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THE EQUIVALENCE BETWEEN MANN AND IMPLICIT MANN ITERATIONS

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Abstract. We shall prove the equivalence bewteen the convergences of Mann and implicit Mann iterations dealing with various classes of non-Lipschitzian operators.

1. Introduction

Let X be a real Banach space, B be a nonempty, convex subset of X, and $T : B \to B$ be an operator. Let $u_0, x_0 \in B$. The following iteration is known as Mann iteration, see [3]:

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n. \tag{1}$$

For $z_0 \in B$, the Ishikawa iteration [2], is defined by

$$\begin{aligned} z_{n+1} &= (1 - \alpha_n)z_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)z_n + \beta_n T z_n. \end{aligned} \tag{2}$$

The sequences $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ satisfy

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty. \tag{3}$$

For the rest of the paper, we suppose that there exists $(I - tT)^{-1}$, for all $t \in (0,1)$ such that the following iteration, known as implicit Mann iteration (see [11]), is well defined:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_{n+1}. \tag{4}$$

REMARK 1.1. In order to have a well defined sequence $\{x_n\}$, the existence of $(I - \lambda T)^{-1}, \forall \lambda \in (0,1)$, is crucial. Take $X = \mathbb{R}$, $Tx = x^4$, $\alpha_0 = 1/3$, $x_0 = 4$, to see that there are no real values for x_1 that satisfy (4), i.e. $x_1 = (1 - \alpha_0)\, x_0 + \alpha_0 x_1^4$.

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The map $J : X \to 2^{X^*}$ given by $J(x) := \{f \in X^* : \langle x,f \rangle = \|x\|^2, \|f\| = \|x\|\}, \forall x \in X,$ is called *the normalized duality mapping*. It is easy to see that

$$\langle y,j(x)\rangle \leqslant \|x\|\,\|y\|\,, \forall x,y \in X, \forall j(x) \in J(x). \tag{5}$$

For sake of simplicity we shall denote by Ψ the following class:

$$\Psi := \{\psi \mid \psi : [0,+\infty) \to [0,+\infty) \text{ is a strictly increasing map such that } \psi(0) = 0\}.$$

DEFINITION 1.2. Let X be a real Banach space. Let B be a nonempty subset of X. A map $T : B \to B$ is called *uniformly pseudocontractive* if there exist maps $\psi \in \Psi$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leqslant \|x-y\|^2 - \psi(\|x-y\|), \forall x,y \in B. \tag{6}$$

Taking $\psi(a) := \psi(a)\cdot a, \forall a \in [0,+\infty),\ (\psi \in \Psi),$ we get the usual definition of a $\psi-$ strongly pseudocontractive map. Taking $\psi(a) := \gamma \cdot a^2, \gamma \in (0,1),\ \forall a \in [0,+\infty),$ $(\psi \in \Psi),$ we get the usual definition of a strongly pseudocontractive map. If $\gamma := 0,$ then we get the definition of a pseudocontractive map.

The convergence to a fixed point of (4), dealing with strongly respectively uniformly pseudocontractive maps was studied in [11] and [9]. Moreover, examples in which Mann iteration does not converge, while implicit Mann iteration converges, and vice versa, were given in [10].

A reasonable conjecture is that the Ishikawa iteration methods satisfying (3) and the corresponding Mann iterations are equivalent for all maps for which either method provides convergence to a fixed point. In an attempt to verify this conjecture, in a series of papers [4], [5], [6], [7], [8] we have shown the equivalence for several classes of maps. We shall prove here the equivalence between Mann and implicit Mann iteration for the most general class of operators (6).

We recall the following result from [1].

