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ON OSTROWSKI AND EULER–GRÜSS TYPE INEQUALITIES INVOLVING MEASURES

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Abstract. Some generalizations of weighted Ostrowski and Euler-Grüss type inequalities are given by using general Euler identities involving real Borel measures.

1. Introduction

For $a, b \in \mathbf{R}$, $a < b$, let $w : [a, b] \to [0, \infty)$ be an integrable function satisfying

$$\int_a^b w(t)dt > 0$$

For $n \geqslant 1$, and $x, t \in [a, b]$ let

$$K_n(x,t) = \begin{cases} \dfrac{1}{(n-1)!} \displaystyle\int_a^t (t-s)^{n-1} w(s)ds, & a \leqslant t < x \\ 0, & t = x \\ \dfrac{1}{(n-1)!} \displaystyle\int_b^t (t-s)^{n-1} w(s)ds, & x < t \leqslant b \end{cases}$$

and $K_0(x,t) = w(t)$. Also let

$$e_n(x,w) = \int_a^b (t-x)^n w(t)dt, \ n \geqslant 0$$

It is easy to see that $K_n(x, \cdot)$ is continuous on $[a, b] \backslash \{x\}$ and has a total jump of

$$K_n(x, x+0) - K_n(x, x-0) = \frac{(-1)^n}{(n-1)!} e_{n-1}(x,w)$$

at x. It is differentiable on $[a, b] \backslash \{x\}$ and

$$K'_{n+1}(x,t) = K_n(x,t)$$

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Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. In paper [3] the following identity has been proved:

$$\int_a^b f(t)w(t)dt = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} e_k(x,w) + R_n(x) \tag{1.1}$$

where $x \in [a,b]$ and

$$R_n(x) = (-1)^n \int_{[a,b]} K_n(x,t)df^{(n-1)}(t) \tag{1.2}$$

This identity has been used in [3] to prove some generalizations of weighted Ostrowski inequalities. The aim of this paper is to generalize formula (1.1), by replacing weight function w by a real Borel measure on $[a,b]$, and using it to prove some further generalizations of weighted Ostrowski inequality.

2. Some integral identities

For $a,b \in \mathbf{R}$, $a < b$, let $C[a,b]$ be the Banach space of all continuous functions $f : [a,b] \to \mathbf{R}$ with the max norm, and $M[a,b]$ the Banach space of all real Borel measures on $[a,b]$ with the total variation norm. For $\mu \in M[a,b]$ define function $\check{\mu}_n : [a,b] \to \mathbf{R}$, $n \geqslant 1$, by

$$\check{\mu}_n(t) = \frac{1}{(n-1)!} \int_{[a,t]} (t-s)^{n-1} d\mu(s)$$

The function $\check{\mu}_n$ is differentiable, $\check{\mu}_n'(t) = \check{\mu}_{n-1}(t)$ and $\check{\mu}_n(a) = 0$, for every $n \geqslant 2$, while for $n = 1$

$$\check{\mu}_1(t) = \int_{[a,t]} d\mu(s) = \mu([a,t])$$

which means that $\check{\mu}_1(t)$ is equal to the distribution function of μ. Note that

$$\check{\mu}_n(t) = \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} \check{\mu}_1(s)ds, \ n \geqslant 2$$

and

$$|\check{\mu}_n(t)| \leqslant \frac{(t-a)^{n-1}}{(n-1)!} \|\mu\|, \ t \in [a,b], \ n \geqslant 1$$

We also write

$$m_n(\mu) = \int_{[a,b]} s^n d\mu(s), \ n \geqslant 0$$

for the n-th moment of μ, and

$$e_n(x,\mu) = \int_{[a,b]} (s-x)^n d\mu(s), \ n \geqslant 0, \ x \in [a,b]$$

for the n-th x-centered moment of μ. We introduce the sequence of functions $P_n : [a,b] \times [a,b] \to \mathbf{R}$, $n \geqslant 1$, by

$$P_n(x,t) = \begin{cases} \check{\mu}_n(t), & a \leqslant t \leqslant x \\ \check{\mu}_n(t) + \frac{(-1)^n}{(n-1)!} e_{n-1}(t,\mu), & x < t \leqslant b \end{cases}$$

for $a \leqslant x < b$, while for $x = b$

$$P_n(b,t) = \begin{cases} \check{\mu}_n(t), & a \leqslant t < b \\ 0, & t = b \end{cases}$$

