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NOTES ON CERTAIN INEQUALITIES BY HÖLDER, LEWENT AND KY FAN

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(*communicated by E. Neuman*)

Abstract. The aim of this paper is twofold. First we show that the famous Hölder inequality (which one, according to [10] should be called as the Rogers inequality) was discovered by Grolous [5] and Besso [3] about 10 years before Rogers. On the other hand, a result obtained by Lewent [9] leads to a new proof of the famous Ky Fan inequality [2]. Related results are pointed out, too.

1. Historical notes

In 1888 Rogers (according to [10]) proved that for $x_i > 0$, $\alpha_i > 0$, $i = \overline{1,n}$ there holds

$$\prod_{i=1}^n x_i^{\alpha_i} \leqslant \left(\frac{\sum\limits_{i=1}^n \alpha_i x_i}{\sum\limits_{i=1}^n \alpha_i}\right)^{\sum\limits_{i=1}^n \alpha_i} . \tag{1}$$

F. Sibirani [15] reported in 1907 that the proof of (1) was already known. Namely, it was published by D. Besso [3] in 1879. We note that Besso's original article was reprinted in 1907, but was never included with a review in JFM ("Jahrbuch der Fortschritte der Mathematik"); see [16].

It is known that Hölder concludes the inequality (1) as a special case of

$$\varphi\left(\frac{\sum\limits_{i=1}^n \alpha_i x_i}{\sum\limits_{i=1}^n \alpha_i}\right) < \frac{\sum\limits_{i=1}^n \alpha_i \varphi(x_i)}{\sum\limits_{i=1}^n \alpha_i}, \tag{2}$$

where φ has an increasing derivative, see [7], [6], [11]. The real importance of this inequality, for continuous, mid-convex ("Jensen-convex") functions φ was discovered, however by Jensen [8].

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It is not widely known today that Hölder's result in the case of equal weights ($\alpha_i = 1/n,\ \varphi'' \geqslant 0$) was proved much earlier by Grolous [5]. He applied the so-called "method of centers" in his proof, compare e.g. [11].

Finally, we wish to mention here the names of the reviewers contributing to JFM, related to the above mentioned articles. These were M. Hamburger, E. Lampe, J. Glaisher, P. Stäckel, R. Hoppe, H. Valentiner, F. Müller, and L. Lewent. It seems that they did not publish in the area of mathematical inequalities, the only exception being [9].

2. Lewent's and Ky Fan's inequalities

By using the power–series method in 1908, Lewent [9] proved the relation

$$\frac{1+\sum\limits_{i=1}^{n}\alpha_i x_i}{1-\sum\limits_{i=1}^{n}\alpha_i x_i} \leqslant \prod_{i=1}^{n}\left(\frac{1+x_i}{1-x_i}\right)^{\alpha_i}, \tag{3}$$

where

$$x_i \in [0,1),\ \ i=\overline{1,n}; \quad \text{and} \quad \sum_{i=1}^{n}\alpha_i = 1\,. \tag{4}$$

We note, that this follows also by inequality (2) applied to the function

$$\varphi(t) = \ln\frac{1+t}{1-t}, \qquad t\in[0,1).$$

The famous Ky Fan inequality (see e.g. [1], [2], [12], [13], [14]) states that if $a_i \in \big(0, 1/2\big]$, $(i = \overline{1,n}$ and $A_n(a) = A_n,\ G_n(a) = G_n$ denote the arithmetic, respectively the geometric means of certain positive $a = (a_1, \cdots, a_n)$, by putting $A'_n = A_n(1-a),\ G'_n = G_n(1-a)$, where $1-a = (1-a_1, \ldots, 1-a_n)$ then one has

$$\frac{G_n}{G'_n} \leqslant \frac{A_n}{A'_n}\,. \tag{5}$$

We want to point out now that, by a method of Sándor [13, Part I], who applied an inequality of Henrici to deduce (5), Lewent's inequality implies Ky Fan's inequality (5). Indeed, let $\alpha_i = 1/n$, and put $x_i = 1-2a_i,\ i=\overline{1,n}$ in (3). As $a_i \in \big(0,1/2\big]$ clearly $x_i \in [0,1)$. Now, simple transformation yields the relation (5), and we are done. A slight modification of the same method used in the proof of (5) would allowed to obtain the weighted version of the Ky Fan inequality.

Let $A^+_n = A_n(1+a)$, $G^+_n = G_n(1+a)$, where $1+a = (1+a_1, \cdots, 1+a_n)$. By letting $\alpha_i = 1/n,\ x_i = a_i$ the inequality (3) may be written in an equivalent form as

$$\frac{G'_n}{G^+_n} \leqslant \frac{A'_n}{A^+_n}, \tag{6}$$

where $0 \leqslant a_i < 1,\ i=\overline{1,n}$, and $G'_n = G'_n(a)$ etc. For this kind inequalities see also [4] and [12, Part II]..

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