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SOME PROPERTIES OF DUAL FORM OF THE HAMY'S SYMMETRIC FUNCTION

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Abstract. The Schur-concavity and the Schur-geometrically convexity of dual form for the Hamy symmetric function are discussed and some analytic inequalities are established by use of the theory of majorization.

1. Introduction

Throughtout the paper we use the sets of n-dimensional vectors over the reals, real number field by $\mathbb{R}^n$, and:

$$\mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geqslant 0, i = 1, \cdots, n\},$$
$$\mathbb{R}^n_{++} = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \cdots, n\}.$$

if $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$ then the Hamy symmetric function ([3],[4]-p.67) is defined as:

$$H_r(x) = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} \left(\prod_{j=1}^{r} x_{i_j} \right)^{\frac{1}{r}}, \quad r = 1, 2, \cdots, n. \tag{1.1}$$

Corresponding to this is the r-th order Hamy mean

$$\sigma_n(x, r) = \frac{1}{\binom{n}{r}} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} \left(\prod_{j=1}^{r} x_{i_j} \right)^{\frac{1}{r}}. \tag{1.2}$$

T. Hara et al. [3] established the following refinement of the classical arithmetic and geometric means inequality:

$$G_n(x) = \sigma_n(x, n) \leqslant \sigma_n(x, n-1) \leqslant \cdots \leqslant \sigma_n(x, 2) \leqslant \sigma_n(x, 1) = A_n(x) \tag{1.3}$$

where $A_n(x) = \frac{1}{n} \sum_{i=1}^{n} x_i, G_n(x) = \left(\prod_{i=1}^{n} x_i \right)^{\frac{1}{n}}$ denote the classical arithmetic and geometric means.

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In [1], the Schur convexity of Hamy's symmetric functions and its generalization are discussed.

The dual form of the Hamy's symmetric functions are

$$H_r^*(x) = \prod_{1\leqslant i_1<\cdots<i_r\leqslant n}\left(\sum_{j=1}^{r} x_{i_j}^{\frac{1}{r}}\right), \quad r=1,2,\cdots,n. \tag{1.4}$$

The aim of this paper is to study the Schur-convexity and Schur-geometric-convexity properties of $H_r^*(x)$ and some interesting results are given.

We need the following definitions and lemmas.

2. Definitions and lemmas

Schur-convexity was introduced by I. Schur in 1923 [5] and has many important applications in analytic inequalities. Hardy, Littlewood, and Pólya were also interested in some inequalities that are related to Schur-convex functions [6]. For a historical development of this class of functions and for some fruitful applications to statistics, economics and other applied fields, reference can be made to the popular book by Marshall and Olkin [5]. The following definitions can be found in many references such as [2, 5, 7].

For fixed $n \geqslant 2$, let

$$x=(x_1,x_2,\cdots,x_n), \quad y=(y_1,y_2,\cdots,y_n)$$

be two n-tuples of real numbers, and let

$$x_{[1]}\geqslant x_{[2]}\geqslant\cdots\geqslant x_{[n]}, \quad y_{[1]}\geqslant y_{[2]}\geqslant\cdots\geqslant y_{[n]},$$

be their components in decreasing order.

DEFINITION 2.1. The n-tuple x is said to be majorized by y (in symbols $x \prec y$), if

$$\sum_{i=1}^{m} x_{[i]} \leqslant \sum_{i=1}^{m} y_{[i]}, \quad m=1,2,\cdots,n-1 \tag{2.1}$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \tag{2.2}$$

DEFINITION 2.2. A real-valued function ϕ defined on a set $\Omega \subset R^n$ is said to be Schur-convex on Ω if

$$x \prec y \Rightarrow \phi(x) \leqslant \phi(y).$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be strictly Schur-convex on Ω. ϕ is a Schur-concave function on Ω if and only if $-\phi$ is a Schur-convex function; ϕ is a strictly Schur-concave function on Ω if and only if $-\phi$ is a strictly Schur-convex function on Ω.

Recall that the following so-called Schur's condition is very useful for determining whether or not a given function is Schur-convex or Schur-concave.