It is easy to see that for $n \geqslant 2$

$$P_n(x,a) = P_n(x,b) = 0$$

and

$$P_1(x,a) = \check{\mu}_1(a) = \mu(\{a\}), \quad P_1(x,b) = 0$$

for every $x \in [a,b]$, and that $P_n(x,\cdot)$, $n \geqslant 2$, is continuous on $[a,b] \backslash \{x\}$, having a jump of

$$\frac{(-1)^n}{(n-1)!} e_{n-1}(x,\mu)$$

at x. Further, $P_n(x,\cdot)$, $n \geqslant 2$, is differentiable on $[a,b] \backslash \{x\}$ and

$$P'_{n+1}(x,t) = P_n(x,t)$$

REMARK 1. Note that

$$|P_n(x,t)| \leqslant \frac{(t-a)^{n-1}}{(n-1)!} \|\mu\|, \ a \leqslant t \leqslant x, \ n \geqslant 1$$

and

$$|P_n(x,t)| \leqslant \frac{(b-t)^{n-1}}{(n-1)!} \|\mu\|, \ x < t \leqslant b, \ n \geqslant 1$$

since for $x < t \leqslant b$ and $n \geqslant 1$ we have

$$\begin{aligned} P_n(x,t) &= \check{\mu}_n(t) - \frac{1}{(n-1)!} \int_{[a,b]} (t-s)^{n-1} d\mu(s) \\ &= -\frac{1}{(n-1)!} \int_{(t,b]} (t-s)^{n-1} d\mu(s) \end{aligned}$$

which can be written, for $n \geqslant 2$, as

$$P_n(x,t) = -\frac{1}{(n-2)!} \int_t^b (t-s)^{n-2} \check{\mu}_1(s) ds$$

REMARK 2. In the special case, when the measure μ has the density w, with respect to Lebesgue measure on $[a,b]$, the sequence $(P_n(x,t), n \geqslant 1)$ reduces to the sequence $(K_n(x,t), n \geqslant 1)$ from Introduction, except for $t = x$. In this case also $P_1(x,\cdot)$ is differentiable a.e. and $P'_1(x,t) = w(t)$, a.e.

LEMMA 1. *For* $n \geqslant 2$, $x \in [a,b]$, *and* $f \in C[a,b]$, *we have*

$$\int_{[a,b]} f(t)dP_n(x,t) = \int_a^b f(t)P_{n-1}(x,t)dt + \frac{(-1)^n}{(n-1)!}e_{n-1}(x,\mu)f(x)$$

while for $n=1$

$$\int_{[a,b]} f(t)dP_1(x,t) = \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a) - \mu([a,b])f(x)$$

Proof. For $n \geqslant 2$, the function $P_n(x,\cdot)$ is differentiable on $[a,b]\setminus\{x\}$ and its derivative is equal to $P_{n-1}(x,\cdot)$. Further, it has a jump of $\frac{(-1)^n}{(n-1)!}e_{n-1}(x,\mu)$ at x, which gives the first formula. Further, $P_1(x,\cdot)$ has a jump of $-\check{\mu}_1(b)$ at x, and by [1, Lemma 1] we have

$$\begin{aligned}\int_{[a,b]} f(t)dP_1(x,t) &= \int_{[a,b]} f(t)d\check{\mu}_1(t) - \check{\mu}_1(b)f(x) \\ &= \int_{[a,b]} f(t)d\mu(t) - \check{\mu}_1(a)f(a) - \check{\mu}_1(b)f(x) \\ &= \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a) - \mu([a,b])f(x),\end{aligned}$$

which proves the second formula. □

THEOREM 1. *Let* $f:[a,b]\to\mathbf{R}$ *be such that* $f^{(n-1)}$ *is a continuous function of bounded variation for some* $n \geqslant 1$. *Then for every* $x \in [a,b]$

$$\int_{[a,b]} f(t)d\mu(t) = S_n(x) + R_n(x) \tag{2.1}$$

where

$$S_n(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}e_k(x,\mu)$$

and

$$R_n(x) = (-1)^n \int_{[a,b]} P_n(x,t)\, df^{(n-1)}(t)$$

Proof. By partial integration, for $k \geqslant 1$, we have

$$R_k(x) = (-1)^k P_k(x,t) f^{(k-1)}(t)\Big|_a^b - (-1)^k \int_{[a,b]} f^{(k-1)}(t)dP_k(x,t) \tag{2.2}$$

Since $P_n(x,a) = P_n(x,b) = 0,$ for $k \geqslant 2,$ by the first formula of Lemma 1,

$$\begin{aligned} R_k(x) &= (-1)^{k-1} \int_{[a,b]} f^{(k-1)}(t) dP_k(x,t) \\ &= (-1)^{k-1} \int_a^b f^{(k-1)}(t) P_{k-1}(x,t)\, dt \\ &\quad + (-1)^{k-1} \frac{(-1)^k}{(k-1)!} e_{k-1}(x,\mu) f^{(k-1)}(x) \\ &= -\frac{f^{(k-1)}(x)}{(k-1)!} e_{k-1}(x,\mu) + R_{k-1}(x) \end{aligned} \tag{2.3}$$

