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EXACT INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. Pachpatte [2] presented linear inequalities involving certain integrals of convex functions defined on a finite closed interval. In the present note we explore the the whole set of possible values of those quantities, thus obtaining sharp inequalities that cannot be further improved. Our results are obtained by the application of the convexity method, a simple but powerful tool which is often used in probability theory for deriving moment-type inequalities.

1. Introduction

Let f be a nonnegative convex function defined on the interval $[a,b]$. Pachpatte presented the following integral inequalities ([2], (5.1)–(5.3), in a slightly rearranged form)

$$\frac{1}{(b-a)^2}\iint\limits_{a\leqslant x<y\leqslant b}\left[\int_0^1 f(tx+(1-t)y)\,dt\right]dx\,dy \geqslant \frac{3}{2}\cdot\frac{1}{(b-a)^2}\int_a^b (b-x)f(x)\,dx-\frac{1}{4}f(a). \tag{1.1}$$

$$\frac{1}{(b-a)^2}\iint\limits_{a\leqslant y<x\leqslant b}\left[\int_0^1 f(tx+(1-t)y)\,dt\right]dx\,dy \geqslant \frac{3}{2}\cdot\frac{1}{(b-a)^2}\int_a^b (x-a)f(x)\,dx-\frac{1}{4}f(b). \tag{1.2}$$

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,dt\,dx\,dy \geqslant \frac{3}{2}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{4}[f(a)+f(b)]. \tag{1.3}$$

All these inequalities hold with equality for linear functions.

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It is clear that the two non-symmetric cases (1.1) and (1.2) are equivalent, and symmetric case (1.3) can be obtained simply by adding them. Inequality (1.1) can be obtained by replacing the function f with its tangent line at x in the innermost integral on the left-hand side.

The aim of the present note is to improve these inequalities by computing exact upper and lower bounds for the triple integral on the left-hand sides, as functions of the two terms on the left-hand sides. When doing so we will apply the convexity method. This method is known to provide exact bounds for integral-type functionals in terms of other quantities of the same kind. A detailed description of the method and some examples of applications can be found in [1].

The bounds we are going to derive are not simply slight improvements on inequalities that were once obtained with short and simply proofs. They are exact in the sense that they cannot be further improved. In addition, they are obtained by a systematic method which, by its applicability, is more useful than any ad hoc methods, whatever short and simple they may be. This property makes them worth computing in spite that they require more effort. However, these exact bounds are often too complicated to replace the much simpler linear estimates.

The main result will be stated and proved in Section 3 (symmetric case) and Section 4 (nonsymmetric case). Section 2 is devoted to a lemma of independent interest.

Before applying the convexity method we transform the quantities in consideration into a standardized form, so as to get rid of some nuissance parameters.

First, we can assume that f is continuous. It surely is on the open interval (a,b), but can have jumps at a or b. If we redefine it at the endpoints so that it becomes continuous, the integrals do not change, but $f(a)+f(b)$ decreases. Therefore the upper bounds obtained for continuous functions remain valid in the general case, but in the lower bounds $f(a)$ and $f(b)$ must be substituted with $f(a+)$ and $f(b-)$, resp.

In the symmetric case (1.3) we can also suppose that $f(a)+f(b)$ is equal to 1, because it appears in the integrals as a multiplicative factor. If there is an upper or lower bound of the form

$$F\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)$$

in this particular case, it can be extended to the general case as

$$[f(a)+f(b)]\,F\left(\frac{1}{f(a)+f(b)}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx\right).$$

Next, let us introduce s by $x=a(1-s)+bs$. Then s runs over $[0,1]$ as x runs over $[a,b]$. Clearly, $dx=(b-a)ds$, and $tx+(1-t)y=a[t(1-s)+(1-t)(1-r)]+b[ts+(1-t)r]$. Let $\widetilde{f}(z)=f(a(1-z)+bz)$, then

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,dt\,dx\,dy$$
$$=\int_0^1\int_0^1\int_0^1 \widetilde{f}(ts+(1-t)r)\,dt\,ds\,dr,$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \int_0^1 \widetilde{f}(s)\,ds,$$
$$\frac{1}{(b-a)^2}\int_a^b (b-x)f(x)\,dx = \int_0^1 (1-s)\widetilde{f}(s)\,ds,$$
$$\frac{1}{(b-a)^2}\int_a^b (x-a)f(x)\,dx = \int_0^1 s\widetilde{f}(s)\,ds.$$

Since $\widetilde{f}$ is a convex function itself, one can suppose that $a = 0$ and $b = 1$ without loss of generality.

