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# GAP FUNCTIONS AND EXISTENCE RESULTS FOR GENERALIZED VECTOR VARIATIONAL INEQUALITIES

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*Abstract.* In this paper, the gap function for a new class of generalized vector variational inequalities with point-to-set mappings (for short, GVVI) is introduced and the necessary and sufficient conditions for the GVVI are established. Furthermore, under certain pseudomonotonicity condition and hemicontinuity condition, we obtain the existence theorems for the GVVI. The results presented in this paper are new and extend corresponding results in this field.

## 1. Introduction

A vector variational inequality (for short, VVI) in a finite-dimensional Euclidean space was introduced first by Giannessi [1] in 1980. This is a generalization of a scalar variational inequality to the vector case by virtue of multi-criterion consideration. Later on, many authors have investigated vector variational inequalities in abstract spaces see [2–6, 8–11, 13, 15–21]. With the development of the theory about vector variational inequalities, it has been seen that vector variational inequalities have many important applications in vector optimization, approximate vector optimization, vector equilibria, vector traffic equilibria and so on, for detail, see [2–8] and the references therein.

Gap function plays a crucial role in transforming a variational inequality problem into an optimization problem. Thus, powerful optimization solution methods and algorithms can be applied to find solutions of variational inequalities. In [9], Yang and Yao introduced the gap function and established necessary and sufficient conditions for the existence of a solution for the VVI, and stated the relation between the VVI and semi-infinite programming problem. In [10], Li and He introduced the gap function for GVVI and established necessary and sufficient conditions for the existence of a solution for the GVVI. They all investigated the existence of solutions for the generalized vector variational inequalities with a point-to-set mapping by virtue of the existence of a solution of the vector variational inequality with a single-valued function and a continuous selection theorem.

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Inspired and motivated by the above research work, in this paper, the gap functions for a new class of generalized vector variational inequalities with point-to-set mappings are introduced and the necessary and sufficient conditions for the GVVI are established. In order to derive the existence of solutions for the GVVI, we introduce the concept of pseudomonotonicity and hemicontinuity, by employing Nadler lemma [12] and KKM-Fan lemma [14], we obtain the existence theorems for the GVVI in this paper, which are new and different from those in [9,10].

## 2. Gap functions for the GVVI

Let $X$ be a Hausdorff topological vector space, $Y$ a topological vector space, $M(X,Y)$ denote the set of all continuous mappings from $X$ to $Y$ and $L(X,Y)$ the set of all continuous linear mappings from $X$ to $Y$. Suppose that $D$ is a nonempty compact subset of $X$, $\{C(x) : x \in D\}$ is a family of closed, pointed, and convex cones of $Y$ with apex at the origin and $\operatorname{int} C(x) \neq \emptyset$. Assume $T : D \to 2^{L(X,Y)}$ is a point-to-set mapping with a compact set $T(x)$ for each $x \in D$; $A : L(X,Y) \to L(X,Y)$ is a continuous single-valued mapping; $\eta : D \times D \to D$ and $h : D \times D \to Y$ are two continuous vector-valued functions satisfying $\eta(x,x) = 0$ and $h(x,x) = 0$ for each $x \in D$, respectively. In this section, we consider the following three generalized vector variational inequalities (for short, GVVI):

find $\bar{x} \in D$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle A\bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -\operatorname{int} C, \quad \forall y \in D; \tag{2.1}$$

find $\bar{x} \in D$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle A\bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -C\backslash\{0\}, \quad \forall y \in D; \tag{2.2}$$

find $\bar{x} \in D$ such that, $\forall y \in D$, $\exists \bar{t}(y) \in T(\bar{x})$ such that

$$\langle A\bar{t}(y), \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -\operatorname{int} C, \quad \forall y \in D. \tag{2.3}$$

The following problems are special cases of the GVVI (2.1–2.3).

(1) If $A(t) = t$ for any $t \in L(X,Y)$, then (2.1)–(2.3) reduce to (2.4)–(2.6), respectively.

Find $\bar{x} \in D$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle \bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -\operatorname{int} C, \quad \forall y \in D; \tag{2.4}$$

find $\bar{x} \in D$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle \bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -C\backslash\{0\}, \quad \forall y \in D; \tag{2.5}$$

find $\bar{x} \in D$ such that, $\forall y \in D$, $\exists \bar{t}(y) \in T(\bar{x})$ such that

$$\langle \bar{t}(y), \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -\operatorname{int} C, \quad \forall y \in D. \tag{2.6}$$

(2) If $h(y,x) = f(y) - f(x)$ for any $x, y \in D$, where $f : D \to Y$ is a vector-valued function, then (2.1)–(2.3) reduce to (2.7)–(2.9).