LEMMA 2.1. [5, p. 57] *Let $\Omega \subset R^n$ be symmetric and convex set with nonempty interior, and let $f : \Omega \to R$ be differentiable in the interior of Ω and continuous on Ω. Then f is Schur-convex on Ω if and only if f is symmetric on Ω and for all $i \neq j$,*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geqslant 0 \tag{2.3}$$

for all $x \in \Omega^0$. It is strictly Schur-convex if (2.3) is a strict inequality for $x_i \neq x_j$, $1 \leqslant i,j \leqslant n$, where Ω^0 is the interior of Ω.

Since $f(x)$ is symmetric, Schur's condition can be reduced to [5, p.57]

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geqslant 0. \tag{2.4}$$

And f is strictly Schur-convex if (2.4) is a strict inequality for $x_1 \neq x_2$. The Schur condition that guarantees a symmetric function being Schur-concave is the same as (2.3) or (2.4) except for the direction of the inequality.

Recently, C. P. Niculescu [8] introduced the multiplicatively convex function, i.e., GG-convex function, which reveals an entire new world of beautiful inequalities. And, Xiao-Ming Zhang [9] stated the Schur-geometrically-convex theory as a parallel one to Schur-convex theory by defining logarithmic majorization and using multiplicatively convex function.

DEFINITION 2.3. (See [9, p. 89]) Let x and y be two n-tuples of nonnegative numbers. Then the n-tuple x is said to be logarithmically majorized by y (in symbols $\ln x \prec \ln y$) if

$$\prod_{i=1}^{m} x_{[i]} \leqslant \prod_{i=1}^{m} y_{[i]}, \quad m = 1, 2, \cdots, n-1 \tag{2.5}$$

and

$$\prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}. \tag{2.6}$$

DEFINITION 2.4. (See [9, p. 107]) Assume that I is a subinterval of $(0,\infty)$. A function $f : I^n \to (0,\infty)$ is called Schur-geometrically-convex if

$$\ln x \prec \ln y \Rightarrow f(x) \leqslant f(y). \tag{2.7}$$

The following Lemma 2.2 is basic and plays an important role in the theory of Schur geometrically-convex function.

LEMMA 2.2. ([9, p. 108]) *Let $f(x) = f(x_1, x_2, \cdots, x_n)$ be symmetric and have continuously partial derivatives on I^n, where I is a subinterval of $(0,\infty)$. Then $f : I^n \to (0,\infty)$ is a Schur-geometrically-convex (concave) function if and only if*

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2}\right) \geqslant (\leqslant) 0. \tag{2.8}$$

LEMMA 2.3. ([5, p. 7]) *A function* $\varphi(x)$ *is increasing if and only if* $\nabla\varphi(x) \geqslant 0$ *for* $x \in \Omega$, *where* $\Omega \in \mathbb{R}^n$ *is an open set,* $\varphi : \Omega \to R$ *is differentiable, and*

$$\nabla\varphi(x) = \left(\frac{\partial\varphi(x)}{\partial x_1}, \cdots, \frac{\partial\varphi(x)}{\partial x_n}\right) \in \mathbb{R}^n.$$

LEMMA 2.4. [2] *Let* $x_i > 0, (i = 1, 2, \cdots, n,\ n \geqslant 2)$, $\sum_{i=1}^n x_i = s$, *then*

$$\frac{s}{n} = \left(\frac{s}{n}, \cdots, \frac{s}{n}\right) \prec (x_1, x_2, \cdots, x_n) = x. \tag{2.9}$$

LEMMA 2.5. [10] *Let* $x_i > 0, (i = 1, 2, \cdots, n,\ n \geqslant 2)$, $\sum_{i=1}^n x_i = s,\ c \geqslant s$, *then*

$$\frac{c+x}{s+nc} = \left(\frac{c+x_1}{s+nc}, \cdots, \frac{c+x_1}{s+nc}\right) \prec \left(\frac{x_1}{s}, \frac{x_2}{s}, \cdots, \frac{x_n}{s}\right) = \frac{x}{s}. \tag{2.10}$$

LEMMA 2.6. [10] *Let* $x_i > 0, (i = 1, 2, \cdots, n,\ n \geqslant 2)$, $\sum_{i=1}^n x_i = s,\ c \geqslant s$, *then*

$$\frac{c-x}{\frac{nc}{s}-1} = \left(\frac{c-x_1}{\frac{nc}{s}-1}, \cdots, \frac{c-x_n}{\frac{nc}{s}-1}\right) \prec (x_1, x_2, \cdots, x_n) = x. \tag{2.11}$$

3. Main results

In this section we investigate the Schur-convexity and Schur-geometrically convexity of $H_r^*(x)$. Some analytic inequalities are established by use of the theory of majorization.