First we deal with the symmetric case, thus we suppose that $f(0) + f(1) = 1$.

By symmetry we have

$$\iint\limits_{0\leqslant r<s\leqslant 1} \left[\int_0^1 f(ts+(1-t)r)\,dt\right] ds\,dr = \iint\limits_{0\leqslant s<r\leqslant 1} \left[\int_0^1 f(ts+(1-t)r)\,dt\right] ds\,dr$$
$$= \frac{1}{2}\int_0^1\int_0^1\int_0^1 f(ts+(1-t)r)\,dt\,ds\,dr. \tag{1.4}$$

Let us substitute $u = ts + (1-t)r$ in the inner integral, then interchange the order of integration to obtain

$$\iint\limits_{0\leqslant r<s\leqslant 1} \left[\int_0^1 f(ts+(1-t)r)\,dt\right] ds\,dr = \iint\limits_{0\leqslant r<s\leqslant 1} \int_r^s \frac{f(u)}{s-r}\,du\,ds\,dr$$
$$= \int_0^1 f(u)\int_0^u\int_u^1 \frac{1}{s-r}\,ds\,dr\,du$$
$$= \int_0^1 f(u)\int_0^u \big[\log(1-r) - \log(u-r)\big]\,dr\,du$$
$$= \int_0^1 f(u)H(u)\,du, \tag{1.5}$$

where $H(u) = -u\log u - (1-u)\log(1-u)$ denotes the entropy function.

Let us introduce the following notations. $\mathcal{F}$ is the set of nonnegative convex functions $f : [0,1] \to \mathbb{R}$, such that $f(0) + f(1) = 1$, and

$$\mathcal{L} = \left\{(\xi,\eta)\;\middle|\; \xi = \int_0^1 f(u)\,du,\ \eta = 2\int_0^1 f(u)H(u)\,du,\ f\in\mathcal{F}\right\}.$$

Thus, we fix ξ and aim at finding

$$\inf/\sup\{\eta \mid (\xi,\eta)\in\mathcal{L}\},$$

or more generally, we want to characterize the whole set $\mathcal{L}$.

For the convexity method we first have to find the extremal points of the set $\mathcal{F}$. Such a result may be of independent interest, therefore it is presented in a separate section.

2. Extremal points of $\mathcal{F}$

For $0 < x \leqslant 1$ let $\varphi_x(t) = \dfrac{(x-t)^+}{x}$, and for $0 \leqslant x < 1$ let $\psi_x(t) = \dfrac{(t-x)^+}{1-x}$.

THEOREM 2.1. *The set of extremal points of $\mathcal{F}$ is equal to*

$$\mathcal{F}_0 = \{\varphi_x \mid 0 < x \leqslant 1\} \cup \{\psi_x \mid 0 \leqslant x < 1\}.$$

Every $f \in \mathcal{F}$ can be expressed as their mixture, that is, there exist measures λ and μ defined on the Borel subsets of $[0,1]$ such that $\lambda(0,1] + \mu[0,1) = 1$, and

$$f(t) = \int_{0+}^{1} \varphi_x(t)\,\lambda(dx) + \int_{0}^{1-} \psi_x(t)\,\mu(dx), \quad 0 \leqslant t \leqslant 1. \tag{2.1}$$

REMARK 2.1. This form of f is not unique, e.g.,

$$\tfrac{1}{2}\big(\varphi_{2/3} + \psi_{1/3}\big) = \tfrac{1}{4}\big(\varphi_{1/3} + \varphi_1 + \psi_0 + \psi_{2/3}\big).$$

Proof. We will prove (2.1) by constructing λ and μ explicitely. Let f'_-, and f'_+ denote the left, resp. right derivative of f. Furthermore, let δ_x denote the degenerate distribution (unit mass) concentrated to x. Let $c = \arg\min f$, then $f'_-(c) \leqslant 0$, if $c > 0$, and $f'_+(c) \geqslant 0$, if $c < 1$.