By the second formula of Lemma 1, for $k = 1,$ (2.2) becomes

$$\begin{aligned} R_1(x) &= \check{\mu}_1(a) f(a) + \int_{[a,b]} f(t) dP_1(x,t) \\ &= \check{\mu}_1(a) f(a) + \int_{[a,b]} f(t) d\mu(t) - \mu(\{a\}) f(a) - \mu([a,b]) f(x) \\ &= \int_{[a,b]} f(t) d\mu(t) - \mu([a,b]) f(x) \end{aligned} \tag{2.4}$$

From (2.3) and (2.4) follows, by iteration

$$\begin{aligned} R_n(x) &= -\sum_{k=2}^{n} \frac{f^{(k-1)}(x)}{(k-1)!} e_{k-1}(x,\mu) + R_1(x) \\ &= -\sum_{k=2}^{n} \frac{f^{(k-1)}(x)}{(k-1)!} e_{k-1}(x,\mu) - \mu([a,b]) f(x) + \int_{[a,b]} f(t) d\mu(t) \\ &= -\sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} e_k(x,\mu) + \int_{[a,b]} f(t) d\mu(t) \end{aligned}$$

which proves our assertion. □

REMARK 3. Note that $R_n(x)$ can be rewritten for $n \geqslant 2$, by Lemma 1, as

$$\begin{aligned} R_n(x) &= (-1)^n \int_{[a,b]} P_n(x,t)\, d\left[f^{(n-1)}(t) - f^{(n-1)}(x)\right] \\ &= (-1)^{n-1} \int_{[a,b]} \left[f^{(n-1)}(t) - f^{(n-1)}(x)\right] dP_n(x,t) \\ &= (-1)^{n-1} \int_{[a,b]} \left[f^{(n-1)}(t) - f^{(n-1)}(x)\right] P_{n-1}(x,t)\, dt \end{aligned}$$

It can be easily seen that the theorem above also holds for functions $f : [a,b] \to \mathbf{R}$ such that $f^{(n-1)}$ is integrable on $[a,b],$ for $n \geqslant 2,$ and

$$R_n(x) = (-1)^{n-1} \int_{[a,b]} \left[f^{(n-1)}(t) - f^{(n-1)}(x)\right] P_{n-1}(x,t)\, dt$$

Note that formula (2.1) is a generalization of (1.1).

3. Generalizations of weighted Ostrowski inequality

In this section we shall use the same notations as above.

THEOREM 2. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geqslant 1$. Then*

$$\begin{aligned} \left| \int_{[a,b]} f(t)d\mu(t) - S_n(x) \right| &\leqslant L \int_a^b |P_n(x,t)|\, dt \\ &\leqslant \frac{1}{n!} \left[(x-a)^n + (b-x)^n \right] L \|\mu\| \end{aligned} \tag{3.1}$$

for every $x \in [a,b]$.

Proof. If $\varphi : [a,b] \to \mathbf{R}$ is L-Lipschitzian on $[a,b]$, then for any integrable function $g : [a,b] \to \mathbf{R}$

$$\left| \int_{[a,b]} g(t)d\varphi(t) \right| \leqslant L \int_a^b |g(t)|\, dt \tag{3.2}$$

Using this estimate and Theorem 1 we get

$$|R_n(x)| = \left| \int_{[a,b]} P_n(x,t)\, df^{(n-1)}(t) \right| \leqslant L \int_a^b |P_n(x,t)|\, dt$$

By Remark 1 we have

$$\begin{aligned} \int_a^b |P_n(x,t)|\, dt &= \int_a^x |P_n(x,t)|\, dt + \int_x^b |P_n(x,t)|\, dt \\ &\leqslant \frac{\|\mu\|}{(n-1)!} \int_a^x (t-a)^{n-1} dt + \frac{\|\mu\|}{(n-1)!} \int_x^b (b-t)^{n-1} dt \\ &= \frac{\|\mu\|}{n!} \left[(x-a)^n + (b-x)^n \right] \end{aligned}$$

which proves our assertion. □

REMARK 4. For positive measure μ we have

$$\begin{aligned} \int_a^b |P_n(x,t)|\, dt &= \int_a^x P_n(x,t)\, dt + (-1)^n \int_x^b P_n(x,t)\, dt \\ &= \frac{1}{n!} \int_{[a,b]} |t-x|^n\, d\mu(t). \end{aligned}$$

