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A VARIANT OF JENSEN'S INEQUALITY FOR CONVEX FUNCTIONS OF SEVERAL VARIABLES

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Abstract. Two Jensen's type inequalities for convex functions defined on $\mathbb{R}^k$ which involve elements of convex hulls are given. As their consequences the comparison theorem for weighted L-conjugate means and a Mercer's result are obtained.

1. Introduction

Let U be a convex subset of $\mathbb{R}^k$ and $f : U \to \mathbb{R}$ a function. We define f to be convex on U if

$$f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leqslant \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y}) \tag{1}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in U$ and $\lambda \in (0,1)$. f is concave if the reversed inequality of (1) holds.

For a convex function $f : U \to \mathbb{R}$, n $k-$tuples $\boldsymbol{x}_i$ in U and n positive real numbers λ_i with $\Lambda_n = \sum_{i=1}^n \lambda_i$, Jensen's inequality

$$f\left(\frac{1}{\Lambda_n}\sum_{i=1}^n \lambda_i \boldsymbol{x}_i\right) \leqslant \frac{1}{\Lambda_n}\sum_{i=1}^n \lambda_i f(\boldsymbol{x}_i) \tag{2}$$

holds.

If we set the following conditions:

$$\lambda_1 > 0, \quad \lambda_i \leqslant 0 \quad (i = 2, \ldots, n), \quad \Lambda_n > 0$$

and

$$\frac{1}{\Lambda_n}\sum_{i=1}^n \lambda_i \boldsymbol{x}_i \in U,$$

then

$$f\left(\frac{1}{\Lambda_n}\sum_{i=1}^n \lambda_i \boldsymbol{x}_i\right) \geqslant \frac{1}{\Lambda_n}\sum_{i=1}^n \lambda_i f(\boldsymbol{x}_i) \tag{3}$$

holds. This inequality is known as the reversed Jensen's inequality and it is a simple consequence of the inequality (2) (see [5]).

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We denote by $H(\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\})$ the convex hull of the set $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\}\subset U$. If $\boldsymbol{y}\in H(\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\})$ then it can be written in a unique way as a convex combination of $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n$ i.e., $\boldsymbol{y}=\sum_{i=1}^{n}\lambda_i\boldsymbol{x}_i$ where $\lambda_i\geqslant 0$ for all $i=1,\dots,n$ and $\sum_{i=1}^{n}\lambda_i=1$ (see [6]).

The aim of this paper is to establish two new Jensen's type inequalities for convex functions defined on convex subsets of the space $\mathbb{R}^k$ which involve elements of convex hulls. As their direct consequences the comparison theorem for weighted L-conjugate means and a Mercer's result are obtained.

2. Main results

The next theorem gives a Jensen's type inequality which involves elements of a convex hull.

THEOREM 1. *Let U be a convex subset of $\mathbb{R}^k$, $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\in U$ and $\boldsymbol{y}_1,\dots,\boldsymbol{y}_m\in H(\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\})$. If $f:U\to\mathbb{R}$ is convex on U then the inequality*

$$f\left(\frac{\sum_{i=1}^{n}p_i\boldsymbol{x}_i-\sum_{j=1}^{m}w_j\boldsymbol{y}_j}{P_n-W_m}\right)\leqslant\frac{\sum_{i=1}^{n}p_if(\boldsymbol{x}_i)-\sum_{j=1}^{m}w_jf(\boldsymbol{y}_j)}{P_n-W_m} \tag{4}$$

holds for all positive real numbers $p_1,\dots,p_n$ and $w_1,\dots,w_m$ satisfying the condition

$$p_i\geqslant W_m \quad \text{for all } i=1,\dots,n, \tag{5}$$

where $P_n=\sum_{i=1}^{n}p_i$ and $W_m=\sum_{j=1}^{m}w_j$. If f is concave on U the inequality (4) is reversed.