If $0 < c < 1$, define

$$\lambda = f(c)\delta_1 - cf'_-(c)\delta_c + \lambda_0, \quad \mu = f(c)\delta_0 + (1-c)f'_+(c)\delta_c + \mu_0,$$

where $\lambda_0(dx) = x\,\mathbf{1}_{(0,c)}(x)\,df'_-(x)$, and $\mu_0(dx) = (1-x)\mathbf{1}_{(c,1)}(x)\,df'_+(x)$.

If $c = 0$, let $\lambda = f(0)\delta_1$, and $\mu = [f(0) + f'_+(0)]\delta_0 + \mu_0$.

Finally, if $c = 1$, let $\lambda = [f(1) - f'_-(1)]\delta_1 + \lambda_0$, and $\mu = f(1)\delta_0$.

It is not too hard to show that in this way λ and μ satisfy (2.1). Here we only deal with the case $0 < c < 1$ in full details. One can proceed similarly in the (simpler) cases $c = 0$ or $c = 1$. In order to avoid lengthy repetitions those cases are left to the reader.

Thus, suppose $0 < c < 1$. Let $I(t)$ denote the sum of integrals on the right-hand side of (2.1). Then

$$I(0) = \int_{0+}^{1} d\lambda = f(c) - cf'_-(c) + \int_{0+}^{c-} x\,df'_-(x).$$

Integrating by parts we get

$$\int_{0+}^{c-} x\,df'_-(x) = \Big[xf'_-(x)\Big]_{0+}^{c-} - \int_{0+}^{c-} f'_-(x)\,dx = cf'_-(c) - f(c) + f(0),$$

that is, $I(0) = f(0)$.

Similarly, we have

$$\begin{aligned} I(1) &= \int_0^{1-} d\mu = f(c) + (1-c)f'_+(c) + \int_{c+}^{1-} (1-x)\, df'_+(x) \\ &= f(c) + (1-c)f'_+(c) + \Big[(1-x)f'_+(x)\Big]_{c+}^{1-} + \int_{c+}^{1-} f'_+(x)\, dx \\ &= f(c) + (1-c)f'_+(c) - (1-c)f'_+(c) + f(1) - f(c) = f(1). \end{aligned}$$

Obviously, $I(c) = (1-c)\lambda(\{1\}) + c\mu(\{0\}) = f(c)$.
Let $0 < t < c$. Integrating by parts again we can write

$$\begin{aligned} I(t) &= \int_{t+}^{1} \frac{x-t}{x}\,\lambda(dx) + \int_0^{t-} \frac{t-x}{1-x}\,\mu(dx) \\ &= f(c)(1-t) + f(c)t + \frac{c-t}{c}(-c)f'_-(c) + \int_{t+}^{c-} \frac{x-t}{x}\, x\, df'_-(x) \\ &= f(c) - (c-t)f'_-(c) + \Big[(x-t)f'_+(x)\Big]_{t+}^{c-} - \int_{t+}^{c-} f'_-(x)\, dx \\ &= f(c) - (c-t)f'_-(c) + (c-t)f'_-(c) - f(c) + f(t) = f(t). \end{aligned}$$