THEOREM 3.1. $H_r^*(x)$ *is an increasing and Schur-concave function in* $\mathbb{R}_+^n$.

Proof. The theorem is obviously true when $r = 1$, so we assume below that $2 \leqslant r \leqslant n$. It is easy to show that:

$$H_r^*(x) = H_r^*(x_1, \cdots, x_n) = H_r^*(x_2, \cdots, x_n) \cdot \prod_{2 \leqslant i_1 < \cdots < i_r \leqslant n} \left(x_1^{\frac{1}{r}} + \sum_{j=1}^{r-1} x_{i_j}^{\frac{1}{r}}\right),$$

$$\ln H_r^*(x) = \ln H_r^*(x_2, \cdots, x_n) + \sum_{2 \leqslant i_1 < \cdots < i_r \leqslant n} \ln\left(x_1^{\frac{1}{r}} + \sum_{j=1}^{r-1} x_{i_j}^{\frac{1}{r}}\right).$$

Differentiating $H_r^*(x)$ with respect to x_1 and x_2, we obtain

$$\begin{aligned}\frac{\partial H_r^*(x)}{\partial x_1} &= H_r^*(x) \sum_{2 \leqslant i_1 < \cdots < i_{r-1} \leqslant n} \left(x_1^{\frac{1}{r}} + \sum_{j=1}^{r-1} x_{i_j}^{\frac{1}{r}}\right)^{-1} \cdot \frac{1}{r} x_1^{\frac{1}{r}-1} \\ &= H_r^*(x) \left[\sum_{3 \leqslant i_1 < \cdots < i_{r-1} \leqslant n} \left(x_1^{\frac{1}{r}} + \sum_{j=1}^{r-1} x_{i_j}^{\frac{1}{r}}\right)^{-1} \frac{1}{r} x_1^{\frac{1}{r}-1}\right.\end{aligned}$$

$$
\begin{aligned}
&+\sum_{3\leqslant i_1<\cdots<i_{i-2}\leqslant n}\left(x_1^{\frac{1}{r}}+x_2^{\frac{1}{r}}+\sum_{j=1}^{r-2}x_{i_j}^{\frac{1}{r}}\right)^{-1}\frac{1}{r}x_1^{\frac{1}{r}-1}\Bigg]\\
&>0\\
\frac{\partial H_r^*(x)}{\partial x_2}&=H_r^*(x)\sum_{2\leqslant i_1<\cdots<i_{r-1}\leqslant n}\left(x_2^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)^{-1}\cdot\frac{1}{r}x_2^{\frac{1}{r}-1}\\
&=H_r^*(x)\Bigg[\sum_{3\leqslant i_1<\cdots<i_{r-1}\leqslant n}\left(x_2^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)^{-1}\frac{1}{r}x_2^{\frac{1}{r}-1}\\
&+\sum_{3\leqslant i_1<\cdots<i_{i-2}\leqslant n}\left(x_1^{\frac{1}{r}}+x_2^{\frac{1}{r}}+\sum_{j=1}^{r-2}x_{i_j}^{\frac{1}{r}}\right)^{-1}\frac{1}{r}x_2^{\frac{1}{r}-1}\Bigg]\\
&>0.
\end{aligned}
$$

From the Lemma 2.3, $H_r^*(x)$ is an increasing function on $\mathbb{R}_+^n$.