Therefore, for every $\mu \in M[a,b]$

$$\int_a^b |P_n(x,t)|\, dt \leqslant \frac{1}{n!} \int_{[a,b]} |t-x|^n \, d|\mu|(t)$$

which gives

$$\begin{aligned} \int_a^b |P_n(x,t)|\, dt &\leqslant \frac{\|\mu\|}{n!} \max_{a\leqslant t\leqslant b} |t-x|^n \\ &= \frac{\|\mu\|}{n!} \max\{(x-a)^n, (b-x)^n\} \\ &= \frac{\|\mu\|}{n!} [\max\{x-a, b-x\}]^n \\ &= \frac{\|\mu\|}{n!} \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^n \end{aligned}$$

COROLLARY 1. *If f is L-Lipschitzian on $[a,b]$, then*

$$\left| \int_{[a,b]} f(t)d\mu(t) - \mu([a,b])f(x) \right| \leqslant L \int_a^b |P_1(x,t)|\, dt \leqslant (b-a)L\,\|\mu\|$$

for every $x \in [a,b]$.

Proof. Put $n=1$ in the theorem above. □

COROLLARY 2. *If f' is L-Lipschitzian on $[a,b]$, then*

$$\begin{aligned} &\left| \int_{[a,b]} f(t)d\mu(t) - f(x)\mu([a,b]) - f'(x)e_1(x,\mu) \right| \\ \leqslant\ & L \int_a^b |P_2(x,t)|\, dt \\ \leqslant\ & \frac{1}{2}\left[(x-a)^2 + (b-x)^2\right] L\,\|\mu\| \end{aligned}$$

for every $x \in [a,b]$.

Proof. Put $n=2$ in the theorem above. □

COROLLARY 3. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geqslant 1$. Then*

$$\begin{aligned} &\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{(k+1)!}\left[(b-x)^{k+1} - (a-x)^{k+1}\right] \right| \\ \leqslant\ & \frac{L}{(n+1)!}\left[(x-a)^{n+1} + (b-x)^{n+1}\right] \end{aligned}$$

for every $x \in [a,b]$.

Proof. Apply the theorem above for the Lebesgue measure on $[a,b]$. □

COROLLARY 4. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n\geqslant 1$. Then*

$$\left|f(y)-\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(y-x)^k\right|\leqslant\frac{L}{n!}|x-y|^n$$

for every $x,y\in[a,b]$.

Proof. Apply the theorem above for $\mu=\delta_y$, where δ_y is the Dirac measure at y. Then

$$e_k(x,\mu)=(y-x)^k,\ k\geqslant 0$$

and

$$|P_n(x,t)|=\frac{|t-y|^{n-1}}{(n-1)!},\ \ t\in[x,y]\text{ or }t\in[y,x]$$

and $P_n(x,t)=0$ for other t. □

COROLLARY 5. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n\geqslant 1$. Further, let $(c_m,m\geqslant 1)$ be a sequence in $\mathbf{R}$ such that*

$$\sum_{m\geqslant 1}|c_m|<\infty$$

and let $\{x_m;m\geqslant 1\}$ be different points in $[a,b]$. Then

$$\begin{aligned}&\left|\sum_{m\geqslant 1}c_m f(x_m)-\sum_{m\geqslant 1}\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}c_m(x_m-x)^k\right|\\ \leqslant\ &\frac{L}{n!}\sum_{m\geqslant 1}|c_m|\,|x-x_m|^n\\ \leqslant\ &\frac{L}{n!}(b-a)^n\sum_{m\geqslant 1}|c_m|\end{aligned}$$

for every $x\in[a,b]$.

Proof. Apply the theorem above for the discrete measure $\mu=\sum_{m\geqslant 1}c_m\delta_{x_m}$. □

THEOREM 3. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n\geqslant 1$. Then*

$$\begin{aligned}\left|\int_{[a,b]}f(t)d\mu(t)-S_n(x)\right|&\leqslant\max_{t\in[a,b]}|P_n(x,t)|\,V_a^b(f^{(n-1)})\\ &\leqslant\frac{1}{(n-1)!}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^{n-1}\|\mu\|\,V_a^b(f^{(n-1)})\end{aligned}$$

for every $x\in[a,b]$, where $V_a^b(f^{(n-1)})$ is the total variation of $f^{(n-1)}$ on $[a,b]$.