Proof. Let $f:U\to\mathbb{R}$ be a convex function on U and let $p_1,\dots,p_n$ and $w_1,\dots,w_m$ be positive real numbers satisfying the condition (5). Since $\boldsymbol{y}_j\in H(\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\})$, there are some $\lambda_i^{(j)}\geqslant 0$ $(i=1,\dots,n)$ such that $\sum_{i=1}^{n}\lambda_i^{(j)}=1$ and

$$\boldsymbol{y}_j=\sum_{i=1}^{n}\lambda_i^{(j)}\boldsymbol{x}_i,$$

for all $j\in\{1,\dots,m\}$. Since f is convex on U,

$$f(\boldsymbol{y}_j)=f\left(\sum_{i=1}^{n}\lambda_i^{(j)}\boldsymbol{x}_i\right)\leqslant\sum_{i=1}^{n}\lambda_i^{(j)}f(\boldsymbol{x}_i)$$

for all $j\in\{1,\dots,m\}$.

Now, we can write

$$\frac{\sum_{i=1}^{n}p_i\boldsymbol{x}_i-\sum_{j=1}^{m}w_j\boldsymbol{y}_j}{P_n-W_m}=\frac{\sum_{i=1}^{n}p_i\boldsymbol{x}_i-\sum_{j=1}^{m}w_j\sum_{i=1}^{n}\lambda_i^{(j)}\boldsymbol{x}_i}{P_n-W_m}$$
$$=\frac{1}{P_n-W_m}\sum_{i=1}^{n}\left(p_i-\sum_{j=1}^{m}w_j\lambda_i^{(j)}\right)\boldsymbol{x}_i.$$

We can easily check that

$$\frac{1}{P_n - W_m}\sum_{i=1}^{n}\left(p_i - \sum_{j=1}^{m} w_j\lambda_i^{(j)}\right) = 1,$$

and since for all $i \in \{1,...,n\}$

$$p_i \geqslant W_m \geqslant \sum_{j=1}^{m} w_j\lambda_i^{(j)},$$

we also have

$$\frac{1}{P_n - W_m}\left(p_i - \sum_{j=1}^{m} w_j\lambda_i^{(j)}\right) \geqslant 0 \quad (i = 1,...,n).$$

Hence, $\frac{\sum_{i=1}^{n} p_i\boldsymbol{x}_i - \sum_{j=1}^{m} w_j\boldsymbol{y}_j}{P_n - W_m}$ is a convex combination of $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n \in U$ and it belongs to U since U is convex. Since f is convex on U, we obtain from (2) the following:

$$\begin{aligned}
f\left(\frac{\sum_{i=1}^{n} p_i\boldsymbol{x}_i - \sum_{j=1}^{m} w_j\boldsymbol{y}_j}{P_n - W_m}\right) &= f\left(\frac{1}{P_n - W_m}\sum_{i=1}^{n}\left(p_i - \sum_{j=1}^{m} w_j\lambda_i^{(j)}\right)\boldsymbol{x}_i\right)\\
&\leqslant \frac{1}{P_n - W_m}\sum_{i=1}^{n}\left(p_i - \sum_{j=1}^{m} w_j\lambda_i^{(j)}\right) f(\boldsymbol{x}_i)\\
&= \frac{\sum_{i=1}^{n} p_i f(\boldsymbol{x}_i) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_j\lambda_i^{(j)} f(\boldsymbol{x}_i)}{P_n - W_m}\\
&= \frac{\sum_{i=1}^{n} p_i f(\boldsymbol{x}_i) - \sum_{j=1}^{m} w_j \sum_{i=1}^{n}\lambda_i^{(j)} f(\boldsymbol{x}_i)}{P_n - W_m}\\
&\leqslant \frac{\sum_{i=1}^{n} p_i f(\boldsymbol{x}_i) - \sum_{j=1}^{m} w_j f(\boldsymbol{y}_j)}{P_n - W_m}.
\end{aligned}$$

It can be easily seen that if f is concave the inequality (4) is reversed. □

The following theorem gives a converse inequality of Jensen's type which involves elements of a convex hull.