Finally, let $c < t < 1$. Then

$$\begin{aligned} I(t) &= \int_{t+}^{1} \frac{x-t}{x}\,\lambda(dx) + \int_0^{t-} \frac{t-x}{1-x}\,\mu(dx) \\ &= f(c)(1-t) + f(c)t + \frac{t-c}{1-c}\,(1-c)f'_+(c) + \int_{c+}^{t-} \frac{t-x}{1-x}\,(1-x)\, df'_+(x) \\ &= f(c) + (t-c)f'_+(c) + \Big[(t-x)f'_+(x)\Big]_{c+}^{t-} + \int_{c+}^{t-} f'_+(x)\, dx \\ &= f(c) + (t-c)f'_+(c) - (t-c)f'_+(c) + f(t) - f(c) = f(t). \end{aligned}$$

In fact, what we have just shown is that all extremal points are among the functions φ_x and ψ_x. We are also expected to prove that none of them can be expressed as a mixture of the others. This is obviously true and not too hard to see, but, since this part of the theorem is not needed for the convexity method, we omit the proof. □

3. Application of the convexity method in the symmetric case

In this section we characterize the set

$$\mathcal{L} = \left\{ (\xi, \eta) \;\Big|\; \xi = \int_0^1 f(u)\, du,\ \eta = 2\int_0^1 f(u)H(u)\, du,\ f \in \mathcal{F} \right\}.$$

THEOREM 3.1. *$\mathcal{L}$ is a convex set, and its projection on the horizontal (ξ) axis is* $[0,\ 1/2]$. *Its boundaries are the following.*

(i) Upper boundary.

$\mathcal{L}$ is bounded from above by the straight line $\eta = \xi$

(ii) Lower boundary.

If $0 \leqslant \xi \leqslant 1/4$, $\mathcal{L}$ is bounded from below by the curve $\eta = \Phi(2\xi)$, where

$$\Phi(x) = -\frac{1}{3}\,x^2\log x - \frac{(1-x)^3}{3x}\,\log(1-x) + \frac{5x-2}{6}\,. \tag{3.1}$$

If $1/4 \leqslant \xi \leqslant 1/2$, $\mathcal{L}$ is bounded from below by the straight line

$$\eta = \frac{5-2\log 2}{3}\,\xi - \frac{1-\log 2}{3}\,. \tag{3.2}$$

Proof. By the linearity of the integral, the set $\mathcal{L}$ is equal to the convex hull of the set $\mathcal{L}_0 = \{(\xi,\ \eta)\ |\ f \in \mathcal{F}_0\}$. $\mathcal{L}_0$ is the union of its two subsets defined by $f = \varphi_x$, $0 < x \leqslant 1$, and $f = \psi_x$, $0 \leqslant x < 1$, resp. Since $\psi_x(s) = \varphi_{1-x}(1-s)$, and $H(s) = H(1-s)$, the two sets of points coincide, thus we can focus on the first one. This set is a curve Γ given in parametrized form

$$\xi = \int_0^x \left(1-\frac{t}{x}\right)dt = \frac{x}{2}, \quad \eta = 2\int_0^x \left(1-\frac{t}{x}\right)H(t)\,dt = \Phi(x).$$

We will first show that the upper boundary of the convex hull of Γ is the chord connecting the origin with the endpoint $(1/2,\ 1/2)$. To this end it is sufficent to prove that Γ is below the chord everywhere over the interval $(0,\ 1/2)$.

By symmetry,

$$\eta = 2\int_0^1 \varphi_x(t)H(t)\,dt = 2\int_0^{1/2} [\varphi_x(t) + \varphi_x(1-t)]H(t)\,dt.$$

Here $\varphi_x(t) + \varphi_x(1-t)$ is decreasing in the interval $[0,\ 1/2]$, while $H(t)$ is inreasing, hence

$$\eta \leqslant \int_0^{1/2} [\varphi_x(t) + \varphi_x(1-t)]\,dt \int_0^{1/2} 4H(t)\,dt = \frac{x}{2} = \xi.$$

This proves the upper bound part of Theorem 3.1.