On the other hand, we have

$$
\begin{aligned}
&(x_1-x_2)\left(\frac{\partial H_r^*(x)}{\partial x_1}-\frac{\partial H_r^*(x)}{\partial x_2}\right)\\
&=(x_1-x_2)H_r^*(x)\Bigg[\sum_{3\leqslant i_1<\cdots<i_{r-1}\leqslant n}\left(x_1^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)^{-1}\frac{1}{r}x_1^{\frac{1}{r}-1}\\
&-\sum_{3\leqslant i_1<\cdots<i_{r-1}\leqslant n}\left(x_2^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)^{-1}\frac{1}{r}x_2^{\frac{1}{r}-1}\Bigg]\\
&+(x_1-x_2)H_r^*(x)\Bigg[\sum_{3\leqslant i_1<\cdots<i_{i-2}\leqslant n}\left(x_1^{\frac{1}{r}}+x_2^{\frac{1}{r}}+\sum_{j=1}^{r-2}x_{i_j}^{\frac{1}{r}}\right)^{-1}\left(\frac{1}{r}x_1^{\frac{1}{r}-1}-\frac{1}{r}x_2^{\frac{1}{r}-1}\right)\Bigg]\\
&=\frac{1}{r}(x_1-x_2)H_r^*(x)\Bigg[\sum_{3\leqslant i_1<\cdots<i_{r-1}\leqslant n}\left(x_1^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)^{-1}\left(x_2^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)^{-1}\cdot M\Bigg]\\
&+(x_1-x_2)H_r^*(x)\Bigg[\sum_{3\leqslant i_1<\cdots<i_{i-2}\leqslant n}\left(x_1^{\frac{1}{r}}+x_2^{\frac{1}{r}}+\sum_{j=1}^{r-2}x_{i_j}^{\frac{1}{r}}\right)^{-1}\left(\frac{1}{r}x_1^{\frac{1}{r}-1}-\frac{1}{r}x_2^{\frac{1}{r}-1}\right)\Bigg];
\end{aligned}
$$

where

$$M=\left(x_2^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)x_1^{\frac{1}{r}-1}-\left(x_1^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)x_2^{\frac{1}{r}-1}$$
$$=x_1^{\frac{1}{r}-1}x_2^{\frac{1}{r}}-x_1^{\frac{1}{r}-1}x_2^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\left(x_1^{\frac{1}{r}-1}-x_2^{\frac{1}{r}-1}\right)$$
$$=x_1^{\frac{1}{r}}x_2^{\frac{1}{r}}\left(\frac{1}{x_1}-\frac{1}{x_2}\right)+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\left(x_1^{\frac{1}{r}-1}-x_2^{\frac{1}{r}-1}\right).$$

Notice that since $\frac{1}{x}, x^{\frac{1}{r}-1}$ are all decreasing in $(0,+\infty)$, it follows that $(x_1-x_2)(\frac{1}{x_1}-\frac{1}{x_2})\leqslant 0$ and $(x_1-x_2)(x_1^{\frac{1}{r}-1}-x_2^{\frac{1}{r}-1})\leqslant 0$. Therefore

$$(x_1-x_2)\left(\frac{\partial H_r^*(x)}{\partial x_1}-\frac{\partial H_r^*(x)}{\partial x_2}\right)\leqslant 0.$$

By Lemma 2.1, Theorem 3.1 follows. □

COROLLARY 3.1. *Let* $x_i>0$, $(i=1,2,\cdots,n,\ n\geqslant 2)$, $\sum_{i=1}^n x_i=s$, *then*

$$H_r^*(x)\leqslant\left[r\left(\frac{s}{n}\right)^{\frac{1}{r}}\right]^{\binom{n}{r}}. \tag{3.1}$$

Proof. From Theorem 3.1 and Lemma 2.4, (3.1) holds. □

COROLLARY 3.2. *Let* $x_i>0$, $(i=1,2,\cdots,n,\ n\geqslant 2)$, $\sum_{i=1}^n x_i=s$, $c\geqslant s$, *then*

$$\frac{H_r^*(c-x)}{H_r^*(x)}\geqslant\left(\frac{nc}{s}-1\right)^{\frac{\binom{n}{r}}{r}}. \tag{3.2}$$

Proof. From Theorem 3.1 and Lemma 2.6, (3.2) holds. □

REMARK 3.1. By Corollary 3.2, let $c=s=1$, $r=1$, the following statements are true

$$\prod_{i=1}^n(x_i^{-1}-1)\geqslant(n-1)^n.$$

(See [1], Weierstrass inequality)

COROLLARY 3.3. *Let* $x_i>0$, $(i=1,2,\cdots,n,\ n\geqslant 2)$, $\sum_{i=1}^n x_i=s$, $c\geqslant s$, *then*

$$\frac{H_r^*(c+x)}{H_r^*(x)}\geqslant\left(\frac{nc}{s}+1\right)^{\frac{\binom{n}{r}}{r}}. \tag{3.3}$$

Proof. From Theorem 3.1 and Lemma 2.5, (3.3) holds. □

REMARK 3.2. By Corollary 3.3, let $c = s = 1$, $r = 1$, we can get the Weierstrass inequality (see [1]):

$$\prod_{i=1}^{n}(x_i^{-1} + 1) \geqslant (n + 1)^n.$$

THEOREM 3.2. *$H_r^*(x)$ is Schur-geometrically convex function in $\mathbb{R}_{++}^n$.*