Proof. If $F : [a,b] \to \mathbf{R}$ is bounded and the Stieltjes integral

$$\int_{[a,b]} F(t)df^{(n-1)}(t)$$

exists, then

$$\left|\int_{[a,b]} F(t)df^{(n-1)}(t)\right| \leqslant \max_{t\in[a,b]} |F(t)| \cdot V_a^b(f^{(n-1)})$$

Let us apply this estimation to formula (2.1)

$$|R_n(x)| = \left|\int_a^b P_n(x,t)\, df^{(n-1)}(t)\right| \leqslant \max_{t\in[a,b]} |P_n(x,t)|\, V_a^b(f^{(n-1)})$$

Further, by Remark 1 we have

$$\begin{aligned}
\max_{t\in[a,b]} |P_n(x,t)| &\leqslant \max\{\frac{(x-a)^{n-1}}{(n-1)!}\|\mu\|, \frac{(b-x)^{n-1}}{(n-1)!}\|\mu\|\} \\
&= \frac{\|\mu\|}{(n-1)!}\max\{(x-a)^{n-1},(b-x)^{n-1}\} \\
&= \frac{\|\mu\|}{(n-1)!}\left[\max\{(x-a),(b-x)\}\right]^{n-1} \\
&= \frac{\|\mu\|}{(n-1)!}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^{n-1}
\end{aligned}$$

which proves our assertion. □

COROLLARY 6. *If f is a continuous function of bounded variation on $[a,b]$, then*

$$\left|\int_{[a,b]} f(t)d\mu(t) - \mu([a,b])f(x)\right| \leqslant \|\mu\|\, V_a^b(f)$$

for every $x \in [a,b]$.

Proof. Put $n = 1$ in the theorem above. □

COROLLARY 7. *If f' is a continuous function of bounded variation on $[a,b]$, then*

$$\begin{aligned}
&\left|\int_{[a,b]} f(t)d\mu(t) - f(x)\mu([a,b]) - f'(x)e_1(x,\mu)\right| \\
\leqslant\ & \max_{t\in[a,b]} |P_2(x,t)|\, V_a^b(f') \\
\leqslant\ & \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\|\mu\|\, V_a^b(f')
\end{aligned}$$

for every $x \in [a,b]$

Proof. Put $n = 2$ in Theorem 3. □

COROLLARY 8. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Then*

$$\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{(k+1)!} \left[(b-x)^{k+1} - (a-x)^{k+1} \right] \right| \leqslant \frac{1}{n!} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^n V_a^b(f^{(n-1)})$$

for every $x \in [a,b]$.

Proof. Apply the theorem above for the Lebesgue measure on $[a,b]$. □

COROLLARY 9. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Then*

$$\left| f(y) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (y-x)^k \right| \leqslant \frac{|x-y|^{n-1}}{(n-1)!} V_a^b(f^{(n-1)})$$

for every $x, y \in [a,b]$.

Proof. Apply the theorem above for $\mu = \delta_y$. □

COROLLARY 10. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Further, let $(c_m, m \geqslant 1)$ be a sequence in $\mathbf{R}$ such that*

$$\sum_{m \geqslant 1} |c_m| < \infty$$

and let $\{x_m; m \geqslant 1\}$ be different points in $[a,b]$. Then

$$\begin{aligned} &\left| \sum_{m \geqslant 1} c_m f(x_m) - \sum_{m \geqslant 1} \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} c_m (x_m - x)^k \right| \\ \leqslant\; & \frac{1}{(n-1)!} \sum_{m \geqslant 1} |c_m| \, |x - x_m|^{n-1} V_a^b(f^{(n-1)}) \\ \leqslant\; & \frac{(b-a)^{n-1}}{(n-1)!} V_a^b(f^{(n-1)}) \sum_{m \geqslant 1} |c_m| \end{aligned}$$

for every $x \in [a,b]$.

Proof. Apply the theorem above for the discrete measure $\mu = \sum_{m \geqslant 1} c_m \delta_{x_m}$. □

THEOREM 4. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)}$ is integrable, for some $n \geqslant 1$. Then*

$$\left|\int_{[a,b]} f(t)d\mu(t) - S_n(x)\right| \leqslant \max_{t\in[a,b]} |P_n(x,t)| \, \|f^{(n)}\|_1 \leqslant \frac{1}{(n-1)!}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^{n-1} \|\mu\| \, \|f^{(n)}\|_1$$

for every $x \in [a,b]$.

Proof. Note that in this case

$$V_a^b(f^{(n-1)}) = \int_a^b \left|f^{(n)}(t)\right| dt = \|f^{(n)}\|_1$$

and apply Theorem 3. □

THEOREM 5. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)} \in L_\infty[a,b]$, for some $n \geqslant 1$. Then*

$$\left|\int_{[a,b]} f(t)d\mu(t) - S_n(x)\right| \leqslant \int_a^b |P_n(x,t)| \, dt \cdot \|f^{(n)}\|_\infty \leqslant \frac{1}{n!}\left[(x-a)^n + (b-x)^n\right] \|\mu\| \, \|f^{(n)}\|_\infty$$

for every $x \in [a,b]$.