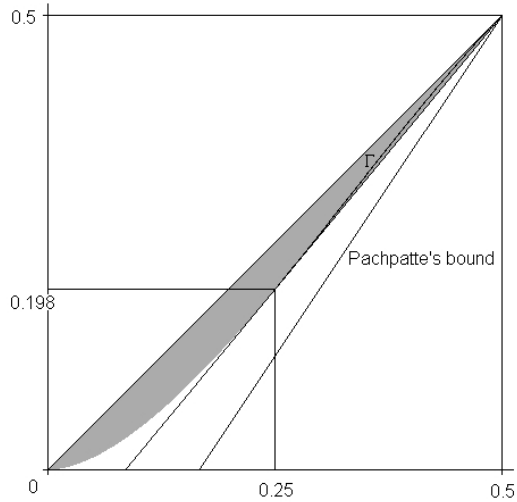

Figure 1. Set $\mathcal{L}$ with curve Γ.

For the lower bound note that Γ is S-shaped: it is convex in the beginning, and concave afterwards.

Consider the derivatives

$$\frac{d\eta}{d\xi} = \frac{d}{d\xi}\int_0^{2\xi}\Big(1-\frac{t}{2\xi}\Big)2H(t)\,dt = \frac{1}{\xi^2}\int_0^{2\xi} tH(t)\,dt = \int_0^1 4sH(2s\xi)\,ds,$$
$$\frac{d^2\eta}{d\xi^2} = \int_0^1 8s^2H'(2s\xi)\,ds.$$

Since H is increasing, the first derivative is positive and increasing for $\xi \leqslant 1/2$. On the other hand, H' is decreasing, thus so is the second derivative. It is negative at $\xi = 1/2$, therefore the second derivative is positive in the beginning, and then it is negative. Hence Γ is S-shaped, indeed.

From this it follows that the largest convex minorant of Γ_{12} coincides with the curve itself for $0 \leqslant \xi \leqslant \xi^*$, and with the tangent line to Γ at ξ^* for $\xi^* < \xi \leqslant 1/2$, where ξ^* is characterized by the property that the tangent line to the curve at ξ^* goes through the upper endpoint $(1/2,\,1/2)$. Thus ξ^* is the only solution to the equation

$$\frac{\frac{1}{2}-\Phi(2\xi)}{\frac{1}{2}-\xi} = 2\,\Phi'(2\xi), \tag{3.3}$$

in the interval $(0,\,1/2)$. One can easily verify that $\xi = 1/4$ satisfies (3.3), and $\Phi(1/2) = (1+2\log 2)/12$, hence the lower bound follows. $\square$

REMARK 3.2. From the proof one can easily derive the conditions of equality in our bounds.

If $\xi < 1/2$, equality in the upper bound cannot be attained for continuous functions f, but one can get arbitrarily close. If the condition of continuity is dropped, equality is attained for functions that are linear inside the unit interval: $f(t) = bt + a(1-t)$ for $0 < t < 1$, where $a, b \geqslant 0$, $a + b = 2\xi$, and $f(0) \geqslant a$, $f(1) \geqslant b$, such that $f(0) + f(1) = 1$ (they are mixtures of φ_1, ψ_0, $\mathbf{1}_{\{0\}}$, and $\mathbf{1}_{\{1\}}$).

$\xi = 1/2$ can only occur for linear functions of the form $f(t) = at + (1-a)(1-t)$, with $0 \leqslant a \leqslant 1$. In this case the upper bound is sharp.

If $\xi \leqslant 1/4$, then in the lower bound equality holds if and only if $f = a\varphi_{2\xi} + (1-a)\psi_{1-2\xi}$, $0 \leqslant a \leqslant 1$ (piecewise linear taking on zero between 2ξ and $1-2\xi$).

Finally, if $\xi > 1/4$, then equality in the lower bound is attained for f that is linear in the intervals $[0, 1/2]$ and $[1/2, 1]$, and $f(0) + f(1) = 1$, while $f(1/2) = 2\xi - 1/2$ (convex combinations of $\varphi_{1/2}$, $\psi_{1/2}$, φ_1, ψ_0).

In the original setting Theorem 3.1 yields the following exact bounds.