LEMMA 1.3. [1] *If X is a real normed space, then the following relation is true*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle\ ,\ \forall x,y \in X, \forall j(x+y) \in J(x+y). \tag{7}$$

2. Sequences supplied by inequalities

LEMMA 2.1. *Let $\{a_n\}$ be a nonnegative bounded sequence which satisfies the following inequality*

$$a_{n+1} \leqslant a_n - 4\alpha_n \psi(a_{n+1}) + 4\alpha_n(\alpha_n + M\sigma_n), \forall n \geqslant n_0, \tag{8}$$

where $M > 0,\ \varepsilon_n, \sigma_n \geqslant 0,\ \forall n \in \mathbb{N},\ \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \varepsilon_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Proof. Since $\{a_n\}$ is bounded, there exists $m > 0$ such that: $a_n \leqslant m, \forall n \in \mathbb{N}$. Set $M := \max\{m, M\}$.

Denote $a := \liminf a_n$. We shall prove that $a = 0$. Suppose that $a > 0$. Thus there exists an $N_1 \in \mathbb{N}$ such that

$$a_n \geqslant \frac{a}{2}, \ \forall n \geqslant N_1.$$

Because $\lim_{n\to\infty} \varepsilon_n = 0$, there exists an $N_2 \in \mathbb{N}$ such that

$$\sigma_n \leqslant \frac{\psi\left(\frac{a}{2}\right)}{3M}, \quad \alpha_n \leqslant \frac{\psi\left(\frac{a}{2}\right)}{3}, \ \forall n \geqslant N_2.$$

Set $N_0 := \max\{N_1, N_2\}$. Using $a_n \geqslant \frac{a}{2}$ we get $-\psi\left(a_{n+1}\right) \leqslant -\psi\left(\frac{a}{2}\right)$, which leads to

$$\begin{aligned} a_{n+1} &\leqslant a_n - 4\alpha_n \psi\left(a_{n+1}\right) + 4\alpha_n\left(\alpha_n + M\sigma_n\right) \\ &\leqslant a_n - 4\alpha_n \psi\left(\frac{a}{2}\right) + 4\alpha_n\left(\alpha_n + M\sigma_n\right) \\ &\leqslant a_n - 4\alpha_n \psi\left(\frac{a}{2}\right) + 4\alpha_n\left(\frac{\psi\left(\frac{a}{2}\right)}{3} + M\frac{\psi\left(\frac{a}{2}\right)}{3M}\right) \\ &= a_n - \frac{8}{3}\alpha_n \psi\left(\frac{a}{2}\right). \end{aligned}$$

Thus, we have $\alpha_n \frac{8}{3}\psi\left(\frac{a}{2}\right) \leqslant a_n - a_{n+1}$, which implies that $\sum \alpha_n < \infty$, contradicting (3). Thus there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $\lim_{j\to\infty} a_{n_j} = 0$. Fix $\varepsilon > 0$. Then there exists an $n_3 \in \mathbb{N}$ such that

$$a_{n_j} < \frac{\varepsilon}{4}, \ \forall j \geqslant n_3.$$

Also, there exists an $n_4 \in \mathbb{N}$ such that

$$\sigma_n < \frac{\psi\left(\frac{\varepsilon}{4}\right)}{2M}, \ \alpha_n < \frac{\psi\left(\frac{\varepsilon}{4}\right)}{2} \ \forall n \geqslant n_{4.}$$

Set $n_0 := \max\{n_3, n_4, N_0\}$. We have $a_{n_j+k} < \frac{\varepsilon}{4}, \forall k > 0$. Otherwise, for a fixed k we have $a_{n_j+k} < \frac{\varepsilon}{4}$ and $a_{n_j+k+1} \geqslant \frac{\varepsilon}{4}$, which leads to the following contradiction:

$$\begin{aligned} \frac{\varepsilon}{4} \leqslant a_{n_j+k+1} &\leqslant a_{n_j+k} - 4\alpha_{n_j+k}\psi\left(a_{n_j+k+1}\right) + 4\alpha_{n_j+k}\left(\alpha_{n_j+k} + M\sigma_{n_j+k}\right) \\ &\leqslant a_{n_j+k} - 4\alpha_{n_j+k}\psi\left(\frac{\varepsilon}{4}\right) + 4\alpha_{n_j+k}\left(\frac{\psi\left(\frac{\varepsilon}{4}\right)}{2} + M\frac{\psi\left(\frac{\varepsilon}{4}\right)}{2M}\right) \\ &= a_{n_j+k} < \frac{\varepsilon}{4}, \end{aligned}$$

and $a_{n_j+k} < \frac{\varepsilon}{4}, \forall k > 0$, so that $\lim_{n\to\infty} a_n = 0$. □

REMARK 2.2. Let $\{a_n\}$ be a nonnegative bounded sequence which satisfies the inequality

$$a_{n+1} \leqslant \frac{(1-\alpha_n)^2}{1-2\alpha_n}a_n - \frac{2\alpha_n}{1-2\alpha_n}\psi\left(a_{n+1}\right) + \frac{2M\alpha_n}{1-2\alpha_n}\varepsilon_n, \ \forall n \geqslant n_0, \tag{9}$$

where $\varepsilon_n \geqslant 0,\ \forall n \in \mathbb{N},\ \sum_{n=0}^{\infty} \alpha_n = \infty,$ and $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \varepsilon_n = 0.$ Then $\lim_{n\to\infty} a_n = 0.$

Proof. Note that (9) is equivalent to

$$\begin{aligned} a_{n+1} &\leqslant \frac{(1-\alpha_n)^2}{1-2\alpha_n} a_n - \frac{2\alpha_n}{1-2\alpha_n}\psi(a_{n+1}) + \frac{2M\alpha_n}{1-2\alpha_n}\varepsilon_n \\ &= a_n - \frac{2\alpha_n}{1-2\alpha_n}\psi(a_{n+1}) + \frac{2\alpha_n}{1-2\alpha_n}(\alpha_n + M\varepsilon_n) \\ &\leqslant a_n - 4\alpha_n\psi(a_{n+1}) + 4\alpha_n(\alpha_n + M\varepsilon_n). \end{aligned}$$

Note that $\frac{1}{1-2\alpha_n} \leqslant 4,\ \forall n \geqslant n_0$. Set $\sigma_n = \varepsilon_n,$ to obtain (8); and by Lemma 2.1 to reach the above conclusion. □

3. Main result

THEOREM 3.1. *Let X be a real Banach space, B be a nonempty, convex subset of X, and let $T : B \to B$ be a uniformly continuous and uniformly pseudocontractive map with $T(B)$ bounded. Let x^* be the fixed point of T. If there exists $(I - tT)^{-1}$, for all $t \in (0,1)$, the sequences $\{\alpha_n\}, \{\beta_n\}$ satisfy (3), and $\{u_n\}, \{x_n\}$ are bounded, then the following are equivalent:*
(i) the Mann iteration (1) converges to x^*,
(ii) the implicit Mann iteration (4) converges to x^*.

Proof. The uniqueness of the fixed point comes from (6). Suppose that $\lim_{n\to\infty}\ u_n = x^*$. Using

$$\lim_{n\to\infty} \|x_n - u_n\| = 0, \tag{10}$$

and

$$0 \leqslant \|x^* - x_n\| \leqslant \|u_n - x^*\| + \|x_n - u_n\|$$

we get $\lim_{n\to\infty} x_n = x^*$. Conversely, suppose that $\lim_{n\to\infty}\ x_n = x^*$. Then

$$0 \leqslant \|x^* - u_n\| \leqslant \|x_n - x^*\| + \|x_n - u_n\| \to 0,$$

leads to $\lim_{n\to\infty} x_n = x^*$. The proof is complete if we prove relation (10).