Proof. In this case $f^{(n-1)}$ is L-Lipschitzian with $L = \|f^{(n)}\|_\infty$. □

THEOREM 6. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)} \in L_p[a,b]$, for some $n \geqslant 1$ and $1 < p < \infty$. Then*

$$\left|\int_{[a,b]} f(t)d\mu(t) - S_n(x)\right| \leqslant \|P_n(x,\cdot)\|_q \|f^{(n)}\|_p \leqslant \left[\frac{(x-a)^{(n-1)q+1} + (b-x)^{(n-1)q+1}}{(n-1)q+1}\right]^{1/q} \frac{\|\mu\| \, \|f^{(n)}\|_p}{(n-1)!}$$

for every $x \in [a,b]$, where $1/p + 1/q = 1$.

Proof. By applying the Hölder inequality we have

$$\left|\int_{[a,b]} f(t)d\mu(t) - S_n(x)\right| \leqslant \int_a^b |P_n(x,t)| \left|f^{(n)}(t)\right| dt \leqslant \left(\int_a^b |P_n(x,t)|^q \, dt\right)^{1/q} \|f^{(n)}\|_p$$

Further, by Remark 1 we have

$$\begin{aligned}\int_a^b |P_n(x,t)|^q\,dt &= \int_a^x |P_n(x,t)|^q\,dt + \int_x^b |P_n(x,t)|^q\,dt \\ &\leqslant \left[\frac{\|\mu\|}{(n-1)!}\right]^q \left[\int_a^x (t-a)^{(n-1)q}dt + \int_x^b (b-t)^{(n-1)q}dt\right] \\ &= \left[\frac{\|\mu\|}{(n-1)!}\right]^q \frac{(x-a)^{(n-1)q+1}+(b-x)^{(n-1)q+1}}{(n-1)q+1}\end{aligned}$$

which proves our assertion. □

COROLLARY 11. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n)}\in L_p[a,b]$, for some $n\geqslant 1$ and $1<p<\infty$. Then*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{(k+1)!}\left[(b-x)^{k+1}-(a-x)^{k+1}\right]\right| \leqslant \left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{1/q}\frac{\|f^{(n)}\|_p}{n!}$$

for every $x\in[a,b]$, where $1/p+1/q=1$.

Proof. Apply the theorem above for the Lebesgue measure on $[a,b]$. □

COROLLARY 12. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n)}\in L_p[a,b]$, for some $n\geqslant 1$ and $1<p<\infty$. Then*

$$\left|f(y)-\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(y-x)^k\right| \leqslant \frac{|x-y|^{n-1+1/q}}{[(n-1)q+1]^{1/q}}\cdot\frac{\|f^{(n)}\|_p}{(n-1)!}$$

for every $x,y\in[a,b]$, where $1/p+1/q=1$.

Proof. Apply the theorem above for $\mu=\delta_y$. □

COROLLARY 13. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n)}\in L_p[a,b]$, for some $n\geqslant 1$ and $1<p<\infty$. Further, let $(c_m, m\geqslant 1)$ be a sequence in $\mathbf{R}$ such that*

$$\sum_{m\geqslant 1}|c_m|<\infty$$

and let $\{x_m; m\geqslant 1\}$ be different points in $[a,b]$. Then

$$\begin{aligned}&\left|\sum_{m\geqslant 1}c_m f(x_m) - \sum_{m\geqslant 1}\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}c_m(x_m-x)^k\right| \\ &\leqslant \frac{\|f^{(n)}\|_p}{(n-1)!\,[(n-1)q+1]^{1/q}}\sum_{m\geqslant 1}|c_m|\,|x-x_m|^{n-1+1/q} \\ &\leqslant \frac{(b-a)^{n-1+1/q}\|f^{(n)}\|_p}{(n-1)!\,[(n-1)q+1]^{1/q}}\sum_{m\geqslant 1}|c_m|\end{aligned}$$

for every $x \in [a,b]$.

Proof. Apply the theorem above for the discrete measure $\mu = \sum_{m \geqslant 1} c_m \delta_{x_m}$. □

Let $\alpha \in (0,1]$ and $L \geqslant 0$. Function $g : [a,b] \to \mathbf{R}$ is called α-Hölder function with constant L if

$$|g(t) - g(s)| \leqslant L |t-s|^{\alpha}, \quad s,t \in [a,b]$$

THEOREM 7. *Let* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is* α*-Hölder function with constant* L, *for some* $n \geqslant 2$. *Then*

$$\begin{aligned}\left|\int_{[a,b]} f(t)d\mu(t) - S_n(x)\right| &\leqslant L \int_a^b |t-x|^{\alpha} |P_{n-1}(x,t)|\, dt \\ &\leqslant \frac{(x-a)^{\alpha+n-1} + (b-x)^{\alpha+n-1}}{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)} L \|\mu\|\end{aligned}$$

for every $x \in [a,b]$.