Find $\bar{x} \in D$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle A\bar{t}, \eta(y,\bar{x})\rangle + f(y) - f(\bar{x}) \notin -\operatorname{int} C, \quad \forall y \in D; \tag{2.7}$$

find $\bar{x} \in D$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle A\bar{t}, \eta(y,\bar{x})\rangle + f(y) - f(\bar{x}) \notin -C\backslash\{0\}, \quad \forall y \in D; \tag{2.8}$$

find $\bar{x} \in D$ such that, $\forall y \in D$, $\exists \bar{t}(y) \in T(\bar{x})$ such that

$$\langle A\bar{t}(y), \eta(y,\bar{x})\rangle + f(y) - f(\bar{x}) \notin -\operatorname{int} C, \quad \forall y \in D. \tag{2.9}$$

REMARK 2.1. For suitable and appropriate choice of $X, Y, \eta, A, T, h$, one can obtain many VVI and GVVI as special cases of the GVVI(2.1)–(2.3), for detail, see [9–11] and the references therein, which is also one of our motivation of this paper.

REMARK 2.2. It is easy to see that any solution of the GVVI (2.2) is a solution of the GVVI (2.1), and any solution of the GVVI (2.1) is a solution of the GVVI (2.3). But the converse is not true in general.

In the rest of this section, let $Y$ be an $l$-dimensional vector space $R^l$, let

$$R^l_+ = \{(x_1, x_2, \cdots, x_l) \in R^l \mid x_i \geqslant 0, i = 1, 2, \cdots, l\}$$

be the non-negative orthant of $R^l$ and let $C = R^l_+$. Next, we will introduce the concept of gap functions for the GVVI with point-to-set mappings.

DEFINITION 2.1. Consider the GVVI. Let $D_1$ be the domain of the GVVI. A function $\phi : D_1 \to R$ is said to be a gap function for the GVVI if it satisfies the following properties:

$(i)$ $\phi(x) \leqslant 0, \forall x \in D_1$;

$(ii)$ $\phi(\bar{x}) = 0$, if and only if $\bar{x}$ solves the GVVI.

Let $x, y \in D$ and $t \in T(x)$. Denote

$$\langle At, \eta(y,x)\rangle + h(y,x) = ([\langle At, \eta(y,x)\rangle + h(y,x)]_1, \cdots, [\langle At, \eta(y,x)\rangle + h(y,x)]_l);$$

i.e., $[\langle At, \eta(y,x)\rangle + h(y,x)]_i$ is the $i$th component of $\langle At, \eta(y,x)\rangle + h(y,x)$, $i = 1, 2, \cdots, l$. Now, we define two mappings $\phi_1 : D \times M(X, R^l) \to R$ and $\phi : D \to R$ as follows:

$$\phi_1(x,t) = \min_{y \in D} \max_{1 \leqslant i \leqslant l} (\langle At, \eta(y,x)\rangle + h(y,x))_i, \tag{2.10}$$

$$\phi(x) = \max\{\phi_1(x,t) \mid t \in T(x)\}. \tag{2.11}$$

Since $D$ is compact, $A$ is a mapping from $L(X,Y)$ to $L(X,Y)$, $\phi_1(x,t)$ is well-defined. Furthermore, $\phi_1(x,t)$ is a lower semicontinuous function in $x$; see corollary 22 in [11]. And since $T(x)$ is a compact set, $\phi(x)$ is well-defined. For $x \in D$ and $t \in T(x)$, it is easy to see that

$$\phi_1(x,t) = \min_{y \in D} \max_{1 \leqslant i \leqslant l} (\langle At, \eta(y,x)\rangle + h(y,x))_i \leqslant 0.$$

THEOREM 2.1. *The function $\phi(x)$ defined by (2.11) is a gap function for the GVVI (2.1).*

*Proof.* It is clear that

$$\phi_1(x,t) \leqslant 0, \forall x \in D, t \in T(x). \tag{2.12}$$

Thus,

$$\phi(x) = \max\{\phi_1(x,t) \mid t \in T(x)\} \leqslant 0, \forall x \in D.$$

If $\phi(\bar{x}) = 0$, then there exists $\bar{t} \in T(\bar{x})$ such that $\phi_1(\bar{x},\bar{t}) = 0$. Consequently, we have

$$\min_{y \in D} \max_{1 \leqslant i \leqslant l} (\langle A\bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}))_i = 0.$$