Proof. From theorem 3.1 we have

$$\begin{aligned} x_1\frac{\partial H_r^*(x)}{\partial x_1} = H_r^*(x)\Bigg[&\sum_{3\leqslant i_1<\cdots<i_{r-1}\leqslant n}\left(x_1^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)^{-1}\frac{1}{r}x_1^{\frac{1}{r}} \\ &+\sum_{3\leqslant i_1<\cdots<i_{i-2}\leqslant n}\left(x_1^{\frac{1}{r}}+x_2^{\frac{1}{r}}+\sum_{j=1}^{r-2}x_{i_j}^{\frac{1}{r}}\right)^{-1}\frac{1}{r}x_1^{\frac{1}{r}}\Bigg] \\ x_2\frac{\partial H_r^*(x)}{\partial x_2} = H_r^*(x)\Bigg[&\sum_{3\leqslant i_1<\cdots<i_{r-1}\leqslant n}\left(x_2^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)^{-1}\frac{1}{r}x_2^{\frac{1}{r}} \\ &+\sum_{3\leqslant i_1<\cdots<i_{i-2}\leqslant n}\left(x_1^{\frac{1}{r}}+x_2^{\frac{1}{r}}+\sum_{j=1}^{r-2}x_{i_j}^{\frac{1}{r}}\right)^{-1}\frac{1}{r}x_2^{\frac{1}{r}}\Bigg]. \end{aligned}$$

Accordingly

$$\begin{aligned} &(\ln x_1-\ln x_2)\left(x_1\frac{\partial H_r^*(x)}{\partial x_1}-x_2\frac{\partial H_r^*(x)}{\partial x_2}\right) \\ &=(\ln x_1-\ln x_2)H_r^*(x)\Bigg[\sum_{3\leqslant i_1<\cdots<i_{r-1}\leqslant n}\left(x_1^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)^{-1}\frac{1}{r}x_1^{\frac{1}{r}} \\ &\quad-\sum_{3\leqslant i_1<\cdots<i_{r-1}\leqslant n}\left(x_2^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)^{-1}\frac{1}{r}x_2^{\frac{1}{r}}\Bigg] \\ &\quad+(\ln x_1-\ln x_2)H_r^*(x)\Bigg[\sum_{3\leqslant i_1<\cdots<i_{i-2}\leqslant n}\left(x_1^{\frac{1}{r}}+x_2^{\frac{1}{r}}+\sum_{j=1}^{r-2}x_{i_j}^{\frac{1}{r}}\right)^{-1}\left(\frac{1}{r}x_1^{\frac{1}{r}}-\frac{1}{r}x_2^{\frac{1}{r}}\right)\Bigg] \\ &=\frac{1}{r}\frac{\ln x_1-\ln x_2}{x_1-x_2}(x_1-x_2)H_r^*(x)\Bigg[\sum_{3\leqslant i_1<\cdots<i_{r-1}\leqslant n}\left(x_1^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)^{-1}\left(x_2^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)^{-1}\cdot M\Bigg] \end{aligned}$$

$$+\frac{\ln x_1-\ln x_2}{x_1-x_2}(x_1-x_2)H_r^*(x)\left[\sum_{3\leqslant i_1<\cdots<i_{i-2}\leqslant n}\left(x_1^{\frac{1}{r}}+x_2^{\frac{1}{r}}+\sum_{j=1}^{r-2}x_{i_j}^{\frac{1}{r}}\right)^{-1}\left(\frac{1}{r}x_1^{\frac{1}{r}}-\frac{1}{r}x_2^{\frac{1}{r}}\right)\right],$$

where

$$\begin{aligned}M&=\left(x_2^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)x_1^{\frac{1}{r}}-\left(x_1^{\frac{1}{r}}+\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\right)x_2^{\frac{1}{r}}\\&=\sum_{j=1}^{r-1}x_{i_j}^{\frac{1}{r}}\left(x_1^{\frac{1}{r}}-x_2^{\frac{1}{r}}\right).\end{aligned}$$

Since $\frac{\ln x_1-\ln x_2}{x_1-x_2}\geqslant 0, (x_1-x_2)(x_1^{\frac{1}{r}}-x_2^{\frac{1}{r}})\geqslant 0$, we have

$$(\ln x_1-\ln x_2)\left(x_1\frac{\partial H_r^*(x)}{\partial x_1}-x_2\frac{\partial H_r^*(x)}{\partial x_2}\right)\geqslant 0.$$

From Lemma 2.2, Theorem 3.2 follows. □

The following result holds in terms of Theorem 3.2 and the fact [9, p. 97] that $\ln(s,\cdots,s)\prec\ln(x_1,\cdots,x_n)$.