Proof. By Remark 3

$$\begin{aligned}|R_n(x)| &\leqslant \int_{[a,b]} \left|\left[f^{(n-1)}(t) - f^{(n-1)}(x)\right]\right| |P_{n-1}(x,t)|\, dt \\ &\leqslant L \int_a^b |t-x|^{\alpha} |P_{n-1}(x,t)|\, dt\end{aligned}$$

Further, by Remark 1 we have

$$\begin{aligned}&\int_a^b |t-x|^{\alpha} |P_{n-1}(x,t)|\, dt \\ \leqslant\ & \|\mu\| \int_a^x (x-t)^{\alpha} \frac{(t-a)^{n-2}}{(n-2)!} dt + \|\mu\| \int_x^b (t-x)^{\alpha} \frac{(b-t)^{n-2}}{(n-2)!} dt \\ =\ & \frac{\|\mu\|}{(n-2)!} \left[\int_a^x (x-t)^{\alpha} (t-a)^{n-2} dt + \int_x^b (t-x)^{\alpha} (b-t)^{n-2} dt\right] \\ =\ & \frac{\|\mu\|}{(n-2)!} B(\alpha+1, n-1) \left[(x-a)^{\alpha+n-1} + (b-x)^{\alpha+n-1}\right] \\ =\ & \frac{(x-a)^{\alpha+n-1} + (b-x)^{\alpha+n-1}}{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)} \|\mu\|\end{aligned}$$

which proves our assertion, where B is the beta function. □

COROLLARY 14. *If* f' *is an* α*-Hölder function with constant* L, *then*

$$\begin{aligned}&\left|\int_{[a,b]} f(t)d\mu(t) - f(x)\mu([a,b]) - f'(x)e_1(x,\mu)\right| \\ \leqslant\ & L \int_a^b |t-x|^{\alpha} |P_1(x,t)|\, dt \\ \leqslant\ & \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\alpha+1} L \|\mu\|\end{aligned}$$

for every $x \in [a,b]$.

Proof. Put $n = 2$ in the theorem above. $\square$

REMARK 5. Applying calculations as in Remark 4 and in the proof of Theorem 7, for positive measure μ we have

$$\int_a^b |t-x|^\alpha |P_n(x,t)|\, dt = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \int_{[a,b]} |t-x|^{\alpha+n}\, d\mu(t)$$

Therefore, for every $\mu \in M[a,b]$

$$\int_a^b |t-x|^\alpha |P_n(x,t)|\, dt \leqslant \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \int_{[a,b]} |t-x|^{\alpha+n}\, d|\mu|(t)$$

which gives

$$\begin{aligned} \int_a^b |t-x|^\alpha |P_n(x,t)|\, dt &\leqslant \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \|\mu\| \max_{a\leqslant t\leqslant b} |t-x|^{\alpha+n} \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \|\mu\| \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^{\alpha+n} \end{aligned}$$

COROLLARY 15. *Let* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is* α*-Hölder function with constant* L*, for some* $n \geqslant 2$*. Then*

$$\left|\int_{[a,b]} f(t)d\mu(t) - S_n(x)\right| \leqslant \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n)} \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^{\alpha+n-1} L\|\mu\|$$

for every $x \in [a,b]$.

Proof. Follows from Theorem 7 and Remark 5. $\square$

COROLLARY 16. *Let* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is* α*-Hölder function with constant* L*, for some* $n \geqslant 2$*. Then*

$$\left| f(y) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(y-x)^k \right| \leqslant \frac{|x-y|^{\alpha+n-1} L}{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}$$

for every $x, y \in [a,b]$.

Proof. Apply the theorem above for $\mu = \delta_y$. $\square$

COROLLARY 17. *Let* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is* α*-Hölder function with constant* L*, for some* $n \geqslant 2$*. Further, let* $(c_m, m \geqslant 1)$ *be a sequence in* $\mathbf{R}$ *such that*

$$\sum_{m\geqslant 1} |c_m| < \infty$$

and let $\{x_m; m \geqslant 1\}$ *be different points in* $[a,b]$. *Then*

$$\begin{aligned}
&\left|\sum_{m\geqslant 1} c_m f(x_m) - \sum_{m\geqslant 1}\sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} c_m (x_m - x)^k\right| \\
\leqslant\ & \frac{L}{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)} \sum_{m\geqslant 1} |c_m|\, |x - x_m|^{\alpha+n-1} \\
\leqslant\ & \frac{L(b-a)^{\alpha+n-1}}{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)} \sum_{m\geqslant 1} |c_m|
\end{aligned}$$

for every $x \in [a,b]$.