It follows that, for any $y \in D$,

$$\max_{1 \leqslant i \leqslant l} (\langle A\bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}))_i \geqslant 0,$$

which implies that, for any $y \in D$,

$$\langle A\bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -\mathrm{int}\, R^l_+,$$

i.e., $\bar{x}$ is a solution of the GVVI (2.1). Conversely, If $\bar{x}$ solves the GVVI (2.1), then there exists $\bar{t} \in T(\bar{x})$ such that

$$\langle A\bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -\mathrm{int}\, R^l_+, \forall y \in D,$$

from which it follows that for any $y \in D$,

$$\max_{1 \leqslant i \leqslant l} (\langle A\bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}))_i \geqslant 0. \tag{2.13}$$

Now, (2.12) and (2.13) imply that

$$\phi_1(\bar{x},\bar{t}) = 0.$$

Again from (2.12), we obtain

$$\phi_1(\bar{x},t) \leqslant 0, \forall t \in T(\bar{x}).$$

Since

$$\phi(\bar{x}) = \max\{\phi_1(\bar{x},t) \mid t \in T(\bar{x})\},$$

it follows from $\phi_1(\bar{x},\bar{t}) = 0$ that $\phi(\bar{x}) = 0$. This completes the proof. □

By Theorem 2.1, the solution of the GVVI (2.1) is equivalent to finding a global solution $\bar{x}$ to the following optimization problem:

$$\max_{x \in D} \phi(x)$$

with $\phi(\bar{x}) = 0$.
Recall that

$$\phi(x) = \max\{\phi_1(x,t) \mid t \in T(x)\}.$$

It is clear that the above optimization problem is equivalent to the following generalized semi-infinite programming problem:

$$\begin{aligned}&\min -s\\&s.t.\ \ \phi_1(x,t)\leqslant s,\ \forall t\in T(x),\\&x\in D,\end{aligned}$$

which is a very important class of optimization problems; see [11].

From remark 2.2 and Theorem 2.1, it is easy to see the following result holds.

COROLLARY 2.1. *If $\bar{x}$ is a solution of the GVVI (2.2), then $\phi(\bar{x})=0$.*

Furthermore, Theorem 2.1 presented in [9,10] is a special case of Theorem 2.1.

COROLLARY 2.2. *The function $\phi(x)$ defined by (2.11) is a gap function for the GVVI (2.4) and VVI (2.7).*

Next, we will consider the gap function for the GVVI (2.3). First, for any $x\in D$, denote

$$B_x=\{t\mid t:D\to T(x)\},$$

i.e., $B_x$ is the set of all operators $t$ from $D$ to $T(x)$. Let $x\in D$ and $t\in B_x$, then $t(y)\in T(x),\forall y\in D$. As functions $\phi_1$ and $\phi$ defined by (2.10) and (2.11), respectively, we defined two mappings $\phi_1^*:D\times M(X,R^l)\to R$ and $\phi^*:D\to R$ as follows:

$$\phi_1^*(x,t)=\min_{y\in D}\max_{1\leqslant i\leqslant l}(\langle At(y),\eta(y,x)\rangle\ +h(y,x))_i,$$

$$\phi^*(x)=\max\{\phi_1^*(x,t)\mid t\in B_x\}. \tag{2.14}$$

THEOREM 2.2. *The function $\phi^*(x)$ defined by (2.14) is a gap function for the GVVI (2.3).*

*Proof.* It is clear that

$$\phi_1^*(x,t)=\min_{y\in D}\max_{1\leqslant i\leqslant l}(\langle At(y),\eta(y,x)\rangle\ +h(y,x))_i\leqslant 0,\forall x\in D,t\in B_x, \tag{2.15}$$

and hence

$$\phi^*(x)=\max\{\phi_1^*(x,t)\mid t\in B_x\}\leqslant 0,\forall x\in D.$$

If $\phi^*(\bar{x})=0$, then there exists $\bar{t}\in B_{\bar{x}}$ such that $\phi_1^*(\bar{x},\bar{t})=0$. Consequently, we have

$$\min_{y\in D}\max_{1\leqslant i\leqslant l}(\langle A\bar{t}(y),\eta(y,\bar{x})\rangle\ +h(y,\bar{x}))_i=0.$$