COROLLARY 3.3.

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b \int_0^1 f(tx + (1-t)y)\, dt\, dx\, dy \leqslant \frac{1}{b-a} \int_a^b f(x)\, dx. \tag{3.4}$$

If $\frac{1}{b-a} \int_a^b f(x)\, dx < \frac{1}{4} [f(a) + f(b)]$, *then*

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b \int_0^1 f(tx + (1-t)y)\, dt\, dx\, dy$$
$$\geqslant [f(a) + f(b)]\, \Phi\left(\frac{1}{f(a) + f(b)} \cdot \frac{1}{b-a} \int_a^b f(x)\, dx \right), \tag{3.4}$$

where $\Phi(x)$ *is the function defined in* (3.1).

On the other hand, if $\frac{1}{4} [f(a) + f(b)] \leqslant \frac{1}{b-a} \int_a^b f(x)\, dx$, *then*

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b \int_0^1 f(tx + (1-t)y)\, dt\, dx\, dy$$
$$\geqslant \frac{5 - 2\log 2}{3} \cdot \frac{1}{b-a} \int_a^b f(x)\, dx - \frac{1 - \log 2}{3} [f(a) + f(b)] \tag{3.5}$$

REMARK 3.4. We can see that Pachpatte's lower bound (1.3) is sharp if and only if f is a linear function. When a linear lower bound is needed, use (3.5) instead.

4. Application of the convexity method in the nonsymmetric case

By the equivalence of (1.1) and (1.2) it is sufficent to deal with one of them, let that be the latter one. Thus, we want to describe the set

$$\mathcal{L}' = \left\{ (\xi, \eta) \;\middle|\; \xi = \int_0^1 t f(t)\, dt,\ \eta = \int_0^1 f(t) H(t)\, dt,\ f \in \mathcal{F}' \right\},$$

where $\mathcal{F}'$ is the set of nonnegative, convex, continuous functions $f : [0,1] \to \mathbb{R}$ such that $f(1)$ is fixed. Such an f can always be decomposed into a sum of $f_0 \in \mathcal{F}'$ and $f_1 \in \mathcal{F}'$, where $f_1(1) = f_0(0) = 0$; namely, with

$$f_1(t) = \min_{[0,t]} f - t \cdot \min_{[0,1]} f, \quad f_0(t) = \min_{[t,1]} f - (1-t) \cdot \min_{[0,1]} f.$$

This observation will be useful in the proof of the following result.

THEOREM 4.1. *$\mathcal{L}'$ is the upward infinite convex region bounded from below by the following inequalities.*

(i) Suppose $0 \leqslant \xi \leqslant \frac{1}{3} f(1)$. Then $\eta \geqslant \frac{1}{2} f(1)\Phi(x)$, where x is given by the equation $\xi = \frac{1}{6} f(1)\, x(3-x)$, and Φ is the function introduced in (3.1).

(ii) Suppose $\xi > \frac{1}{3} f(1)$. Then $\eta \geqslant \frac{3}{2}\xi - \frac{1}{4} f(1)$.

Proof. By defining

$$\mathcal{L}_0 = \big\{(\xi,\eta) \,\big|\, f \in \mathcal{F}', f(0) = 1, f(1) = 0\big\},$$
$$\mathcal{L}_1 = \big\{(\xi,\eta) \,\big|\, f \in \mathcal{F}', f(0) = 0, f(1) = 1\big\}$$

we have $\mathcal{L}' = \widetilde{\mathcal{L}}_0 + f(1)\mathcal{L}_1$, where $\widetilde{\mathcal{L}}_0 = \bigcup_{c>0} c\mathcal{L}_0$ is the convex cone defined by $\mathcal{L}_0$. In order to characterize $\mathcal{L}'$ we first study $\mathcal{L}_0$ and $\mathcal{L}_1$.

