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AN EXTENSION ON THE HARDY–HILBERT INTEGRAL INEQUALITY AND ITS APPLICATIONS

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Abstract. In this article, it is shown that some new extensions on the Hardy-Hilbert inequality related to exponent function can be established by introducing a parameter λ $(1-\frac{q}{p}<\lambda\leqslant 2$, $\frac{1}{p}+\frac{1}{q}=1$ and $p\geqslant q>1)$ and two exponent functions Aa^x $(A>0, a>1)$ and Bb^y $(B>0, b>1)$. In particular, for the case $p=2$, an extension of the Hilbert integral inequality is built. As an application, a new Hardy-Littlewood integral inequality is given.

1. Introduction and lemmas

Let $f(x), g(x)\geqslant 0, f(x)\in L^p(0,+\infty), g(x)\in L^q(0,+\infty), \frac{1}{p}+\frac{1}{q}=1$ and $p>1$. Then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy\leqslant \frac{\pi}{\sin\pi/p}\left\{\int_0^\infty f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty g^q(y)dy\right\}^{1/q} \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin\pi/p}$ is the best possible. And the equality in (1.1) holds if and only if $f(x)=0$ or $g(x)=0$. This is the famous Hardy-Hilbert integral inequality (see [1]). In particular, when $p=2$, the classical Hilbert's integral inequality is that

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy\leqslant \pi\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(y)\,dy\right\}^{1/2} \tag{1.2}$$

where the constant factor π is the best possible.

The both inequalities (1.1) and (1.2) are important in analysis and its applications (see [2]). Recently, the various extensions on the inequality (1.1) and (1.2) appeared in some papers (such as [3]-[7] etc.). They focalize on changing the denominator of the function of the left-hand side of (1.1). Such as the denominator $(x+y)$ is replaced by $(Ax+By)^\lambda$ in paper [4]; the denominator $(x+y)$ is replaced by x^t+y^t (t is a parameter

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which is independent of x and y) in the paper [5]; Generally, the denominator $(x+y)$ is replaced by $(xu(x)+yv(y))^{\lambda}$ in the paper [6] etc. Some new results in these papers were yielded. If the denominator of the function of the left-hand side of (1.1) is replaced by $(Aa^x+Bb^y)^{\lambda}$ and the integral area of it is replaced by $(-\infty,+\infty)\times(-\infty,+\infty)$,a new inequality established is significant in theory and applications. The main purpose of the present paper is to establishthe following inequality of the form

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(x)g(y)}{(Aa^x+Bb^y)^{\lambda}}dxdy \leqslant k(\lambda)\left\{\int_{-\infty}^{+\infty}\omega_p(\lambda,x)f^p(x)dx\right\}^{1/p}\left\{\int_{-\infty}^{+\infty}\omega_q(\lambda,x)g^q(x)dx\right\}^{1/q} \tag{1.3}$$

and to decide the coefficient $k(\lambda)$ and to prove $k(\lambda)$ to be the best possible, at same time to find the specific expression of the weight function $\omega_r(\lambda,x),(r=p,q)$. Evidently, a special kernel of exponential type and the interval of integration in (1.3) are different from those in the above -mentioned papers. In particular, it should be mentionable that Aa^x can not be replaced by $xu(x)$ in paper [6] and that Bb^y can not be also replaced by $yv(y)$ in paper [6], where u and v are non-negative differentiable functions, because $xu(x)$ and $yv(y)$ have no definitions in paper [6] when $x\leqslant 0$ and $y\leqslant 0$. In general, the method adopted by us has trait itself, explicitly, the proof of the optimization $k(\lambda)$ possesses new meanings.

For convenience, let $\alpha_p=1-\frac{2-\lambda}{p}$. Throughout this paper, we will frequently use the beta function of the form $B^*=B(\lambda-\alpha_p,\alpha_p)$. In order to prove our assertions we need the following lemmas.

LEMMA 1.1. *Let $r>1, 0\leqslant rs<1$ and $\lambda>1-rs$. Then*

$$\int_0^{\infty}\frac{1}{(1+t)^{\lambda}}\left(\frac{1}{t}\right)^{rs}dt=B(\lambda-(1-rs),1-rs), \tag{1.4}$$

where $B(p,q)$ is the beta function.