It follows that, for any $y\in D$,

$$\max_{1\leqslant i\leqslant l}(\langle A\bar{t}(y),\eta(y,x)\rangle\ +h(y,\bar{x}))_i\geqslant 0,$$

which implies that, for any $y\in D$,

$$\langle A\bar{t}(y),\eta(y,\bar{x})\rangle\ +h(y,\bar{x})\ \notin\ -\mathrm{int}\,R^l_+,$$

i.e., $\bar{x}$ is a solution of the GVVI (2.3). Conversely, assume $\bar{x}$ solves the GVVI (2.3). Since $\bar{x}$ is a solution of the GVVI (2.3), for any $y \in D$, there is a $\bar{t}(y) \in T(\bar{x})$ such that

$$\langle A\bar{t}(y), \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -\mathrm{int}\, R^l_+,$$

from which it follows that

$$\max_{1 \leqslant i \leqslant l}(\langle A\bar{t}(y), \eta(y,\bar{x})\rangle + h(y,\bar{x}))_i \geqslant 0, \forall y \in D.$$

Thus, an operator $\bar{t}$ from $D$ to $T(\bar{x})$ has been defined. Then, $\bar{t} \in B_{\bar{x}}$ and

$$\max_{1 \leqslant i \leqslant l}(\langle A\bar{t}(y), \eta(y,\bar{x})\rangle + h(y,\bar{x}))_i \geqslant 0, \forall y \in D.$$

Hence,

$$\phi_1^*(\bar{x},\bar{t}) = \min_{y \in D} \max_{1 \leqslant i \leqslant l}(\langle A\bar{t}(y), \eta(y,\bar{x})\rangle + h(y,\bar{x}))_i \geqslant 0. \tag{2.16}$$

Thus, (2.15) and (2.16) imply that

$$\phi_1^*(\bar{x},\bar{t}) = 0.$$

Again by (2.15), we have

$$\phi_1^*(\bar{x},t) \leqslant 0, \forall t \in B_{\bar{x}}.$$

Since

$$\phi^*(\bar{x}) = \max\{\phi_1^*(\bar{x},t) \mid t \in B_{\bar{x}}\},$$

it follows from $\phi_1^*(\bar{x},\bar{t}) = 0$ that $\phi^*(\bar{x}) = 0$. This completes the proof. □

From Remark 2.2 and Theorem 2.2, we can easily have the following result.

COROLLARY 2.3. *If $\bar{x}$ is a solution of the GVVI (2.1) or the GVVI (2.2), then $\phi^*(\bar{x}) = 0$.*

Furthermore, Theorem 2.2 presented in [9,10] is a special case of Theorem 2.2.

COROLLARY 2.4. *The function $\phi^*(\bar{x})$ defined by (2.14) is a gap function for the GVVI (2.6) and VVI (2.9).*

## 3. Existence of a solutions for the GVVI

Let $X$ and $Y$ be two real Banach spaces, and let $D$ be a nonempty compact and convex subset of $X$. Let $L(X,Y)$ be as in section 2. Let $C : D \to 2^Y$ be a point-to-set mapping such that for each $x \in D$, $C(x)$ is a point, closed and convex cone in $Y$ with $\mathrm{int}\, C(x) \neq \emptyset$. Assume that $T : D \to 2^{L(X,Y)}$ is a point-to-set mapping, $A : L(X,Y) \to L(X,Y)$, $\eta : D \times D \to D$ and $h : D \times D \to Y$ are single-valued mappings. In this section, we consider the GVVI with moving cone $C(x)$: find $\bar{x} \in D$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle A\bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -\mathrm{int}\, C(\bar{x}), \forall y \in D. \tag{3.1}$$

If $At = t$, for all $t \in L(X,Y)$, then the GVVI (3.1) reduces to the following GVVI:

find $\bar{x} \in D$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle \bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -\mathrm{int}\, C(\bar{x}), \forall y \in D, \tag{3.2}$$

which has been studied by Li and He in [10].

If $\eta(y,x) = y - x$ and $h(y,x) = 0$ for all $x, y \in D$, then GVVI (3.2) reduces to the following VVI:

find $\bar{x} \in D$ and $\bar{t} \in T(\bar{x})$ such that

$$\langle \bar{t}, y - \bar{x}\rangle \notin -\mathrm{int}\, C(\bar{x}), \forall y \in D,$$

which has been studied by Yang and Yao in [9].

Now, let's recall some definitions and results.