Set $M = \sup_n (\|u_n\|, \|Tu_n\|, \|x_n\|)$. Observe that $M < \infty$. Using (1), (4) and (5) we get

$$\begin{aligned} \|x_{n+1} - u_{n+1}\|^2 &= \|(1-\alpha_n)(x_n - u_n) + \alpha_n(Tx_{n+1} - Tu_n)\|^2 \\ &\leqslant (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n \langle Tx_{n+1} - Tu_n, j(x_{n+1} - u_{n+1})\rangle \\ &= (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n \langle Tx_{n+1} - Tu_{n+1}, j(x_{n+1} - u_{n+1})\rangle \\ &\quad + 2\alpha_n \langle Tu_{n+1} - Tu_n, j(x_{n+1} - u_{n+1})\rangle \\ &\leqslant (1-\alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n \|x_{n+1} - u_{n+1}\|^2 - 2\alpha_n\psi(\|x_{n+1} - u_{n+1}\|) \\ &\quad + \alpha_n \|Tu_{n+1} - Tu_n\| \|x_{n+1} - u_{n+1}\|. \end{aligned}$$

Thus, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have $\alpha_n < \frac{1}{2}$, and hence,

$$\|x_{n+1} - u_{n+1}\|^2 \leqslant \frac{(1-\alpha_n)^2}{1-2\alpha_n} \|x_n - u_n\|^2 - \frac{2\alpha_n}{1-2\alpha_n} \psi\left(\|x_{n+1} - u_{n+1}\|\right) + \frac{2M\alpha_n}{1-2\alpha_n} \|Tu_{n+1} - Tu_n\|.$$

Note that

$$\lim_{n\to\infty} \|Tu_{n+1} - Tu_n\| = 0 \tag{11}$$

holds independently of (i) or (ii), because

$$\|u_{n+1} - u_n\| = \alpha_n \|-u_n + Tu_n\| \leqslant 2M\alpha_n \to 0,\ n \to \infty. \tag{12}$$

Observe that $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$ and the uniform continuity of T lead to (11). Denote by $a_n := \|x_n - u_n\|$ and use Remark 2.2 , to obtain $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \|x_n - u_n\| = 0.$ □

4. Further results

Let I denote the identity map.

DEFINITION 4.1. The map $S : X \to X$ is called *uniformly accretive* if there exist the maps $\psi \in \Psi$ and respectively $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y)\rangle \geqslant \psi(\|x-y\|),\ \forall\, x, y \in X. \tag{13}$$

Taking $\psi(a) := \psi(a) \cdot a, \forall a \in [0,+\infty),$ $(\psi \in \Psi),$ we get the usual definition of a $\psi-$ strongly accretive map. Taking $\psi(a) := \gamma \cdot a^2, \gamma \in (0,1),$ $\forall a \in [0,+\infty),$ $(\psi \in \Psi),$ we get the usual definition of the strongly accretive map. If $\gamma := 0,$ then we get the definition of an accretive map.

REMARK 4.2.

1. The operator T is a (uniformly, $\psi-$ strongly, strongly) pseudocontractive map if and only if $(I-T)$ is a (uniformly, $\psi-$ strongly, strongly) accretive map.

2. Let $T, S : X \to X,$ and $f \in X$ be given. A fixed point for the map $Tx = f + (I-S)x, \forall x \in X$ is a solution for $Sx = f,$ and vice versa.

3. Consider (1) and (4) with $Tx = f + (I-S)x$ and x^* the solution of $Sx = f,$ in order to obtain the equivalence result for the (uniformly, $\psi-$ strongly, strongly) accretive.

4. Let $f \in X$ be given. If the operator S is accretive, then $f - S$ is a strongly pseudocontractive map.

5. Let $T, S : X \to X$. A fixed point for the map $Tx = f - Sx, \forall x \in X$ is a solution for $x + Sx = f$ and conversely.

6. Consider (1) and (4) with $Tx = f - Sx,$ and x^* the solution of $x + Sx = f,$ in order to obtain the equivalence result for the accretive case.

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