LEMMA 1.2. *Let $0\leqslant ps<1$ and $1-qs<\lambda\leqslant 2$. Define a function Φ by*

$$\Phi(s)=\{B(\lambda-(1-ps),1-ps)\}^{1/p}\{B(\lambda-(1-qs),1-qs)\}^{1/q} \tag{1.5}$$

where $B(m,n)$ is beta function. Then $\Phi(s)$ attains the minimum B^*, when $s=\frac{2-\lambda}{pq}$.

The proofs of the lemmas 1.1 and 1.2 have given in the paper [8]. It is omitted here.

2. Theorem and its corollaries

In the section, we will apply the above lemmas to build some new inequalities. For sake of convenience, we need also to define some functions.

$$\omega_p(\lambda,x) = (Aa^x)^{1-\lambda}(Aa^x \ln a)^{1-p} \text{ and} \omega_q(\lambda,x) = (Bb^x)^{1-\lambda}(Bb^x \ln b)^{1-q} \tag{2.1}$$

THEOREM 2.1. *Let $A,B>0$, $a,b>1$, $f(x),g(x)\geqslant 0$, $\frac{1}{p}+\frac{1}{q}=1$, $p\geqslant q>1$, $1-\frac{q}{p}<\lambda\leqslant 2$. If $\int\limits_{-\infty}^{+\infty}\left\{(a^x)^{2-\lambda-p}\right\}f^p(x)\,dx<+\infty$ and $\int\limits_{-\infty}^{+\infty}\left\{(b^x)^{2-\lambda-q}\right\}g^q(x)\,dx<+\infty$, then*

$$\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\frac{f(x)g(y)}{(Aa^x+Bb^y)^\lambda}dxdy \leqslant \mu B^*\left\{\int\limits_{-\infty}^{+\infty}\left\{(a^x)^{2-\lambda-p}\right\}f^p(x)dx\right\}^{1/p} \times\left\{\int\limits_{-\infty}^{+\infty}\left\{(b^x)^{2-\lambda-q}\right\}g^q(x)\,dx\right\}^{1/q}, \tag{2.2}$$

where $\mu=\left(\frac{A^{1-\lambda}}{B\ln b}\right)^{\frac{1}{p}}\left(\frac{B^{1-\lambda}}{B\ln a}\right)^{\frac{1}{q}}$, B^* is beta function and the constant factor μB^* is the best possible. The equality in (2.2) holds if and only if $f(x)=0$ or $g(x)=0$.

Proof. Let $s>p$, $f(x)=F(x)\{Aa^x\ln a\}^{1/q}$ and $g(y)=G(y)\{Bb^y\ln b\}^{1/p}$. We firstly define two functions:

$$\alpha=\frac{F(x)\{Bb^y\ln b\}^{1/p}}{(Aa^x+Bb^y)^{\lambda/p}}\left(\frac{Aa^x}{Bb^y}\right)^s \text{ and } \beta=\frac{G(y)\{Aa^x\ln a\}^{1/q}}{(Aa^x+Bb^y)^{\lambda/q}}\left(\frac{Bb^y}{Aa^x}\right)^s \tag{2.3}$$

Let's apply Hölder's inequality to estimate the right hand side of (2.2) as follows:

$$\begin{aligned}
&\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\frac{f(x)g(y)}{(Aa^x+Bb^y)^\lambda}dxdy \\
&=\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\frac{F(x)\{Bb^y\ln b\}^{1/p}}{(Aa^x+Bb^y)^{\lambda/p}}\left(\frac{Aa^x}{Bb^y}\right)^s\frac{G(y)\{Aa^x\ln a\}^{1/q}}{(Aa^x+Bb^y)^{\lambda/q}}\left(\frac{Bb^y}{Aa^x}\right)^s dxdy \\
&=\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\alpha\beta dxdy \leqslant \left\{\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\alpha^p dxdy\right\}^{1/p}\left\{\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\beta^q dxdy\right\}^{1/q}
\end{aligned} \tag{2.4}$$

It is easy to deduce that

$$\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\alpha^p dxdy=\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\frac{F^p(x)(Bb^y\ln b)}{(Aa^x+Bb^y)^\lambda}\left(\frac{Aa^x}{Bb^y}\right)^{ps}dxdy=\int\limits_{-\infty}^{+\infty}\varpi(p,\lambda,x)\,F^p(x)dx$$