THEOREM 2. *Let U be a convex subset of $\mathbb{R}^k$, $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n \in U$, $\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m \in H(\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\})$ and let $p_1,\ldots,p_n$ and $w_1,\ldots,w_m$ be positive real numbers such that $P_n - W_m > 0$ and*

$$\frac{\sum_{i=1}^{n} p_i\boldsymbol{x}_i - \sum_{j=1}^{m} w_j\boldsymbol{y}_j}{P_n - W_m} \in U. \tag{6}$$

If $f: U \to \mathbb{R}$ is convex on U then

$$f\left(\frac{\sum_{i=1}^{n} p_i \boldsymbol{x}_i - \sum_{j=1}^{m} w_j \boldsymbol{y}_j}{P_n - W_m}\right) \geqslant \frac{P_n f(\overline{\boldsymbol{x}}) - W_m f(\overline{\boldsymbol{y}})}{P_n - W_m}$$
$$\geqslant \frac{P_n f(\overline{\boldsymbol{x}}) - \sum_{j=1}^{m} w_j f(\boldsymbol{y}_j)}{P_n - W_m}, \tag{7}$$

where

$$\overline{\boldsymbol{x}} = \frac{1}{P_n} \sum_{i=1}^{n} p_i \boldsymbol{x}_i, \quad \overline{\boldsymbol{y}} = \frac{1}{W_m} \sum_{j=1}^{m} w_j \boldsymbol{y}_j.$$

If f is concave on U the inequalities (7) are reversed.

Proof. From (3) and then (2) we immediately obtain

$$f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i \boldsymbol{x}_i\right) - W_m\left(\frac{1}{W_m}\sum_{j=1}^{m} w_j \boldsymbol{y}_j\right)}{P_n - W_m}\right) \geqslant \frac{P_n f(\overline{\boldsymbol{x}}) - W_m f(\overline{\boldsymbol{y}})}{P_n - W_m}$$
$$\geqslant \frac{P_n f(\overline{\boldsymbol{x}}) - W_m \frac{1}{W_m}\sum_{j=1}^{m} w_j f(\boldsymbol{y}_j)}{P_n - W_m}. \qquad \square$$

REMARK 1. *If positive real numbers $p_1, \ldots, p_n$ and $w_1, \ldots, w_m$ satisfy the condition (5) then they obviously satisfy the condition $P_n - W_m > 0$. Also, since U is a convex set and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in U$ they satisfy the condition (6). Hence, in this case the inequality (4) from Theorem 1 can be extended in the following way*

$$\frac{P_n f(\overline{\boldsymbol{x}}) - \sum_{j=1}^{m} w_j f(\boldsymbol{y}_j)}{P_n - W_m} \leqslant \frac{P_n f(\overline{\boldsymbol{x}}) - W_m f(\overline{\boldsymbol{y}})}{P_n - W_m}$$
$$\leqslant f\left(\frac{\sum_{i=1}^{n} p_i \boldsymbol{x}_i - \sum_{j=1}^{m} w_j \boldsymbol{y}_j}{P_n - W_m}\right)$$
$$\leqslant \frac{\sum_{i=1}^{n} p_i f(\boldsymbol{x}_i) - \sum_{j=1}^{m} w_j f(\boldsymbol{y}_j)}{P_n - W_m}.$$

If we consider real valued functions of one variable then the direct consequences of Theorems 1 and 2 are some results in [3] related to means of n variables.

COROLLARY 1. *Let I be an interval in $\mathbb{R}$ and let $M_1,\dots,M_m$ be fixed means of n variables $x_1,\dots,x_n \in I$. If $f: I \to \mathbb{R}$ is convex on I then the inequality*

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{P_n - W_m}\right) \leqslant \frac{\sum_{i=1}^{n} p_i f(x_i) - \sum_{j=1}^{m} w_j f(M_j(\boldsymbol{x}))}{P_n - W_m} \tag{8}$$

holds for all positive real numbers $p_1,\dots,p_n$ and $w_1,\dots,w_m$ satisfying the condition (5). If f is concave on I the inequality (8) is reversed.