DEFINITION 3.1. Let $T : D \to 2^{L(X,Y)}$, $A : L(X,Y) \to L(X,Y)$, $\eta : D \times D \to D$, $h : D \times D \to Y$.

$(i)$ $T$ is said to be $\eta$-$h$-$C(x)$-weakly pseudomonotonic with respect to $A$ if, for every pair of points $x, y \in D$, we have that $\exists t' \in T(x)$, $\langle At', \eta(y,x)\rangle + h(y,x)\rangle \notin -\mathrm{int}\, C(x)$ implies $\langle At'', \eta(y,x)\rangle + h(y,x) \notin -\mathrm{int}\, C(x)$, for all $t'' \in T(y)$. If $T$ is a single-valued mapping, then the above definition reduces to the following:

$(ii)$ $T$ is said to be $\eta$-$h$-$C(x)$-pseudomonotonic with respect to $A$ if, for every pair of points $x, y \in D$, we have that $\langle A(Tx), \eta(y,x)\rangle + h(y,x)\rangle \notin -\mathrm{int}\, C(x)$ implies $\langle A(Ty), \eta(y,x)\rangle + h(y,x) \notin -\mathrm{int}\, C(x)$.

$(iii)$ $h(x,y)$ is said to be $C(x)$-convex with respect to $x$ if, for any given $y \in D$,

$$h(tx_1 + (1-t)x_2, y) \in th(x_1,y) + (1-t)h(x_2,y) - C(y), \forall x_1, x_2 \in D, t \in [0,1];$$

$(iv)$ $\eta(x,y)$ is affine with respect to $x$ if, for any given $y \in D$,

$$\eta(tx_1 + (1-t)x_2, y) = t\eta(x_1,y) + (1-t)\eta(x_2,y), \forall x_1, x_2 \in D, t \in R,$$

with $x = tx_1 + (1-t)x_2 \in D$.

REMARK 3.1. It is easy to prove that $h(x,y)$ is $C(x)$-convex with respect to $x$ if and only if for any given $y \in D$,

$$h(\sum_{i=1}^{n} t_i x_i, y) \in \sum_{i}^{n} t_i h(x_i, y) - C(y)$$

for all $x_i \in D$ and $t_i \in [0,1]$ $(i = 1, \cdots, n)$ with $\sum_{i=1}^{n} t_i = 1$.

And $\eta(x,y)$ is affine with respect to $x$ if and only if for any given $y \in D$,

$$\eta(\sum_{i=1}^{n} t_i x_i, y) = \sum_{i=1}^{n} t_i \eta(x_i, y)$$

for all $x_i \in D$ and $t_i \in R$ $(i = 1, \cdots, n)$ with $\sum_{i=1}^{n} t_i x_i \in D$ and $\sum_{i=1}^{n} t_i = 1$.

DEFINITION 3.2. Let $W : X \to 2^Y$ be a point-to-set mapping. The graph of $W$, denoted by $\mathrm{gph}(W)$ is

$$\mathrm{gph}(W) = \{(x,z) \in X \times Y \mid x \in X, z \in W(x)\}.$$

DEFINITION 3.3. Let $D$ be a nonempty subset of topological vector space $X$. A point-to-set mapping $T : D \to 2^X$ is called KKM-mapping if, for every finite subset $\{x_1, x_2, \cdots, x_n\}$ of $D$, $Co\{x_1, \cdots, x_n\} \subset \bigcup_{i=1}^n T(x_i)$, where $Co$ denotes the convex hull.

LEMMA 3.1. *([14]) Let $D$ be a nonempty subset of Hausdorff topological vector space $X$. Let $G : D \to 2^X$ be a KKM-mapping, such that for any $y \in D$, $G(y)$ is closed and $G(y_0)$ is compact for some $y_0 \in D$. Then there exists $x^* \in D$ such that $x^* \in G(y)$ for all $y \in D$, i.e., $\bigcap_{y \in D} G(y) \neq \emptyset$.*

LEMMA 3.2. *([12]) Let $(X, \|\cdot\|)$ be a normed vector space and $H$ be a Hausdorff metric on the collection $C(X)$ of all closed and bounded subset of $X$, induced by a metric $d$ in term of $d(u,v) = \|u - v\|$, which is defined by*

$$H(A,B) = \max(\sup_{u \in A} \inf_{v \in B} \|u - v\|, \sup_{v \in B} \inf_{u \in A} \|u - v\|),$$

*for $A$ and $B$ in $C(X)$. If $A$ and $B$ are compact sets in $X$, then for each $u \in A$, there exists $v \in B$ such that*

$$\|u - v\| \leqslant H(A,B).$$

DEFINITION 3.4. A point-to-set mapping $T : D \to 2^{L(X,Y)}$ is said to be hemicontinuous on $D$, if for all $x, y \in D$, $H(T(x + \lambda(y - x)), T(x)) \to 0$ as $\lambda \to 0^+$, where $H$ is a Hausdorff metric defined on $L(X,Y)$.