COROLLARY 3.4. *Let* $x_i>0$, $(i=1,2,\cdots,n,\ n\geqslant 2)$, $\sqrt[n]{\prod_{i=1}^n x_i}=s$, *then*

$$H_r^*(x)=\prod_{1\leqslant i_1<\cdots<i_r\leqslant n}\left(\sum_{j=1}^{r}x_{i_j}^{\frac{1}{r}}\right)\geqslant\left(rs^{\frac{1}{r}}\right)^{\binom{n}{r}}. \tag{3.4}$$

4. Applications

In this section, by using our results, we establish some interesting matrix and geometric inequalities. In what follows $A=(a_{ij})_{m\times n}$ is a complex matrix, and let $d(A)=(a_{11},\cdots,a_{nn}),\lambda(A)=(\lambda_1,\lambda_2,\cdots,\lambda_n)$, where the components λ_i is eigenvalues of A respectively.

THEOREM 4.1. *Let* $A=(a_{ij})_{m\times n}$ $(n\geqslant 3)$ *be a positive definite Hermitian matrix, and* I *denotes* $n\times n$ *matrix, then*

$$\prod_{1\leqslant i_1<\cdots<i_r\leqslant n}\sum_{j=1}^{r}(\lambda_{i_j})^{\frac{1}{r}}\geqslant\left(r(\sqrt[n]{\det(A)})^{\frac{1}{r}}\right)^{\binom{n}{r}}. \tag{4.1}$$

Proof. Since $\lambda_i(1\leqslant i\leqslant n)$ is an eigenvalue of matrix A, one can easily find that $\operatorname{tr}(A)=\sum_{i=1}^n$, $\det(A)=\lambda_1\cdots\lambda_n$. It is not difficult to see that

$$\ln\left(\frac{\sqrt[n]{\det(A)}}{\operatorname{tr}(A)},\cdots,\frac{\sqrt[n]{\det(A)}}{\operatorname{tr}(A)}\right)\prec\ln\left(\frac{\lambda_1}{\operatorname{tr}(A)},\cdots,\frac{\lambda_n}{\operatorname{tr}(A)}\right), \tag{4.2}$$

where $\operatorname{tr}(A)$ is the trace of matrix A.

From Theorem 3.2, it follows that inequality (4.1) holds. □

THEOREM 4.2. *Let A be an n-dimensional simplex in n-dimensional Euclidean space $\mathbb{E}^n (n \geqslant 3)$ and $\{A_1, \cdots, A_{n+1}\}$ be the set of vertices. Let P be an arbitrary point in the interior of A. If B_i is the intersection point of the extension line of A_iP and the $(n-1)$-dimensional hyperplane opposite to the point A, then we have*

$$H_r^*\left(\frac{PB_1}{A_1B_1}, \cdots, \frac{PB_{n+1}}{A_{n+1}B_{n+1}}\right) \leqslant \left[r\left(\frac{1}{n+1}\right)^{\frac{1}{r}}\right]^{\binom{n+1}{r}}, \tag{4.3}$$

and

$$H_r^*\left(\frac{PA_1}{A_1B_1}, \cdots, \frac{PA_{n+1}}{A_{n+1}B_{n+1}}\right) \leqslant \left[r\left(\frac{n}{n+1}\right)^{\frac{1}{r}}\right]^{\binom{n+1}{r}}. \tag{4.4}$$

Proof. It is easy to see that

$$\sum_{i=1}^{n+1} \frac{PB_i}{A_iB_i} = 1, \ \frac{A_iP}{A_iB_i} = 1 - \frac{B_iP}{A_iB_i}, \ i = 1, 2, \cdots, n+1, \ \sum_{i=1}^{n+1} \frac{A_iP}{A_iB_i} = 1.$$

Taking $s = 1$ and $s = n$ in (3.1), it follows that (4.3) and (4.4) holds respectively. □

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