Based on Lemma 1.1 we compute the weight function ϖ as follows:

$$\varpi(p,\lambda,x) = \int_{-\infty}^{+\infty} \frac{Bb^y \ln b}{(Aa^x + Bb^y)^\lambda} \left(\frac{Aa^x}{Bb^y}\right)^{ps} dy = \int_0^{+\infty} \frac{(Aa^x)^{1-\lambda}}{(1+t)^\lambda} \left(\frac{1}{t}\right)^{ps} dt$$
$$= (Aa^x)^{1-\lambda} B(\lambda - (1-ps), 1-ps)$$

Notice that $F(x) = \{Aa^x \ln a\}^{-1/q} f(x)$, hence we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \alpha^p dxdy = B(\lambda - (1-ps), 1-ps) \int_{-\infty}^{+\infty} \omega_p(\lambda,x) f^p(x) dx \tag{2.5}$$

where $\omega_p(\lambda,x)$ is defined by (2.1).

Similarly, we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \beta^q dxdy = B(\lambda - (1-qs), 1-qs) \int_{-\infty}^{\infty} \omega_q(\lambda,x)\, g^q(x) dx \tag{2.6}$$

where $\omega_q(\lambda,x)$ is defined by (2.1). Substituting (2.5) and (2.6) into (2.4), we obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{f(x)g(y)}{(Aa^x + Bb^y)^\lambda} \leqslant \Phi(s) \left\{\int_{-\infty}^{+\infty} \omega_p(\lambda,x) f^p(x) dx\right\}^{1/p} \times \left\{\int_{-\infty}^{+\infty} \omega_q(\lambda,x)\, g^q(x) dx\right\}^{1/q} \tag{2.7}$$

where $\Phi(s)$ is defined by (1.5).

It follows from Lemma 1.2 that the minimum of $\Phi(s)$ is B^*, when $s = \frac{2-\lambda}{pq}$, where λ satisfies the constraint $1 - \frac{q}{p} < \lambda \leqslant 2$. So we obtain from (2.7) that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{f(x)g(y)}{(Aa^x+Bb^y)^\lambda} dxdy \leqslant B^* \left\{\int_{-\infty}^{+\infty} \omega_p(\lambda,x) f^p(x) dx\right\}^{1/p} \times \left\{\int_{-\infty}^{+\infty} \omega_q(\lambda,x)\, g^q(x) dx\right\}^{1/q}. \tag{2.8}$$

where $\omega_r(\lambda,x)$ is defined by (2.1), $r = p,q$.

It remains to show that B^* in (2.8) is the best possible. Define two functions by

$$\tilde{f}(x) = \begin{cases} 0, & x \in (-\infty, 1) \\ (Aa^x)^{-(2-\lambda+\varepsilon)/p} (Aa^x \ln a), & x \in [1, +\infty) \end{cases}$$

$$\tilde{g}(y) = \begin{cases} 0, & y \in (-\infty, 1) \\ (Bb^y)^{-(2-\lambda+\varepsilon)/q} (Bb^y \ln b), & y \in [1, +\infty) \end{cases}$$

Assume that $0 < \varepsilon < (\lambda - 1) + \frac{q}{2p}$, then

$$\int_{-\infty}^{+\infty} (Aa^x)^{1-\lambda} (Aa^x \ln a)^{1-p} \tilde{f}^p(x)dx = \int_1^{+\infty} (Aa^x)^{-1-\varepsilon} d(Aa^x) = \frac{1}{\varepsilon}.$$

Similarly, we have

$$\int_{-\infty}^{\infty} (Bb^y)^{1-\lambda} (Bb^y \ln b)^{1-q} \tilde{g}^q(y)dy = \frac{1}{\varepsilon}.$$