Proof. Follows from Theorem 1, since the convex hull of the set $\{x_1,\dots,x_n\} \subset I$ is the interval $\left[\min_{i\in\{1,\dots,n\}}\{x_i\}, \max_{i\in\{1,\dots,n\}}\{x_i\}\right]$ and for each $j \in \{1,\dots,m\}$ $M_j(\boldsymbol{x})$ is in that interval. $\square$

REMARK 2. *The notion of L-conjugate means was introduced in [2]. It was generalized to the notion of the weighted L-conjugate means in [3]. In the same paper the comparison theorem for these means, which immediately follows from Corollary 1, was proved.*

COROLLARY 2. *Let I be an interval in $\mathbb{R}$, let $M_1,\dots,M_m$ be fixed means of n variables $x_1,\dots,x_n \in I$ and let $p_1,\dots,p_n$ and $w_1,\dots,w_m$ be positive real numbers such that $P_n - W_m > 0$ and*

$$\frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{P_n - W_m} \in I.$$

If $f: I \to \mathbb{R}$ is convex on I then

$$\begin{aligned} f\left(\frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{P_n - W_m}\right) &\geqslant \frac{P_n f(\overline{x}) - W_m f(\overline{M})}{P_n - W_m} \\ &\geqslant \frac{P_n f(\overline{x}) - \sum_{j=1}^{m} w_j f(M_j(\boldsymbol{x}))}{P_n - W_m}, \end{aligned} \tag{9}$$

where

$$\overline{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \quad \overline{M} = \frac{1}{W_m}\sum_{j=1}^{m} w_j M_j(\boldsymbol{x}).$$

If f is concave on I the inequalities (9) are reversed.

Also, the direct consequence of Theorem 1 is a variant of Jensen's inequality proved in [4].

COROLLARY 3. *Let* $[a,b]$ *be an interval in* $\mathbb{R}$, $y_1,\ldots,y_m \in [a,b]$ *and* $w_1,\ldots,w_m$ *positive real numbers such that* $W_m = 1$. *If* $f : [a,b] \to \mathbb{R}$ *is convex on* $[a,b]$ *then*

$$f\left(a+b-\sum_{j=1}^{m} w_j y_j\right) \leqslant f(a)+f(b)-\sum_{j=1}^{m} w_j f(y_j).$$

Proof. Follows from Theorem 1 by setting $n=2$, $x_1=a$, $x_2=b$ and $p_1 = p_2 = 1$. □

REMARK 3. *Corrolary 3 was first proved by Mercer in [4, Theorem 1.2.] and later it was generalized in [1, Theorem 2.].*

COROLLARY 4. *Let* $[a,b]$ *be an interval in* $\mathbb{R}$, $y_1,\ldots,y_m \in [a,b]$ *and* $w_1,\ldots,w_m$ *positive real numbers such that* $W_m = 1$. *If* $f : [a,b] \to \mathbb{R}$ *is convex on* $[a,b]$ *then*

$$\begin{aligned} f\left(a+b-\sum_{j=1}^{m} w_j y_j\right) &\geqslant 2f\left(\frac{a+b}{2}\right) - f\left(\sum_{j=1}^{m} w_j y_j\right) \\ &\geqslant 2f\left(\frac{a+b}{2}\right) - \sum_{j=1}^{m} w_j f(y_j). \end{aligned}$$

We can also obtain natural generalizations of Corrolaries 3 and 4 to convex functions defined on $\mathbb{R}^k$.

COROLLARY 5. *Let* U *be a simplex with vertices* $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n \in \mathbb{R}^k$ $(n \geqslant 2)$, $\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m \in U$ *and* $w_1,\ldots,w_m$ *positive real numbers such that* $W_m = 1$. *If* $f : U \to \mathbb{R}$ *is convex on* U *then*

$$\begin{aligned} \frac{nf\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\right) - \sum_{j=1}^{m} w_j f(\boldsymbol{y}_j)}{n-1} &\leqslant \frac{nf\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\right) - f\left(\sum_{j=1}^{m} w_j \boldsymbol{y}_j\right)}{n-1} \\ &\leqslant f\left(\frac{\sum_{i=1}^{n}\boldsymbol{x}_i - \sum_{j=1}^{m} w_j \boldsymbol{y}_j}{n-1}\right) \\ &\leqslant \frac{\sum_{i=1}^{n} f(\boldsymbol{x}_i) - \sum_{j=1}^{m} w_j f(\boldsymbol{y}_j)}{n-1}. \end{aligned}$$

Proof. Follows from Theorems 1 and 2 by setting $p_1 = \cdots = p_n = 1$ since in this case $U = H(\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\})$. □

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