LEMMA 3.3. *Let $Y$ be a topological vector space with a pointed, closed and convex cone $C$ such that $\operatorname{int} C \neq \emptyset$. Then $\forall x, y, z \in Y$, we have $x - y \in -C$ and $x \notin -\operatorname{int} C \Longrightarrow y \notin -\operatorname{int} C$.*

*Proof.* If $y \in -\operatorname{int} C$, then $x = x - y + y \in -C - \operatorname{int} C \subseteq -\operatorname{int} C$, a contradiction of our assumption. $\square$

THEOREM 3.1. *Let $X$ and $Y$ be two real Banach spaces, and $D$ be a nonempty compact and convex subset of $X$. Let $C : D \to 2^Y$ be a point-to-set mapping such that for each $x \in D$, $C(x)$ is a pointed, closed and convex cone in $Y$ with $\operatorname{int} C(x) \neq \emptyset$. Let $T : D \to 2^{L(X,Y)}$, $\eta : D \times D \to D$, $h : D \times D \to Y$ and $A : L(X,Y) \to L(X,Y)$. Assume that the following conditions hold:*

- *(i) $A : L(X,Y) \to L(X,Y)$ is a continuous mapping;*
- *(ii) $\eta(x,x) = h(x,x) = 0$ for each $x \in D$;*
- *(iii) $\eta(y,x)$ is affine with respect to $y$, $h(y,x)$ is $C(y)$-convex with respect to $y$;*
- *(iv) $\eta(y,x)$ and $h(y,x)$ are continuous with respect to $x$;*
- *(v) $T$ is a nonempty compact -valued point-to-set mapping, i.e., $\forall x \in D$, $T(x)$ is compact set of $L(X,Y)$;*
- *(vi) $T$ is $\eta$-$h$-$C(x)-$ weakly pseudomonotonic with respect to $A$ and hemicontinuous on $D$;*
- *(vii) the graph of $W : D \to Y$, say $\operatorname{gph}(W)$, is closed in $X \times Y$, where $W : D \to Y$ is defined by $W(x) = Y \backslash (-\operatorname{int} C(x)), \forall x \in D$. Then, the GVVI (3.1) has a solution.*

First, we give the following Lemma.

LEMMA 3.4. *If all conditions in Theorem 3.1 hold, then the GVVI (3.1) is equivalent to the following GVVI: find* $\bar{x} \in D$ *such that*

$$\langle A\bar{s}, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -\operatorname{int} C(\bar{x}), \forall y \in D \text{ and } \forall \bar{s} \in T(y). \tag{3.3}$$

*Proof.* It follows from $(vi)$ that it is easy to see that every solution of the GVVI (3.1) is also a solution of the GVVI (3.3). Conversely, let $\bar{x} \in D$ be a solution of the GVVI (3.3), then

$$\langle A\bar{s}, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -\operatorname{int} C(\bar{x}), \forall y \in D \text{ and } \forall \bar{s} \in T(y).$$

For any given $y \in D$ and $\lambda \in (0,1)$, set $y_\lambda = (1-\lambda)\bar{x} + \lambda y$. It follows that for all $s_\lambda \in T(y_\lambda)$

$$\langle As_\lambda, \eta(y_\lambda,\bar{x})\rangle + h(y_\lambda,\bar{x}) \notin -\operatorname{int} C(\bar{x}). \tag{3.4}$$

Since $\eta(y,x)$ is affine with respect to $y$, $h(y,x)$ is $C(y)-$ convex with respect to $y$, and $\eta(x,x) = h(x,x) = 0$ for each $x \in D$, we get

$$\begin{aligned}
&\{\langle As_\lambda, \eta(y_\lambda,\bar{x})\rangle + h(y_\lambda,\bar{x})\} - \lambda\{\langle As_\lambda, \eta(y,\bar{x})\rangle + h(y,\bar{x})\} \\
&\quad \in \{(1-\lambda)\langle As_\lambda, \eta(\bar{x},\bar{x})\rangle + \lambda\langle As_\lambda, \eta(y,\bar{x})\rangle + (1-\lambda)h(\bar{x},\bar{x}) + \lambda h(y,\bar{x}) \\
&\qquad -\lambda\{\langle As_\lambda, \eta(y,\bar{x})\rangle + h(y,\bar{x})\} - C(\bar{x}) \\
&\quad = -C(\bar{x}).
\end{aligned} \tag{3.5}$$