Proof. Apply the theorem above for the discrete measure $\mu = \sum_{m\geqslant 1} c_m \delta_{x_m}$. □

4. Some Grüss-type inequalities

Measure $\mu \in M[a,b]$ is called balanced if $\mu([a,b]) = 0$. It is called n-balanced if $\check{\mu}_n(b) = 0$. We see that 1 -balanced measure is the same as balanced measure. Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)} \in L_\infty[a,b]$, for some $n \geqslant 1$. Then

$$m_n \leqslant f^{(n)}(t) \leqslant M_n,\ t \in [a,b],\ a.e.$$

for some real constants m_n and M_n.

THEOREM 8. *Let* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)} \in L_\infty[a,b]$, *for some* $n \geqslant 2$. *If* $x \in [a,b]$ *and* $\mu \in M[a,b]$ *are such that*

$$e_{n-1}(x,\mu) = 0$$

then

$$\left|\int_{[a,b]} f(t)d\mu(t) - S_n(x)\right| \leqslant \frac{M_{n-1} - m_{n-1}}{2(n-1)!} \left[(x-a)^{n-1} + (b-x)^{n-1}\right] \|\mu\|$$

Proof. By Remark 3 we have

$$R_n(x) = (-1)^{n-1} \int_{[a,b]} \left[f^{(n-1)}(t) - f^{(n-1)}(x)\right] P_{n-1}(x,t)\, dt$$

Define measure ν_{n-1} by

$$d\nu_{n-1}(t) = (-1)^{n-1} P_{n-1}(x,t)\, dt$$

Then

$$\begin{aligned}
\nu_{n-1}([a,b]) &= (-1)^{n-1} \int_a^b P_{n-1}(x,t)\, dt \\
&= (-1)^{n-1} \frac{(-1)^{n-1}}{(n-1)!} e_{n-1}(x,\mu) \\
&= \frac{1}{(n-1)!} e_{n-1}(x,\mu)
\end{aligned}$$

and by our condition $\nu_{n-1}([a,b]) = 0$, which means that ν_{n-1} is balanced measure. Further,

$$\|\nu_{n-1}\| = \int_a^b |P_{n-1}(x,t)|\, dt \leqslant \frac{\|\mu\|}{(n-1)!}\left[(x-a)^{n-1} + (b-x)^{n-1}\right]$$

Therefore, by [2, Theorem 1] we have

$$\begin{aligned} |R_n(x)| &\leqslant \frac{M_{n-1} - m_{n-1}}{2}\|\nu_{n-1}\| \\ &\leqslant \frac{M_{n-1} - m_{n-1}}{2}\frac{\|\mu\|}{(n-1)!}\left[(x-a)^{n-1} + (b-x)^{n-1}\right] \\ &= \frac{M_{n-1} - m_{n-1}}{2(n-1)!}\left[(x-a)^{n-1} + (b-x)^{n-1}\right]\|\mu\| \end{aligned}$$

which proves our assertion. □

COROLLARY 18. *For $f' \in L_\infty[a,b]$ let $x \in [a,b]$ and $\mu \in M[a,b]$ be such that*

$$\int_{[a,b]} (t-x)\, d\mu(t) = 0$$

Then

$$\left|\int_{[a,b]} f(t)\, d\mu(t) - \mu([a,b]) f(x)\right| \leqslant \frac{M_1 - m_1}{2}(b-a)\|\mu\|$$

Proof. Put $n = 2$ in the theorem above. □

COROLLARY 19. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)} \in L_\infty[a,b]$, for some $n \geqslant 2$. If $\mu \in M[a,b]$ is such that*

$$m_0(\mu) = m_1(\mu) = \cdots = m_{n-1}(\mu) = 0$$

then

$$\left|\int_{[a,b]} f(t)\, d\mu(t)\right| \leqslant \frac{M_{n-1} - m_{n-1}}{2(n-1)!}\left[(x-a)^{n-1} + (b-x)^{n-1}\right]\|\mu\|$$

for every $x \in [a,b]$.

Proof. Apply the theorem above and note that in this case we have $e_k(x,\mu) = 0$, for $k = 0, 1, ..., n-1$, and also $S_n(x) = 0$, for every $x \in [a,b]$. □

COROLLARY 20. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)} \in L_\infty[a,b]$, for some $n \geqslant 2$. If $\mu \in M[a,b]$ is k-balanced, for $k = 1, .., n$, then*

$$\left|\int_{[a,b]} f(t)\, d\mu(t)\right| \leqslant \frac{M_{n-1} - m_{n-1}}{2(n-1)!}\left[(x-a)^{n-1} + (b-x)^{n-1}\right]\|\mu\|$$

for every $x \in [a,b]$.

Proof. Note that in this case, by [2, Theorem 4] we have

$$m_0(\mu) = m_1(\mu) = \cdots = m_{n-1}(\mu) = 0$$

Apply now Corollary 19. □

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