x}_n \in U\), \(\mathbf{y}_1, \ldots, \mathbf{y}_m \in H (\{\mathbf{x}_1, \ldots, \mathbf{x}_n\})\) and let \(p_1, \ldots, p_n\) and \(w_1, \ldots, w_m\) be positive real numbers such that \(P_n - W_m > 0\) and
\[
\sum_{i=1}^{n} p_i \mathbf{x}_i - \sum_{j=1}^{m} w_j \mathbf{y}_j
\]
\[
\frac{P_n - W_m}{P_n - W_m} \in U.
\]
If \( f : U \rightarrow \mathbb{R} \) is convex on \( U \) then

\[
f \left( \frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j y_j}{P_n - W_m} \right) \geq \frac{P_n f (\mathbf{x}) - W_m f (\mathbf{y})}{P_n - W_m}
\]

\[
P_n f (\mathbf{x}) - W_m f (\mathbf{y}) \geq P_n f (\mathbf{x}) - \sum_{j=1}^{m} w_j f (y_j)
\]

where

\[
\mathbf{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i, \quad \mathbf{y} = \frac{1}{W_m} \sum_{j=1}^{m} w_j y_j.
\]

If \( f \) is concave on \( U \) the inequalities (7) are reversed.

**Proof.** From (3) and then (2) we immediately obtain

\[
f \left( \frac{P_n \left( \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) - W_m \left( \frac{1}{W_m} \sum_{j=1}^{m} w_j y_j \right)}{P_n - W_m} \right) \geq \frac{P_n f (\mathbf{x}) - W_m f (\mathbf{y})}{P_n - W_m}
\]

\[
P_n f (\mathbf{x}) - W_m \frac{1}{W_m} \sum_{j=1}^{m} w_j f (y_j) \geq \frac{P_n f (\mathbf{x}) - W_m f (\mathbf{y})}{P_n - W_m}.
\]

**Remark 1.** If positive real numbers \( p_1, \ldots, p_n \) and \( w_1, \ldots, w_m \) satisfy the condition (5) then they obviously satisfy the condition \( P_n - W_m > 0 \). Also, since \( U \) is a convex set and \( x_1, \ldots, x_n \in U \) they satisfy the condition (6). Hence, in this case the inequality (4) from Theorem 1 can be extended in the following way

\[
\frac{P_n f (\mathbf{x}) - \sum_{j=1}^{m} w_j f (y_j)}{P_n - W_m} \leq \frac{P_n f (\mathbf{x}) - W_m f (\mathbf{y})}{P_n - W_m}
\]

\[
\leq f \left( \frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j y_j}{P_n - W_m} \right)
\]

\[
\leq \frac{\sum_{i=1}^{n} p_i f (x_i) - \sum_{j=1}^{m} w_j f (y_j)}{P_n - W_m}.
\]
If we consider real valued functions of one variable then the direct consequences of Theorems 1 and 2 are some results in \([3]\) related to means of \(n\) variables.

**Corollary 1.** Let \(I\) be an interval in \(\mathbb{R}\) and let \(M_1, \ldots, M_m\) be fixed means of \(n\) variables \(x_1, \ldots, x_n \in I\). If \(f : I \to \mathbb{R}\) is convex on \(I\) then the inequality

\[
\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(x) \leq \frac{\sum_{i=1}^{n} p_i f(x_i) - \sum_{j=1}^{m} w_j f(M_j(x))}{P_n - W_m},
\]

holds for all positive real numbers \(p_1, \ldots, p_n\) and \(w_1, \ldots, w_m\) satisfying the condition (5). If \(f\) is concave on \(I\) the inequality (8) is reversed.

**Proof.** Follows from Theorem 1, since the convex hull of the set \(\{x_1, \ldots, x_n\} \subset I\) is the interval \(\left[ \min_{i \in \{1, \ldots, n\}} \{x_i\}, \max_{i \in \{1, \ldots, n\}} \{x_i\} \right]\) and for each \(j \in \{1, \ldots, m\}\) \(M_j(x)\) is in that interval. \(\square\)

**Remark 2.** The notion of \(L\)-conjugate means was introduced in \([2]\). It was generalized to the notion of the weighted \(L\)-conjugate means in \([3]\). In the same paper the comparison theorem for these means, which immediately follows from Corollary 1, was proved.