If B^* is not the best possible, then there exists $k > 0$ and k less than B^* such that

$$\begin{aligned} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\tilde{f}(x)\tilde{g}(y)}{(Aa^x + Bb^y)^\lambda} dxdy &< k \left(\int_{-\infty}^{\infty} \omega_p(\lambda, x)\tilde{f}^p(x)dx \right)^{1/p} \left(\int_{-\infty}^{\infty} \omega_q(\lambda, x)\tilde{g}^q(x)dx \right)^{1/q} \\ &= k \left\{ \int_{-\infty}^{+\infty} (Aa^x)^{1-\lambda} (Aa^x \ln a)^{1-p} \tilde{f}^p(x)dx \right\}^{1/p} \\ &\quad \times \left\{ \int_{-\infty}^{\infty} (Bb^y)^{1-\lambda} (Bb^y \ln b)^{1-q} \tilde{g}^q(y)dy \right\}^{1/q} = \frac{k}{\varepsilon} \end{aligned} \tag{2.9}$$

where $\omega_r(\lambda, x)$ is defined by (2.1), $r = p, q$.

On the other hand, we have

$$\begin{aligned} &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\tilde{f}(x)\tilde{g}(y)}{(Aa^x + Bb^y)^\lambda} dxdy \\ &= \int_1^{\infty}\int_1^{\infty} \frac{\left\{(Aa^x)^{-(2-\lambda+\varepsilon)/p} (Aa^x \ln a)\right\} \left\{(Bb^y)^{-(2-\lambda+\varepsilon)/q} (Bb^y \ln b)\right\}}{(Aa^x + Bb^y)^\lambda} dxdy \\ &= \int_1^{\infty} \left\{ \int_1^{\infty} \frac{(Bb^y)^{-(2-\lambda+\varepsilon)/q} (Bb^y \ln b)}{(Aa^x + Bb^y)^\lambda} dy \right\} \left\{ (Aa^x)^{-(2-\lambda+\varepsilon)/p} (Aa^x \ln a) \right\} dx \\ &= \int_1^{+\infty} \left\{ \int_{Bb/Aa^x}^{+\infty} \frac{1}{(1+t)^\lambda} \left(\frac{1}{t}\right)^{-(2-\lambda+\varepsilon)/q} dt \right\} \left\{ (Aa^x)^{-1-\varepsilon} (Aa^x \ln a) \right\} dx \\ &= \frac{1}{\varepsilon} \int_{Bb/Aa^x}^{\infty} \frac{1}{(1+t)^\lambda} \left(\tfrac{1}{t}\right)^{(2-\lambda+\varepsilon)/q} dt \end{aligned}$$

If the lower limit Bb/Aa^x of the integral is replaced by zero, thenthe resulting error is smaller than $\frac{(Bb/Aa^x)^\alpha}{\alpha}$, where α is positive and independent of ε. In fact, we have

$$\int_0^{Bb/Aa^x} \frac{1}{(1+t)^\lambda}\left(\frac{1}{t}\right)^{(2-\lambda+\varepsilon)/q} dt < \int_0^{Bb/Aa^x} t^{-(2-\lambda+\varepsilon)/q}dt = \frac{(Bb/Aa^x)^\beta}{\beta}$$

where $\beta = 1-(2-\lambda+\varepsilon)/q$. If $0<\varepsilon<(\lambda-1)+\frac{q}{2p}$, then we may take α such that

$$\alpha = 1-\frac{(2-\lambda)+((\lambda-1)+q/2p)}{q} = \frac{1}{2p}.$$

Consequently, we get

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\tilde{f}(x)\tilde{g}(y)}{(Aa^x+Bb^y)^\lambda}dxdy > \frac{1}{\varepsilon}\left\{B^* + \circ(1)\right\}. \quad (\varepsilon\to 0) \tag{2.10}$$

Clearly, when ε is small enough, the inequality (2.9) is in contradiction with (2.10). Therefore, B^* is the best possible value of which the inequality (2.8) keeps valid.

As a result, it follows from (2.8) that the inequality (2.2) is yielded after simplifications and the constant factor μB^* is the best possible. And it is obvious that the equality in (2.2) holds if and only if $f(x)=0$ or $g(x)=0$. The proof of Theorem is completed.