In view of (3.4), (3.5) and Lemma 3.3, we get

$$\lambda\{\langle As_\lambda, \eta(y,\bar{x})\rangle + h(y,\bar{x})\} \notin -\operatorname{int} C(\bar{x}).$$

Since $C(\bar{x})$ is a cone, we have

$$\langle As_\lambda, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \notin -\operatorname{int} C(\bar{x}).$$

i.e.,

$$\langle As_\lambda, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \in W(\bar{x}). \tag{3.6}$$

Since $A$ is continuous and $T(y_\lambda)$, $T(\bar{x})$ are compact, by Lemma 3.2, for each $s_\lambda \in T(y_\lambda)$, we can find an $t_\lambda \in T(\bar{x})$ such that

$$\|s_\lambda - t_\lambda\| \leqslant H(T(y_\lambda), T(\bar{x})).$$

Since $T(\bar{x})$ is compact, without loss of generality, we may assume that $t_\lambda \to \bar{t} \in T(\bar{x})$ as $\lambda \to 0^+$. Moreover, we have

$$\begin{aligned}
\|s_\lambda - \bar{t}\| &\leqslant \|s_\lambda - t_\lambda\| + \|t_\lambda - \bar{t}\| \\
&\leqslant H(T(y_\lambda), T(\bar{x})) + \|t_\lambda - \bar{t}\|.
\end{aligned}$$

Since $H(T(y_\lambda), T(\bar{x})) \to 0$, as $\lambda \to 0^+$, then $s_\lambda \to \bar{t}$ as $\lambda \to 0^+$. Thus $\langle As_\lambda, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \to \langle A\bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x})$, as $\lambda \to 0^+$. It follows from (3.6) and the closedness of $W(\bar{x})$ that

$$\langle A\bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \in W(\bar{x}),$$

that is to say, $\bar{x}$ is a solution of the GVVI (3.1). This completes the proof.

The proof of Theorem 3.1. Define the point-set-set mappings $F_1, F_2 : D \to 2^D$ by

$$F_1(y) = \{x \in D \mid \exists t \in T(x), \langle At, \eta(y,x)\rangle + h(y,x) \notin -\operatorname{int} C(x)\}$$

and

$$F_2(y) = \{x \in D \mid \forall s \in T(y), \langle As, \eta(y,x)\rangle + h(y,x) \notin -\operatorname{int} C(x)\}$$

for each $y \in D$, respectively. The proof of which consists of four steps.

*Step 1.* We show that $F_1$ is a KKM-mapping. Note that $F_1(y) \neq \emptyset$ for each $y \in D$, since $y \in F_1(y)$. Let $z$ be in the convex hull of any finite subset $\{y_1, \cdots, y_n\}$ of $D$. Then, $z = \sum_{i=1}^n \lambda_i y_i \in D$ for some non-negative $\lambda_i$, $1 \leqslant i \leqslant n$, with $\sum_{i=1}^n \lambda_i = 1$. Suppose that $z \notin \bigcup_{i=1}^n F_1(y_i)$. Then $z \notin F_1(y_i)$, $\forall i = 1, \cdots, n$, and thus $\forall t \in T(z)$

$$\langle At, \eta(y_i,z)\rangle + h(y_i,z) \in -\operatorname{int} C(z), \forall i = 1, \cdots, n.$$

Since $C(z)$ is a convex cone, we have

$$\sum_{i=1}^n \lambda_i \{\langle At, \eta(y_i,z)\rangle + h(y_i,z)\} \in -\operatorname{int} C(z).$$

Again since $\eta(y,x)$ is affine with respect to $y$, $h(y,x)$ is $C(y)-$ convex with respect to $y$, and $\eta(x,x) = h(x,x) = 0$ for each $x \in D$, one has

$$\begin{array}{rcl} 0 & = & \langle At, \eta(z,z)\rangle + h(z,z) \\ & = & \langle At, \sum_{i=1}^n \lambda_i \eta(y_i,z)\rangle + h(\sum_{i=1}^n \lambda_i y_i, z) \\ & \in & \sum_{i=1}^n \lambda_i \langle At, \eta(y_i,z)\rangle + \sum_{i=1}^n \lambda_i h(y_i,z) - C(z) \\ & \in & -\operatorname{int} C(z) - C(z) \\ & = & -\operatorname{int} C(z). \end{array}$$

Thus, $0 \in -\operatorname{int} C(z)$, which is a contradiction. Therefore, $F_1$ is a KKM-mapping.