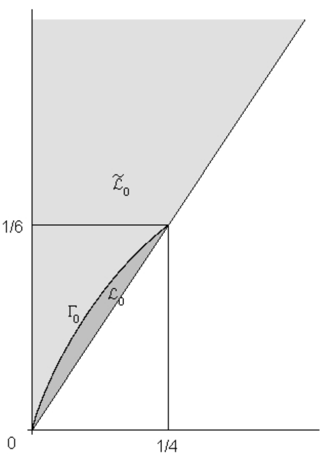

Figure 2. $\mathcal{L}_0$ *and* $\widetilde{\mathcal{L}}_0$.

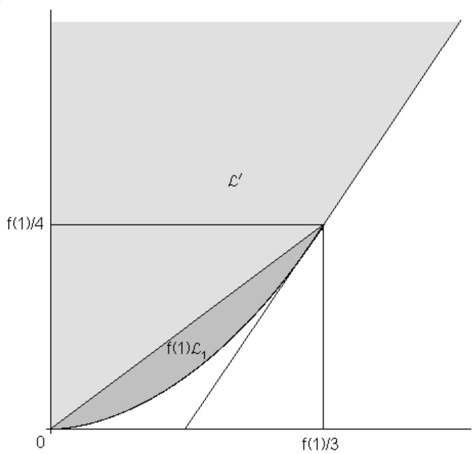

Figure 3. $f(1)\mathcal{L}_1$ *and* $\mathcal{L}'$.

From Theorem 2.1 it follows that $\mathcal{L}_0$ is the convex hull of the curve Γ_0 given in the following parametrized form.

$$\xi = \int_0^1 t\varphi_x(t)\,dt, \quad \eta = \int_0^1 \varphi_x(t)H(t)\,dt, \quad x \in (0,1],$$

that is,

$$\xi = \int_0^x \Big(1 - \frac{t}{x}\Big) t\,dt = \frac{x^2}{6}, \quad \eta = \int_0^x \Big(1 - \frac{t}{x}\Big) H(t)\,dt = x \int_0^1 (1-s)H(sx)\,ds.$$

The endpoints of the curve are $(0,\,0)$ (as $x \to 0$), and $(1/6,\,1/4)$ (for $x = 1$). Since

$$\frac{d\eta}{d\xi} = \frac{d\eta/dx}{d\xi/dx} = \frac{3}{x}\int_0^x \frac{t}{x^2}\,H(t)\,dt = \frac{3}{x}\int_0^1 sH(sx)\,ds \geqslant 0, \tag{4.1}$$

η, as a function of ξ, is increasing.

Let us examine the sign of the second derivative.

$$\begin{aligned} \frac{d^2\eta}{d\xi^2} &= \frac{3}{x}\Big[-\frac{9}{x^4}\int_0^x tH(t)\,dt + \frac{3}{x^2}\,H(x)\Big] = \frac{9}{x^5}\Big[x^2H(x) - 3\int_0^x tH(t)\,dt\Big] \\ &= \frac{9}{x^5}\Big[2\int_0^x tH(t)\,dt + \int_0^x t^2H'(t)\,dt - 3\int_0^x tH(t)\,dt\Big] = \frac{9}{x^5}\int_0^x t^3\Big(\frac{H(t)}{t}\Big)'\,dt. \end{aligned}$$

Here $\big(H(t)/t\big)' = t^{-2}\log(1-t) < 0$, hence Γ_0 is concave. Thus $\mathcal{L}_0$ is equal to the domain between Γ_0 and the chord connecting its endpoints. The vertical axis is tangent to Γ_0 at the origin, because from (4.1) we get $\lim\limits_{x\to 0}(d\eta/d\xi) = +\infty$. Therefore the cone $\widetilde{\mathcal{L}}_0$ is just the angular domain between the positive half of the vertical axis and the ray $\eta = 3\xi/2$, $\xi \geqslant 0$.