In particular, when $\lambda = 1$, B^* is reduced to $\frac{\pi}{\sin\pi/p}$, we have the following result:

COROLLARY 2.2. *Let* $A,B>0$, $a,b>1$, $f(x),g(x)\geqslant 0$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $p>1$. *If* $\int_{-\infty}^{+\infty}\left\{(a^x)^{1-p}\right\}f^p(x)dx<+\infty$ *and* $\int_{-\infty}^{+\infty}\left\{(b^x)^{1-q}\right\}g^q(x)dx<+\infty$, *then*

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{f(x)g(y)}{Aa^x+Bb^y}dxdy \leqslant \frac{\pi}{\mu_1\sin\pi/p}\left\{\int_{-\infty}^{+\infty}\left\{(a^x)^{1-p}\right\}f^p(x)dx\right\}^{1/p} \times\left\{\int_{-\infty}^{+\infty}\left\{(b^x)^{1-q}\right\}g^q(x)dx\right\}^{1/q}. \tag{2.11}$$

where $\mu_1 = (A\ln b)^{1/q}(B\ln a)^{1/p}$, *and the constant factor* $\frac{\pi}{\mu_1\sin\pi/p}$ *is the best possible.*

When $p=2$, we get a new Hilbert integral type inequality.

COROLLARY 2.3. *Let* $A,B>0$, $a,b>1$, $0<\lambda\leqslant 2$. *If* $\int_{-\infty}^{+\infty} a^{-\lambda x} f^2(x)dx<+\infty$ *and* $\int_{-\infty}^{+\infty} b^{-\lambda x} g^2(x)dx<+\infty$, *then*

$$(i)\qquad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(x)g(y)}{(Aa^x+Bb^y)^\lambda}dxdy\leqslant\frac{B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)}{\mu_2}\left\{\int_{-\infty}^{+\infty}a^{-\lambda x}f^2(x)dx\right\}^{1/2}\times\left\{\int_{-\infty}^{+\infty}b^{-\lambda x}g^2(x)\,dx\right\}^{1/2}. \tag{2.12}$$

where $\mu_2=\left((AB)^\lambda\ln a\ln b\right)^{1/2}$ *and* $B(\frac{\lambda}{2},\frac{\lambda}{2})$ *is beta function, and the constant factor* $\frac{B(\frac{\lambda}{2},\frac{\lambda}{2})}{\mu_2}$ *is the best possible. In particular,*

(ii) For $\lambda=1$, *we have*

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(x)g(y)}{Aa^x+Bb^y}dxdy\leqslant\frac{\pi}{(AB\ln a\ln b)^{1/2}}\left\{\int_{-\infty}^{+\infty}a^{-x}f^2(x)dx\right\}^{1/2}\times\left\{\int_{-\infty}^{+\infty}b^{-x}g^2(x)dx\right\}^{1/2}. \tag{2.13}$$

In particular, when $A=B=1$ *and* $a=b$, *we have*

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(x)g(y)}{a^x+a^y}dxdy\leqslant\frac{\pi}{\ln a}\left\{\int_{-\infty}^{+\infty}a^{-x}f^2(x)dx\right\}^{1/2}\left\{\int_{-\infty}^{+\infty}a^{-x}g^2(x)dx\right\}^{1/2}. \tag{2.14}$$

(iii) For $\lambda=2$, *we have*

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(x)g(y)}{(Aa^x+Bb^y)^2}dxdy\leqslant\frac{1}{AB\,(\ln a\ln b)^{1/2}}\left\{\int_{-\infty}^{+\infty}a^{-2x}f^2(x)dx\right\}^{1/2}\times\left\{\int_{-\infty}^{+\infty}b^{-2x}g^2(x)dx\right\}^{1/2}. \tag{2.15}$$

And the constant factors $\frac{\pi}{(AB\ln a\ln b)^{1/2}}$ *in (2.13),* $\frac{\pi}{\ln a}$ *in (2.14) and* $\frac{1}{AB(\ln a\ln b)^{1/2}}$ *in (2.15) are the best possible.*

3. Application

In this section, we will give a new Hardy-Littlewood integral type inequality.

Let $f(x) \in L^2(0,1)$ and $f(x) \neq 0$. If $a_n = \int_0^1 x^n f(x)dx$, $n = 0,1,2,\cdots$, then we have the Hardy-Littlewood's inequality (see [1]) of the form

$$\sum_{n=0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x)\,dx \tag{3.1}$$

where π is the best constant that the inequality (3.1) keeps valid. In our previous paper [9], the inequality (3.1) was extended and established the following inequality:

$$\int_0^{\infty} f^2(x)dx < \pi \int_0^1 h^2(x)dx \tag{3.2}$$

where $f(x) = \int_0^1 t^x h(x)dx$, $x \in [0,+\infty)$

The inequality (3.2) is called the Hardy-Littlewood integral inequality. Afterwards the inequality (3.2) was refined into the following form (see [10]):

$$\int_0^{\infty} f^2(x)dx \leqslant \pi \int_0^1 th^2(t)\,dt. \tag{3.3}$$

We will further extend the inequality (3.3) here.