*Step 2.* By condition $(vi)$, we obtain that $F_1(y) \subseteq F_2(y)$ for all $y \in D$, and hence, $F_2$ is also a KKM-mapping.

*Step 3.* We show that $F_2(y)$ is closed and compact for all $y \in D$ and $\bigcap_{y \in D} F_2(y) \neq \emptyset$. In fact, let $\{x_\alpha\}$ be a net of $F_2(y)$ such that $x_\alpha$ converges to $x \in D$. Since $x_\alpha \in F_2(y)$, for all $\alpha$, and for all $s \in T(y)$, then

$$\langle As, \eta(y,x_\alpha)\rangle + h(y,x_\alpha) \notin -\operatorname{int} C(x_\alpha),$$

i.e.

$$(x_\alpha, \langle As, \eta(y,x_\alpha)\rangle + h(y,x_\alpha)) \in \operatorname{gph}(W).$$

By the conditions $(i)$ and $(iv)$, for all $s \in T(y)$ we have

$$\langle As, \eta(y,x_\alpha)\rangle + h(y,x_\alpha) \to \langle As, \eta(y,x)\rangle + h(y,x).$$

Thus

$$(x_\alpha, \langle As, \eta(y,x_\alpha)\rangle + h(y,x_\alpha)) \to (x, \langle As, \eta(y,x)\rangle + h(y,x)).$$

Therefore, $(x, \langle As, \eta(y,x)\rangle + h(y,x) \in \mathrm{gph}(W)$, which is closed in $X \times Y$. Hence $\langle As, \eta(y,x)\rangle + h(y,x) \notin -\mathrm{int}\, C(x)$ and $F_2(y)$ is closed. Since $D$ is compact, $F_2(y)$ is also compact for all $y \in D$. By Step 2, we know that $F_2$ is a KKM-mapping. It follows from Lemma 3.1, we have that $\bigcap_{y\in D} F_2(y) \neq \emptyset$.

*Step 4.* We prove that the GVVI (3.1) has a solution. From Lemma 3.4, we have $\bigcap_{y\in D} F_1(y) = \bigcap_{y\in D} F_2(y)$, and by Step 3, we obtain $\bigcap_{y\in D} F_2(y) \neq \emptyset$. Then $\bigcap_{y\in D} F_1(y) \neq \emptyset$, that is to say, the GVVI (3.1) has a solution. This completes the proof.

REMARK 3.2. From Theorem 3.1, it is easy to see that the set of solution for the GVVI (3.1) is $\bigcap_{y\in D} F_1(y) = \bigcap_{y\in D} F_2(y)$, which is nonempty closed and compact.

THEOREM 3.2. *Let $X, Y, C$ and $W$ be as in Theorem 3.1 and $D$ be a nonempty, closed and convex subset of $X$. Let $A : L(X,Y) \to L(X,Y)$, $T : D \to 2^{L(X,Y)}$, $\eta : D \times D \to D$ and $h : D \times D \to Y$. Assume that conditions $(i)$–$(vii)$ in Theorem 3.1 hold and the following coercive condition on $D$ is satisfied: there exists a compact subset $K$ of $X$ such that $y_0 \in K \cap D$ and for all $x \in D\backslash K$, $\exists t \in T(y_0)$,*

$$\langle At, \eta(y_0,x)\rangle + h(y_0,x) \in -\mathrm{int}\, C(x).$$

*Then, the GVVI (3.1) has a solution.*

*Proof.* As the proof in Theorem 3.1, we only need to prove that $F_2(y_0)$ is compact. From the coercive condition, it is clear that $F_2(y_0) \subseteq K$. Consider Step 3 in the proof of Theorem 3.1, $F_2(y_0)$ is closed. Since $K$ is compact, $F_2(y_0)$ is also compact. This completes the proof. □

COROLLARY 3.1. *Our results of Theorems 3.1 and 3.2 are different from the corresponding results in [9,10].*

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