**Corollary 2.** Let \(I\) be an interval in \(\mathbb{R}\), let \(M_1, \ldots, M_m\) be fixed means of \(n\) variables \(x_1, \ldots, x_n \in I\) and let \(p_1, \ldots, p_n\) and \(w_1, \ldots, w_m\) be positive real numbers such that \(P_n - W_m > 0\) and

\[
\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(x) \in I.
\]

If \(f : I \to \mathbb{R}\) is convex on \(I\) then

\[
f\left(\frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(x)}{P_n - W_m}\right) \geq \frac{P_n f(\bar{x}) - W_m f(M)}{P_n - W_m}
\]

\[
\geq \frac{P_n f(\bar{x}) - \sum_{j=1}^{m} w_j f(M_j(x))}{P_n - W_m},
\]

where

\[
\bar{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i, \quad M = \frac{1}{W_m} \sum_{j=1}^{m} w_j M_j(x).
\]

If \(f\) is concave on \(I\) the inequalities (9) are reversed.
Also, the direct consequence of Theorem 1 is a variant of Jensen’s inequality proved in [4].

**COROLLARY 3.** Let \([a, b]\) be an interval in \(\mathbb{R}\), \(y_1, \ldots, y_m \in [a, b]\) and \(w_1, \ldots, w_m\) positive real numbers such that \(W_m = 1\). If \(f : [a, b] \to \mathbb{R}\) is convex on \([a, b]\) then

\[
f(a + b - \sum_{j=1}^{m} w_j y_j) \leq f(a) + f(b) - \sum_{j=1}^{m} w_j f(y_j).
\]

**Proof.** Follows from Theorem 1 by setting \(n = 2, x_1 = a, x_2 = b\) and \(p_1 = p_2 = 1\). □

**REMARK 3.** Corrolary 3 was first proved by Mercer in [4, Theorem 1.2.] and later it was generalized in [1, Theorem 2.].

**COROLLARY 4.** Let \([a, b]\) be an interval in \(\mathbb{R}\), \(y_1, \ldots, y_m \in [a, b]\) and \(w_1, \ldots, w_m\) positive real numbers such that \(W_m = 1\). If \(f : [a, b] \to \mathbb{R}\) is convex on \([a, b]\) then

\[
f(a + b - \sum_{j=1}^{m} w_j y_j) \geq 2 f\left(\frac{a + b}{2}\right) - \sum_{j=1}^{m} w_j f(y_j)
\]

We can also obtain natural generalizations of Corrolaries 3 and 4 to convex functions defined on \(\mathbb{R}^k\).

**COROLLARY 5.** Let \(U\) be a simplex with vertices \(x_1, \ldots, x_n \in \mathbb{R}^k\) \((n \geq 2)\), \(y_1, \ldots, y_m \in U\) and \(w_1, \ldots, w_m\) positive real numbers such that \(W_m = 1\). If \(f : U \to \mathbb{R}\) is convex on \(U\) then

\[
\frac{nf\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) - \sum_{j=1}^{m} w_j f(y_j)}{n - 1} \leq \frac{nf\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) - f\left(\frac{1}{n} \sum_{j=1}^{m} w_j y_j\right)}{n - 1}
\]
\[
\leq f\left(\frac{\sum_{i=1}^{n} x_i - \sum_{j=1}^{m} w_j y_j}{n - 1}\right)
\]
\[
\leq \frac{\sum_{i=1}^{n} f(x_i) - \sum_{j=1}^{m} w_j f(y_j)}{n - 1}.
\]

**Proof.** Follows from Theorems 1 and 2 by setting \(p_1 = \cdots = p_n = 1\) since in this case \(U = H\left(\{x_1, \ldots, x_n\}\right)\). □
A variant of Jensen’s inequality

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