THEOREM 3.1. *Let $h(t) \in L^2(0,1)$, $h(t) \neq 0$ and $a > 1$. Define a function by*

$$f(x) = \int_0^1 t^{a^x} |h(t)|\,dt \quad x \in (-\infty,+\infty)$$

If $0 < \int_{-\infty}^{+\infty} (a^x)^{1-r} f^r(x)dx < +\infty$, $(r = p,q)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p \geqslant q > 1$, then

$$\left(\int_{-\infty}^{+\infty} f^2(x)dx\right)^2 < \frac{\pi}{(\sin \pi/p)\ln a} \left(\int_{-\infty}^{+\infty} (a^x)^{1-p} f^p(x)dx\right)^{1/p} \times \left(\int_{-\infty}^{+\infty} (a^x)^{1-q} f^q(x)dx\right)^{1/q} \int_0^1 th^2(t)\,dt \tag{3.4}$$

where the constant factor $\frac{\pi}{(\sin\pi/p)\ln a}$ *in (3.4) is the best possible.*

Proof. Let us write $f^2(x)$ in form:

$$f^2(x) = \int_0^1 f(x)t^{a^x}\,|h(t)|\,dt.$$

Apply, in turn, Schwarz's inequality and Corollary 2.2, we have

$$\begin{aligned}
\left(\int_{-\infty}^{+\infty} f^2(x)dx\right)^2 &= \left\{\int_{-\infty}^{\infty}\left(\int_0^1 f(x)t^{a^x}\,|h(t)|\,dt\right)dx\right\}^2 \\
&= \left\{\int_0^1\left(\int_{-\infty}^{+\infty} f(x)t^{a^x-1/2}dx\right)t^{1/2}\,|h(t)|\,dt\right\}^2 \\
&\leqslant \int_0^1\left(\int_{-\infty}^{+\infty} f(x)t^{a^x-1/2}dx\right)^2 dt\int_0^1 th^2(t)\,dt \\
&= \int_0^1\left(\int_{-\infty}^{+\infty} f(x)t^{a^x-1/2}dx\right)\left(\int_{-\infty}^{+\infty} f(y)t^{a^y-1/2}dy\right)dt\int_0^1 th^2(t)\,dt \\
&= \int_0^1\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x)f(y)t^{a^x+a^y-1}dxdy\right)dt\int_0^1 th^2(t)\,dt \\
&= \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(x)f(y)}{a^x+a^y}dxdy\right)\int_0^1 th^2(t)\,dt \\
&\leqslant \frac{\pi}{(\sin\pi/p)\ln a}\left\{\int_{-\infty}^{+\infty}(a^x)^{1-p}f^p(x)dx\right\}^{1/p} \\
&\quad\times\left\{\int_{-\infty}^{+\infty}(a^y)^{1-q}f^q(y)dy\right\}^{1/q}\int_0^1 th^2(t)\,dt.
\end{aligned}\tag{3.5}$$

Since $h(t)\neq 0$, $f^2(x)\neq 0$, it is impossible to take equality in (3.5). The proof of the theorem is completed. □

In particular, when $p=2$, we have the following result:

COROLLARY 3.2. *Let* $a>1$, $h(t)$ *and* $f(x)$ *be the functions with the assumptions* *as the theorem 3.1. If* $0 < \int\limits_0^\infty (a^x)^{-1} f^2(x)dx < +\infty$*, then*

$$\left(\int\limits_{-\infty}^{+\infty} f^2(x)dx\right)^2 < \frac{\pi}{\ln a}\left(\int\limits_{-\infty}^{+\infty} (a^x)^{-1} f^2(x)dx\right)\int\limits_0^1 th^2(t)\,dt \tag{3.6}$$

where the constant factor $\frac{\pi}{\ln a}$ *in (3.6) is the best possible.*

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