Let us turn to $\mathcal{L}_1$. It is the convex hull of the curve Γ_1 given by

$$\xi = \int_0^1 t\psi_y(t)\,dt, \quad \eta = \int_0^1 \psi_y(t)H(t)\,dt, \quad y \in [0,1).$$

By substituting $x = 1 - x$ and $s = 1 - t$ we get

$$\xi = \int_0^x \Big(1 - \frac{s}{x}\Big)(1-s)\,ds = \frac{x(3-x)}{6},$$

$$\eta = \int_0^x \Big(1 - \frac{s}{x}\Big)H(s)\,ds = \int_0^x \varphi_x(s)H(s)\,ds = \frac{1}{2}\,\Phi(x).$$

This time we have

$$\begin{aligned} \frac{d\eta}{d\xi} &= \frac{d\eta/dx}{d\xi/dx} = \frac{6}{3-2x}\int_0^x \frac{s}{x^2}\,H(s)\,ds, \\ \frac{d^2\eta}{d\xi^2} &= \frac{6}{3-2x}\Big[\Big(\frac{12}{(3-2x)^2x^2} - \frac{12}{(3-2x)x^3}\Big)\int_0^x sH(s)\,ds + \frac{6}{(3-2x)x}\,H(x)\Big] \\ &= \frac{36\,A(x)}{(3-2x)^3x^3}, \end{aligned}$$

where

$$
\begin{aligned}
A(x) &= (3-2x)x^2H(x) - 6(1-x)\int_0^x sH(s)\,ds \\
&= (3-2x)x^2\Big(-x\log x - (1-x)\log(1-x)\Big) - \\
&\quad - 6(1-x)\Big(-\frac{x^3}{3}\log x + \frac{(1-x)^2(1+2x)}{6}\log(1-x) + \frac{x^2+2x}{12}\Big) \\
&= -x^3\log x - (1-x)\log(1-x) - (1-x)\,\frac{x^2+2x}{2}.
\end{aligned}
$$

This is positive, because $-\log(1-x) \geqslant x + \dfrac{x^2}{2}$. Hence Γ_1 is convex from below, and $\mathcal{L}_1$ is equal to the domain between Γ_1 and the chord connecting its endpoints, $(0,\,0)$ and $(1/3,\,1/4)$. If we move a point along Γ_1 from the origin, the slope of the tangent line drawn to Γ_1 at that point increases, starting from 0 and arriving at $3/2$ at $\xi = 1/3$. Therefore $\mathcal{L}'$ is bounded from below by the curve $f(1)\Gamma_1$ itself, if $0 \leqslant \xi \leqslant f(1)/3$, and by its tangent line at the endpoint $\big(f(1)/3, f(1)/4\big)$, if $\xi \geqslant f(1)/3$. The equation of the tangent line is

$$\eta = \frac{3}{2}\Big(\xi - \frac{1}{3}f(1)\Big) + \frac{1}{4}f(1) = \frac{3}{2}\,\xi - \frac{1}{4}f(1)\,;$$

this is the Pachpatte bound, which now proves to be sharp not only at a single point, but over a whole halfline, unlike in the symmetric case.

Hence Theorem 4.1 follows. □

Summarizing our results in the original setting of (1.2) we obtain the following exact lower bounds.

COROLLARY 4.2. *Suppose* $\dfrac{1}{(b-a)^2}\displaystyle\int_a^b (x-a)f(x)\,dx < \frac{1}{3}f(b)$. *Then*

$$\frac{1}{(b-a)^2}\iint\limits_{a\leqslant y<x\leqslant b}\Big[\int_0^1 f(tx+(1-t)y)\,dt\Big]dx\,dy \geqslant \frac{1}{2}f(b)\Phi(z),$$

where z *is given by the equation*

$$\frac{1}{(b-a)^2}\int_a^b (x-a)f(x)\,dx = \frac{1}{6}f(b)\,z(3-z),$$

and Φ *is the function introduced in* (3.1).

If $\dfrac{1}{(b-a)^2}\displaystyle\int_a^b (x-a)f(x)\,dx \geqslant \frac{1}{3}f(b)$, *the best possible lower bound is that of Pachpatte* (1.2).

On the other hand, no upper bounds can be set. □

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